Math 2270-2, Quiz 2

Tuesday June 14, 2016

1. (10 points) Let $\boldsymbol{A}$ be a 3×3 matrix such that

$$\boldsymbol{A}\begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Find 3 different solutions to the linear system

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution. Let

$$\boldsymbol{a} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Then $\boldsymbol{x} = \alpha\boldsymbol{a} + \boldsymbol{b}$ solves $\boldsymbol{A}\boldsymbol{x} = [1, 1, 1]^T$ for every $\alpha \in \mathbb{R}$. Three different examples are $\alpha = -1$, $\alpha = 1$, and $\alpha = 2$ which respectively yield

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}.$$

2. (10 points) Are the following vectors linearly independent? Justify your answer.

$$\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \begin{bmatrix} -3 \\ 9 \end{bmatrix}.$$

Solution. Let $\boldsymbol{A} = \begin{bmatrix} 1 & -3 \\ -3 & 9 \end{bmatrix}$. We begin by finding all solutions to $\boldsymbol{Ax} = \mathbf{0}$. If the only solution is $\boldsymbol{x} = \mathbf{0}$ then we have linear independence. Otherwise, the two vectors are linearly dependent. That is, we reduce

$$\begin{bmatrix} 1 & -3 & 0 \\ -3 & 9 & 0 \end{bmatrix} \xrightarrow{\mathcal{R}_3' = 3\mathcal{R}_1 + \mathcal{R}_3} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, there are nonzero solutions to $\boldsymbol{Ax} = \mathbf{0}$, which means that our two vectors are not linearly independent.

3. (10 points) Compute $\boldsymbol{A}\boldsymbol{B}^T$ and $(\boldsymbol{A}\boldsymbol{B})^T$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}.$$

Solution. First,

$$\boldsymbol{A}\boldsymbol{B} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \\ 17 & 39 \end{bmatrix} \Longrightarrow (\boldsymbol{A}\boldsymbol{B})^T = \begin{bmatrix} 5 & 11 & 17 \\ 11 & 25 & 39 \end{bmatrix}.$$

Next, note that

$$\boldsymbol{B}^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Therefore,

$$\boldsymbol{A}\boldsymbol{B}^T = \begin{bmatrix} 7 & 10 \\ 15 & 22 \\ 23 & 34 \end{bmatrix}.$$