Math 2270–2, Midterm 2
Summer 2016

Your Name: ______________________________________

June 23, 2016

This midterm exam is made up of 4 problems on 4 sheets of paper, for a total of 20 points. You have one hour for this exam. Good luck.

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1. (4 points) Invert

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}.$$

Solution. Row reduce to find that $[\boldsymbol{A} \mid \boldsymbol{I}] \sim [\boldsymbol{I} \mid \boldsymbol{A}^{-1}]$ where

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}.$$

2. (5 points) Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies the following:

$$T(\boldsymbol{e}_1) = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix}, \qquad T(\boldsymbol{e}_2) = \begin{bmatrix} -5/2 \\ 5/2 \end{bmatrix}.$$

Compute $T(\boldsymbol{x})$ where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution. The matrix of T is

$$\boldsymbol{A} = \begin{bmatrix} 5/2 & -5/2 \\ -5/2 & 5/2 \end{bmatrix}.$$

Therefore,

$$T(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} 5/2 & -5/2 \\ -5/2 & 5/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 5/2 \end{bmatrix}.$$

3. (6 points total; 3 points each) True or false; fully justify your answer in each case:

(a) If $\boldsymbol{A}$ is a $k \times n$ matrix and $\boldsymbol{x} \in \mathbb{R}^n$ satisfies $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$, then we always have $\boldsymbol{x} = \mathbf{0}$.

Solution. False. This happens if and only if the columns of $\boldsymbol{A}$ are linearly independent. For example, set

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

to see that $\boldsymbol{A}$ is not invertible ($\det(\boldsymbol{A}) = 0$) and $\boldsymbol{x} \neq \mathbf{0}$, but $\boldsymbol{A}\boldsymbol{x} = \mathbf{0}$.

(b) If $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are respectively $k \times n$, $n \times m$, and $n \times m$ matrices, and if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{C}$ then $\boldsymbol{B} = \boldsymbol{C}$.

Solution. False. For instance, take $\boldsymbol{A}$ to be from the solution of the above problem, and $\boldsymbol{B} := \boldsymbol{x}$ from the above, and $\boldsymbol{C} = \mathbf{0}$. Then, $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{C}$ but $\boldsymbol{B} \neq \boldsymbol{C}$.

4. (5 points) Suppose $\boldsymbol{A}$ is an $n \times n$ matrix such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ has a solution for every $\boldsymbol{b} \in \mathbb{R}^n$. Prove that $\boldsymbol{A}$ is invertible.
(Hint. First find $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ such that $\boldsymbol{A}\boldsymbol{x}_j = \boldsymbol{e}_j$ for every $1 \leq j \leq n$.)

Solution. Find $\boldsymbol{x}_i, \ldots, \boldsymbol{x}_n \in \mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{x}_j = \boldsymbol{e}_j$ for all $1 \leq j \leq n$. Append these equations to find that $\boldsymbol{A}\boldsymbol{X} = \boldsymbol{I}$ where $\boldsymbol{X} = [\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n]$. That is, $\boldsymbol{X}$ is the inverse to $\boldsymbol{A}$.