Math 2270-2, Quiz 2

Tuesday June 21, 2016

This is a 20-minute quiz. It has 3 questions on 3 pages for a total of 30 points.

1. (10 points) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

If possible then find a 3×3 matrix $\boldsymbol{X}$ such that $\boldsymbol{AX} = \boldsymbol{B}$. If it is not possible then explain why it is not possible.

Solution. First of all,

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\mathcal{R}_3' = \mathcal{R}_3/2]{\mathcal{R}_1' = \mathcal{R}_1 - \mathcal{R}_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_2' = \mathcal{R}_2 - \mathcal{R}_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right].$$

This proves that $\boldsymbol{A}$ is invertible and

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & -1 & -0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix}.$$

Thus,

$$\boldsymbol{X} = \boldsymbol{A}^{-1}\boldsymbol{B} = \begin{bmatrix} 1 & -1 & -0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1 & 0 \\ 1/2 & 0 & 0 \end{bmatrix}.$$

Or you can directly row-reduce the following matrix

$$[\boldsymbol{A} \mid \boldsymbol{B}] = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 1 & 0 \\ 0 & 0 & 1 & 1/2 & 0 & 0 \end{array}\right] = [\boldsymbol{I} \mid \boldsymbol{X}].$$

2. (10 points) Suppose $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices. Show that if $\boldsymbol{AB}$ is invertible, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are both invertible.

Solution. Let $\boldsymbol{C}$ denote the inverse to $\boldsymbol{AB}$. That is, $\boldsymbol{I} = \boldsymbol{ABC} = \boldsymbol{A}(\boldsymbol{BC})$. It follows that $\boldsymbol{A}$ is invertible and $\boldsymbol{A}^{-1} = \boldsymbol{BC}$. Since $\boldsymbol{C}$ is also invertible, we multiply both sides by $\boldsymbol{C}^{-1}$ to see that $(\boldsymbol{CA})^{-1} = \boldsymbol{A}^{-1}\boldsymbol{C}^{-1} = \boldsymbol{B}$. This shows that $\boldsymbol{B}$ is invertible and $\boldsymbol{B}^{-1} = \boldsymbol{CA}$.

3. (10 points) Suppose $\boldsymbol{A}$ is a 4×4 invertible matrix whose inverse is

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}.$$

Let

$$\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Solve $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

Solution. Clearly,

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \\ 20 \\ 28 \end{bmatrix}.$$