Math 2270-2, Quiz 7

Tuesday July 26, 2016

Instructions

This is a 20-minute quiz. It has 3 questions on 3 pages for a total of 30 points. There is an additional extra-credit problem on page 4 that is worth 10 points.

Questions 1, 2, and 3 concern the matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

1. (10 points) Are the columns of $\boldsymbol{A}$ orthogonal?

Solution. Let

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \qquad \boldsymbol{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \qquad \boldsymbol{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Then,

$$\begin{aligned} \boldsymbol{u}_1 \cdot \boldsymbol{u}_2 &= -2 + 0 + 2 = 0, \\ \boldsymbol{u}_1 \cdot \boldsymbol{u}_3 &= 0 + 0 + 0 = 0, \\ \boldsymbol{u}_2 \cdot \boldsymbol{u}_3 &= 0 + 0 + 0 = 0. \end{aligned}$$

Therefore, $\boxed{\text{yes}}$; the columns are orthogonal.

2. (10 points) Are the columns of $\boldsymbol{A}$ orthonormal?

Solution. No. For example, $\|\boldsymbol{u}_1\|^2 = 3 \neq 1$.

3. (10 points) Let $\boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the columns of $\boldsymbol{A}$.

Solution. We would like to write $\boldsymbol{b} = x_1\boldsymbol{u}_1 + x_2\boldsymbol{u}_2 + x_3\boldsymbol{u}_3$ for some $x_1, x_2, x_3 \in \mathbb{R}$. Equivalently, we wish to solve

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}.$$

Let us use Gaussian elimination:

$$\left[\begin{array}{ccc:c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 \end{array}\right] \xrightarrow{\mathcal{R}_3' = \mathcal{R}_3 - 2\mathcal{R}_1} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 5 & 0 & 1 \end{array}\right] \xrightarrow{\mathcal{R}_2 \leftrightarrow \mathcal{R}_3} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 1 \\ 0 & 5 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_2' = \mathcal{R}_2/5} \left[\begin{array}{ccc:c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2 \end{array}\right] \xrightarrow{\mathcal{R}_1' = \mathcal{R}_1 + 2\mathcal{R}_2} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 7/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 2 \end{array}\right].$$

Therefore, $\boldsymbol{x} = \begin{bmatrix} 7/5 \\ 1/5 \\ 1 \end{bmatrix}$. That is,

$$\boldsymbol{b} = \tfrac{7}{5}\boldsymbol{u}_1 + \tfrac{1}{5}\boldsymbol{u}_2 + \boldsymbol{u}_3.$$

4. (Extra Credit; 10 points total) Consider an orthogonal 10×10 matrix $\boldsymbol{A}$ whose $(i\,,j)$th entry is denoted by $a_{i,j}$. Compute the $(4\,,3)$th entry of $\boldsymbol{A}^{-1}$ in terms of the $a_{i,j}$'s.

   Solution. Recall that the orthogonality of $\boldsymbol{A}$ is synonymous with $\boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$. Since $\boldsymbol{A}$ is a square matrix, this implies that $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$. Therefore, the $(4\,,3)$th entry of $\boldsymbol{A}^{-1}$ is $a_{3,4}$.