# Math 2270–2, Solutions to Midterm 1
# Summer 2016

June 2, 2016

1. *Let* $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. *Is* $\boldsymbol{b}$ *in the span of the columns of* $\boldsymbol{A}$*? Justify your answer.*

**Solution.** We create the augmented matrix and reduce it toward echelon form. After a few steps we arrive at the reduced echelon form of $\boldsymbol{A}$:

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The last column is pivotal. Therefore, $\boldsymbol{b}$ is not in the span of the columns of $\boldsymbol{A}$.

2. *Let* $\boldsymbol{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. *Can* $\boldsymbol{c}$ *be written as a linear combination of* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*? Justify your answer.*

**Solution.** We create the augmented matrix and reduce:

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The last column is pivotal; therefore, $\boldsymbol{c} \notin \operatorname{Span}(\boldsymbol{a}, \boldsymbol{b})$.

3. *Consider the following matrices.* $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

   (a) *Identify all of those which are in echelon form. Justify your answers.*

   **Solution.** $\boldsymbol{A}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ are in echelon form. $\boldsymbol{B}$ is not because the second row is all zeros, yet lies above the third row.

   (b) *Identify all of those which are in reduced echelon form. Justify your answers.*

   **Solution.** Only $\boldsymbol{D}$ is in reduced echelon form.

   (c) *For the matrices that are in reduced echelon form, identify the pivots and pivot columns as is done in the lectures and your textbook.*

   **Solution.** The pivotal columns are columns 1, 3, and 4. The pivots are boxed below.

$$\boldsymbol{D} = \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

4. *Find all, if any, of the solutions $x, y, z, w$ to the following system linear of equations:*

$$\begin{array}{ccccl} x & +y & +2z & -5w & = 3 \\ 2x & +5y & -z & -9w & = -3 \\ 2x & -3y & +2z & +7w & = -11 \\ x & -3y & +2z & +7w & = -5. \end{array}$$

**Solution.** The augmented matrix is

$$\begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 2 & 5 & -1 & -9 & -3 \\ 2 & -3 & 2 & 7 & -11 \\ 1 & -3 & 2 & 7 & -5 \end{bmatrix}.$$

We write this in row echelon form. This task requires a little patience and care. There are of course many ways of doing this. Here is one way:

- Replace row 2 by 2(Row 1) − (Row 2); row 3 by (Row 2)−(Row 3); and row 4 by (Row 1)−(Row 4):

$$\begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 0 & -3 & 5 & -1 & 9 \\ 0 & 8 & -3 & -16 & 8 \\ 0 & 4 & 0 & -12 & 8 \end{bmatrix} \xrightarrow{\text{Divide row 4 by 4}} \begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 0 & -3 & 5 & -1 & 9 \\ 0 & 8 & -3 & -16 & 8 \\ 0 & 1 & 0 & -3 & 2 \end{bmatrix}$$

- Exchange rows 2 and 4:

$$\begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 8 & -3 & -16 & 8 \\ 0 & -3 & 5 & -1 & 9 \end{bmatrix}$$

- Replace row 3 by 8(Row 2) − (Row 3) and row 4 by 3(Row 2) + (Row 4):

$$\begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -8 & 8 \\ 0 & 0 & 5 & -10 & 15 \end{bmatrix} \xrightarrow[\text{replace row 4 by 3(Row 4)}]{\text{replace row 3 by 5(Row 3)}} \begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 15 & -40 & 40 \\ 0 & 0 & 15 & -30 & 45 \end{bmatrix}$$

- Divide rows 3 and 4 both by 5:

$$\begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -8 & 8 \\ 0 & 0 & 3 & -6 & 9 \end{bmatrix} \xrightarrow{\text{replace row 4 by (Row 4)−(Row 3):}} \begin{bmatrix} 1 & 1 & 2 & -5 & 3 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -8 & 8 \\ 0 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

- Replace row 1 by (Row 1)−(Row 2), divide row 3 by 3, and divide row 4 by 2:

$$\begin{bmatrix} 1 & 0 & 2 & -2 & 1 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -8/3 & 8/3 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix} \xrightarrow{\text{replace row 1 by (Row 1)}-2\text{(Row 3)}} \begin{bmatrix} 1 & 0 & 0 & 10/3 & -13/3 \\ 0 & 1 & 0 & -3 & 2 \\ 0 & 0 & 1 & -8/3 & 8/3 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}.$$

- Replace row 1 by (Row 1)$-\frac{10}{3}$(Row 4), row 2 by (Row 2)+3(Row 4), and row 3 by (Row 3)+$\frac{8}{3}$(Row 4):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -6 \\ 0 & 1 & 0 & 0 & 7/2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix}.$$

The preceding is in reduced row echelon form, and yields the following unique solution to our linear system:

$$x = -6, \; y = 3.5, \; z = 4, \; w = 0.5.$$