Math 2270-2, Quiz 6

Tuesday July 19, 2016

This is a 20-minute quiz. It has XXX questions on XXX pages for a total of XXX points.

1. (10 points) Find the eigenvalues and a basis for each eigenspace in $\mathbb{C}^2$ of

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}.$$

Solution. To find eigenvalues first: Solve for $\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = 0$; i.e.,

$$= \det \begin{bmatrix} -\lambda & 1 \\ -8 & 4-\lambda \end{bmatrix} = -\lambda(4-\lambda) + 8 = \lambda^2 - 4\lambda + 8.$$

This yields

$$\lambda = \frac{4 \pm \sqrt{16 - (4 \times 8)}}{2} = \frac{4 \pm \sqrt{-16}}{2} = \frac{4 \pm 4i}{2} = 2 \pm 2i.$$

$\boxed{\text{For } \lambda = 2 + 2i\text{:}}$ Solve $(\boldsymbol{A} - (2+2i)\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$; equivalently,

$$\begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix} \sim \begin{bmatrix} 2+2i & -1 \\ -0 & 0 \end{bmatrix}.$$

Therefore, $(2+2i)v_1 - v_2 = 0$ and v_2 is a free variable. This means that all corresponding eigenvectors are of the form

$$\boldsymbol{v} = \begin{bmatrix} \alpha \\ (2+2i)\alpha \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2+2i \end{bmatrix} \qquad \text{for } \alpha \in \mathbb{C}.$$

A basis for this eigenspace is

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2+2i \end{bmatrix},$$

say.

$\boxed{\text{For } \lambda = 2 - 2i\text{:}}$ Recall that if $\boldsymbol{x}$ is an eigenvalue for λ then $\bar{\boldsymbol{x}}$ is an eigenvalue for $\bar{\lambda}$, and vice versa. Therefore, all eigenvalues of $2 - 2i$ are of the form

$$\boldsymbol{v} = \alpha \begin{bmatrix} 1 \\ 2-2i \end{bmatrix} \qquad \text{for } \alpha \in \mathbb{C}.$$

A basis for this eigenspace is

$$\boldsymbol{v} = \begin{bmatrix} 1 \\ 2-2i \end{bmatrix},$$

say.

2. (10 points) Diagonalize the matrix $\boldsymbol{A}$ of the previous problem. That is, write $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where $\boldsymbol{D}$ is a diagonal matrix. Compute $\boldsymbol{P}$ and $\boldsymbol{P}^{-1}$.

Solution. Since $\boldsymbol{A}$ has two distinct eigenvalues, $\boldsymbol{A}$ has two linearly independent eigenvectors $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 2+2i \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 1 \\ 2-2i \end{bmatrix}$. Therefore, $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$, where

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 \\ 2+2i & 2-2i \end{bmatrix} \quad \text{and} \quad \boldsymbol{D} = \begin{bmatrix} 2+2i & 0 \\ 0 & 2-2i \end{bmatrix}.$$

It remains to invert $\boldsymbol{P}$:

$$\left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2+2i & 2-2i & 0 & 1 \end{array}\right] \xrightarrow{\mathcal{R}_2' = (2+2i)\mathcal{R}_1 - \mathcal{R}_2} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 4i & 2+2i & -1 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_2' = \mathcal{R}_2/4i} \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} - \frac{i}{2} & \frac{i}{4} \end{array}\right] \xrightarrow{\mathcal{R}_1' = \mathcal{R}_1 - \mathcal{R}_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} + \frac{i}{2} & -\frac{i}{4} \\ 0 & 1 & \frac{1}{2} - \frac{i}{2} & \frac{i}{4} \end{array}\right].$$

Therefore,

$$\boldsymbol{P}^{-1} = \begin{bmatrix} \frac{1}{2} + \frac{i}{2} & -\frac{i}{4} \\ \frac{1}{2} - \frac{i}{2} & \frac{i}{4} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1+i & -i/2 \\ 1-i & i/2 \end{bmatrix}.$$