Math 2270-2, Quiz 5

Tuesday July 12, 2016

This is a 20-minute quiz. It has 3 questions on 3 pages for a total of 30 points.
We plan to diagonalize the matrix

$$\boldsymbol{A} = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix},$$

as $\boldsymbol{A} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}$ where $\boldsymbol{D}$ is a 3×3 diagonal matrix and $\boldsymbol{P}$ is a 3×3 invertible matrix.

1. (10 points) Find $\boldsymbol{D}$.

Solution. The eigenvalues of $\boldsymbol{A}$ are $\lambda_1 = 5$, $\lambda_2 = 2$, and $\lambda_3 = 3$. Therefore,

$$\boldsymbol{D} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

2. (10 points) Find $\boldsymbol{P}$.

Solution. Next we compute the corresponding eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$.

Computation of $\boldsymbol{v}_1$: To solve $(\boldsymbol{A} - 5\boldsymbol{I})\boldsymbol{v}_1 = \boldsymbol{0}$,

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow{\mathcal{R}_2' = \frac{1}{2}(3\mathcal{R}_1 + \mathcal{R}_3)} \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Computation of $\boldsymbol{v}_2$: To solve $(\boldsymbol{A} - 2\boldsymbol{I})\boldsymbol{v}_2 = \boldsymbol{0}$,

$$\begin{bmatrix} 3 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}.$$

Computation of $\boldsymbol{v}_3$: To solve $(\boldsymbol{A} - 3\boldsymbol{I})\boldsymbol{v}_3 = \boldsymbol{0}$,

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -1 \end{bmatrix}.$$

3. (10 points) Compute $\boldsymbol{P}^{-1}$.

Solution. Perform Gaussian elimination:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -3 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1/3 & 1/3 \\ 0 & 1 & 0 & 0 & -1/3 & -1/3 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right].$$