Math 2270-2, Quiz 4

Tuesday June 28, 2016

This is a 20-minute quiz. It has 2 questions on 2 pages for a total of 20 points.

1. (10 points) Compute the determinant of the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} -4 & 2 & 0 & 0 \\ 2 & 3 & 1 & 0 \\ 3 & 1 & 0 & 2 \\ 1 & 3 & 0 & 3 \end{bmatrix}.$$

Solution. We will compute

$$|\boldsymbol{A}| = |\boldsymbol{A}^T| = \begin{vmatrix} -4 & 2 & 3 & 1 \\ 2 & 3 & 1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ -4 & 2 & 3 & 1 \\ 2 & 3 & 1 & 3 \end{vmatrix} = - \begin{vmatrix} 0 & 2 & 3 \\ -4 & 3 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 2 \begin{vmatrix} -4 & 1 \\ 2 & 3 \end{vmatrix} - 3 \begin{vmatrix} -4 & 3 \\ 2 & 1 \end{vmatrix}$$
$$= 2(-12-2) - 3(-4-6) = -28 + 30 = 2.$$

2. (10 points total) Suppose we have a matrix $\boldsymbol{A}$ that satisfies

$$\boldsymbol{A} \sim \begin{bmatrix} 1 & 0 & 0 & 9 & 19/2 \\ 0 & 1 & 0 & -17/4 & -5/2 \\ 0 & 0 & 1 & 3/2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

in reduced row echelon form.

(a) (8 points; 4 points each) Find a basis for $\mathrm{Col}(\boldsymbol{A})$ and a basis for $\mathrm{Nul}(\boldsymbol{A})$.

Solution. A basis for the column space of $\boldsymbol{A}$ is

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

To find the nullity you solve $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, which is equivalent to the augmented-matrix computation for the augmented matrix,

$$\begin{bmatrix} 1 & 0 & 0 & 9 & 19/2 & 0 \\ 0 & 1 & 0 & -17/4 & -5/2 & 0 \\ 0 & 0 & 1 & 3/2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now $\alpha = x_4$ and $\beta = x_5$ are free variables, and

$$x_1 = -9\alpha - \tfrac{19}{2}\beta, \qquad x_2 = \tfrac{17}{4}\alpha + \tfrac{5}{2}\beta, \qquad x_3 = -\tfrac{3}{2}\alpha - 2\beta.$$

So $\mathrm{Nul}(\boldsymbol{A})$ is the span of all vectors of the form

$$\begin{bmatrix} -9\alpha - 19/2\beta \\ 17/4\alpha + 5/2\beta \\ -3/2\alpha - 2\beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -9 \\ 17/4 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -19/2 \\ 5/2 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

So a basis for $\mathrm{Nul}(\boldsymbol{A})$ is

$$\begin{bmatrix} -9 \\ 17/4 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -19/2 \\ 5/2 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

(b) (2 points; 1 point each) Compute $\dim(\mathrm{Col}(\boldsymbol{A}))$ and $\dim(\mathrm{Nul}(\boldsymbol{A}))$.

Solution. $\dim(\mathrm{Col}(\boldsymbol{A})) = 3$ and $\dim(\mathrm{Nul}(\boldsymbol{A})) = 2$.