Math 2270–2, Midterm 3
Summer 2016

Your Name: ______________________________

July 14, 2016

This midterm exam is made up of 5 problems on 5 sheets of paper, for a total of 40 points. You have one hour for this exam. Good luck.

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1. (5 points) Let $\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}$ and $\boldsymbol{w} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$. Is $\boldsymbol{w}$ in $\operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$? Justify your answer.

Solution. The question asks if there are scalars x_1 and x_2 such that $\boldsymbol{w} = x_1\boldsymbol{v}_1 + x_2\boldsymbol{v}_2$. Equivalently, $\boldsymbol{Ax} = \boldsymbol{w}$ for some $\boldsymbol{x} \in \mathbb{R}^2$ where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ -1 & 5 \\ 2 & -5 \end{bmatrix}.$$

We can study the augmented matrix

$$\begin{bmatrix} 1 & 0 & 2 \\ -1 & 5 & 3 \\ 2 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 5 \\ 2 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 5 \\ 2 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore, $[\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}]$ is invertible; this is another way to say that $\boldsymbol{v}_1, \boldsymbol{v}_2\, \boldsymbol{w}$ are linearly independent; this is yet another way to say that $\boldsymbol{w} \notin \operatorname{Span}\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$.

2. (5 points) Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 4 \end{bmatrix}.$$

If it is possible, then compute a 3×3 matrix $\boldsymbol{B}$ such that

$$\boldsymbol{AB} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

If it is not possible, then explain why not.

Solution. Row reduce

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 4 & 1 & 1 & 0 \\ 3 & 4 & 4 & 1 & 1 & 1 \end{array}\right] \xrightarrow[\mathcal{R}_3'=\mathcal{R}_3-3\mathcal{R}_1]{\mathcal{R}_2'=\mathcal{R}_2-2\mathcal{R}_1} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & -2 & -5 & -2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 2 & 5 & 2 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_3'=\mathcal{R}_3-2\mathcal{R}_2} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\mathcal{R}_2'=\mathcal{R}_2-2\mathcal{R}_3} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{\mathcal{R}_1'=\mathcal{R}_1-2\mathcal{R}_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & -1 & 6 & 4 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \xrightarrow{\mathcal{R}_1'=\mathcal{R}_1-3\mathcal{R}_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 3 & 7 \\ 0 & 1 & 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

Therefore, $\boldsymbol{B} = \begin{bmatrix} -1 & 3 & 7 \\ 1 & -3 & 2 \\ 0 & -1 & -1 \end{bmatrix}$.

3. (15 points total) Let $\boldsymbol{A}$ be a 10×10 matrix with determinant $\det(\boldsymbol{A}) = -3$.

(a) (5 points) Compute $\det(2\boldsymbol{A})$.

Solution. $\det(2\boldsymbol{A}) = 2^{10} \times \det(\boldsymbol{A}) = -3 \times 2^{10}$.

(b) (5 points) Compute $\det(\boldsymbol{A}^{-1})$.

Solution. $\det(\boldsymbol{A}^{-1}) = 1/\det(\boldsymbol{A}) = -1/3$.

(c) (5 points) Compute $\det(\boldsymbol{A}^2)$.

Solution. $\det(\boldsymbol{A}^2) = [\det(\boldsymbol{A})]^2 = 9$.

4. (5 points) Use Cramer's rule to solve the linear system,

$$\begin{aligned} 5x_1 \quad &+7x_2 &= 3 \\ 2x_1 \quad &+4x_2 &= 1. \end{aligned}$$

Solution. Let

$$\boldsymbol{A} = \begin{bmatrix} 5 & 7 \\ 2 & 4 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

Then,

$$\boldsymbol{A}_1(\boldsymbol{b}) = \begin{bmatrix} 3 & 7 \\ 1 & 4 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}_2(\boldsymbol{b}) = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}.$$

Note that $\det(\boldsymbol{A}) = 6$, $\det(\boldsymbol{A}_1(\boldsymbol{b})) = 5$, and $\det(\boldsymbol{A}_2(\boldsymbol{b})) = -1$. Therefore,

$$x_1 = \frac{\det(\boldsymbol{A}_1(\boldsymbol{b}))}{\det(\boldsymbol{A})} = \frac{5}{6} \quad \text{and} \quad x_2 = \frac{\det(\boldsymbol{A}_2(\boldsymbol{b}))}{\det(\boldsymbol{A})} = -\frac{1}{6}.$$

That is, $\boldsymbol{x} = \begin{bmatrix} 5/6 \\ -1/6 \end{bmatrix}$.

5. (10 points) Let $\boldsymbol{A}$ be an arbitrary 4×4 matrix.

(a) (5 points) Write a 4×4 matrix $\boldsymbol{E}$ such that $\boldsymbol{EA}$ is the matrix $\boldsymbol{A}$ with its first two rows interchanged.

Solution. We have

$$\boldsymbol{E} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(b) (5 points) Compute $\det(\boldsymbol{E})$.

Solution. $\boldsymbol{E}$ is obtained from $\boldsymbol{I}$ by interchanging 2 rows. Therefore, $\det(\boldsymbol{E}) = -\det(\boldsymbol{I}) = -1$.