Math 2270-2, Quiz 8

Tuesday August 2, 2016

Instructions

This is a 20-minute quiz. It has 2 questions on 3 pages for a total of 20 points.

The next two problems concern the matrix

$$\boldsymbol{A} := \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & 0 \end{bmatrix}.$$

1. (10 points) Use the Gram–Schmidt method in order to find an orthonormal basis for $\mathrm{Col}(\boldsymbol{A})$.

   Solution. Write $\boldsymbol{A} = [\boldsymbol{x}_1 , \boldsymbol{x}_2]$, and apply Gram–Schmidt to obtain

$$\boldsymbol{v}_1 = \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},$$

   and

$$\boldsymbol{v}_2 = \boldsymbol{x}_2 - \frac{\boldsymbol{x}_2 \cdot \boldsymbol{v}_1}{\|\boldsymbol{v}_1\|^2}\boldsymbol{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{0-1+0}{1+1+1}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0+\frac{1}{3} \\ 1-\frac{1}{3} \\ 0+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}.$$

   $\{\boldsymbol{v}_1 , \boldsymbol{v}_2\}$ is a basis for $\mathrm{Col}(\boldsymbol{A})$. To find an O.N. basis, we normalize:

$$\boldsymbol{u}_1 := \frac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Similarly,

$$\boldsymbol{u}_2 = \frac{\boldsymbol{v}_2}{\|\boldsymbol{v}_2\|} = \sqrt{\frac{3}{2}} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \frac{\sqrt{3}}{3\sqrt{2}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

2. (10 points) Compute the projection $\text{proj}_W(\boldsymbol{y})$ of $\boldsymbol{y} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ on to $\text{Col}(\boldsymbol{A})$.

Solution. Let

$$\boldsymbol{U} = [\boldsymbol{u}_1\,, \boldsymbol{u}_2] = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Then, the project we seek is $\boldsymbol{U}\boldsymbol{U}^T\boldsymbol{y}$. Now,

$$\boldsymbol{U}\boldsymbol{U}^T = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} = \begin{bmatrix} \frac{1}{3}+\frac{1}{6} & -\frac{1}{3}+\frac{2}{6} & \frac{1}{3}+\frac{1}{6} \\ -\frac{1}{3}+\frac{2}{6} & \frac{1}{3}+\frac{4}{6} & \frac{1}{3}+\frac{2}{6} \\ \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{2}{6} & \frac{1}{3}+\frac{1}{6} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 2/3 \\ 1/2 & 2/3 & 1/2 \end{bmatrix}.$$

Therefore,

$$\text{proj}_W(\boldsymbol{y}) = \boldsymbol{U}\boldsymbol{U}^T\boldsymbol{y} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 2/3 \\ 1/2 & 2/3 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 5/3 \\ 13/6 \end{bmatrix}.$$