# ON DYNAMICAL GAUSSIAN RANDOM WALKS

DAVAR KHOSHNEVISAN, DAVID A. LEVIN, AND PEDRO J. MÉNDEZ-HERNÁNDEZ

Abstract. Motivated by the recent work of Benjamini, Häggström, Peres, and Steif (2003) on dynamical random walks, we: (i) Prove that, after a suitable normalization, the dynamical Gaussian walk converges weakly to the Ornstein–Uhlenbeck process in classical Wiener space; (ii) derive sharp tail-asymptotics for the probabilities of large deviations of the said dynamical walk; and (iii) characterize (by way of an integral test) the minimal envelop(es) for the growth-rate of the dynamical Gaussian walk. This development also implies the tail capacity-estimates of Mountford (1992) for large deviations in classical Wiener space.

The results of this paper give a partial affirmative answer to the problem, raised in Benjamini et al. (2003, Question 4) of whether there are precise connections between the OU process in classical Wiener space and dynamical random walks.

## 1. Introduction and Main Results

Let $\{\omega_j\}_{j=1}^\infty$ denote a sequence of i.i.d. random variables, and to each $\omega_j$ we associate a rate-one Poisson process with jump times $0 < \tau_j(1) < \tau_j(2) < \ldots$. (All of the said processes are assumed to be independent from one another.) Now at every jump-time of the $j$th Poisson process, we replace the existing $\omega$-value by an independent copy. In symbols, let $\{\omega_j^k\}_{j,k=1}^\infty$ be a double-array of i.i.d. copies of the $\omega_j$'s—all independent of the Poisson clocks—and define the process $X := \{X_j(t);\ t \geq 0\}_{j=1}^\infty$ as follows: For all $j \geq 1$,

$$X_j(0) := \omega_j, \qquad X_j(t) := \omega_j^k, \quad \forall t \in [\tau_j(k), \tau_j(k+1)). \tag{1.1}$$

We remark that, as a process indexed by $t$, $t \mapsto (X_1(t), X_2(t), \cdots)$ is a stationary Markov process in $\mathbb{R}^{\mathbb{N}}$ whose invariant measure is the product measure $\mu^\infty$, where $\mu$ denotes the law of $\omega_1$.

Recently, Benjamini, Häggström, Peres, and Steif (2003) have introduced *dynamical random walks* as the partial-sum processes that are associated to the Markov process $X$. In other words, the dynamical walk associated to the distribution $\mu$ is defined as the two-parameter process $S := \{S_n(t)\}_{n\geq 1, t\geq 0}$ that is defined by

$$S_n(t) := X_1(t) + \cdots + X_n(t), \qquad \forall n \geq 1,\ t \geq 0. \tag{1.2}$$

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From now on, we specialize our dynamical walks by assuming that the incremental distribution $\mu$ is standard normal, i.e., for all $x \in \mathbb{R}$,

$$\mu([x,\infty)) = 1 - \Phi(x) := \bar{\Phi}(x) := \int_x^\infty \frac{e^{-z^2/2}}{\sqrt{2\pi}}\,dz. \tag{1.3}$$

Our forthcoming analysis depends on this simplification in a critical way.

Now consider the following rescaled dynamical Gaussian walk $U^n$:

$$U_t^n(s) := \frac{S_{\lfloor nt\rfloor}(s)}{\sqrt{n}}, \qquad \forall s,t \in [0,1]). \tag{1.4}$$

Our first contribution is the following large-sample result on dynamical Gaussian walks.

**Theorem 1.1.** *As $n$ tends to infinity, the random field $U^n$ converges weakly in $D([0,1]^2)$ to the continuous centered Gaussian random field $U$ whose covariance is*

$$\mathsf{E}\{U_s(t)U_{s'}(t')\} = e^{-|s-s'|}\min(t,t'), \qquad \forall s,s',t,t' \in [0,1]. \tag{1.5}$$

(For information on $D([0,1]^2)$ consult Section 4.)

Before proceeding further, we make two tangential remarks.

*Remark* 1.2. The limiting random field $U$ has the following interpretation:

$$U_t(s) := e^{-s}B\left(e^{2s},t\right), \qquad \forall s,t \in [0,1], \tag{1.6}$$

where $B$ is the two-parameter Brownian sheet. Standard arguments then show that $\mathcal{U} := \{U_t\}_{t\geq 0}$ is an infinite-dimensional stationary diffusion on the classical Wiener space $C([0,1])$, and the invariant measure of $\mathcal{U}$ is, in fact, the Wiener measure on $C([0,1])$. The process $\mathcal{U}$ is the so-called *Ornstein–Uhlenbeck* (OU) process in classical Wiener space. Theorem 1.1, in conjunction with this observation, gives a partial affirmative answer to Benjamini et al. (2003, Question 4), where it is asked whether there are precise potential-theoretic connections between the dynamical (here, Gaussian) walks, and the OU process in $C([0,1])$.

*Remark* 1.3. Theorem 1.1 can be viewed as a construction of the OU process in $C([0,1])$. This is an interesting process in and of itself, and arises independently in diverse areas in stochastic analysis. For three samples, see Kuelbs (1973), Malliavin (1979), and Walsh (1986). The elegant relation (1.6) to the Brownian sheet was noted by David Williams; cf. Meyer (1982, appendix).

Our next result elaborates further on the connection between the dynamical Gaussian walk and the process $\mathcal{U}$.

**Theorem 1.4.** *Choose and fix a sequence $\{z_j\}_{j=1}^\infty$ that satisfies*

$$\inf_n z_n \geq 1, \quad \lim_{n\to\infty} z_n = +\infty, \quad \textit{and} \quad \lim_{n\to\infty}\sqrt{\frac{\log n}{n}}\,z_n = 0. \tag{1.7}$$

*Then, as $n \to \infty$,*

$$\frac{1+o(1)}{9} z_n^2\bar{\Phi}(z_n) \leq \mathsf{P}\left\{\sup_{t\in[0,1]} S_n(t) \geq z_n\sqrt{n}\right\} \leq (2+o(1))z_n^2\bar{\Phi}(z_n). \tag{1.8}$$

The following reformulation of a theorem of Mountford (1992) provides the analogue for the standard OU process $U := \{U_1(s);\ s \geq 0\}$: *There exists a constant* $K_{1.9} > 1$ *such that*

$$K_{1.9}^{-1} z^2 \bar{\Phi}(z) \leq \mathsf{P}\left\{ \sup_{s\in[0,1]} U_1(s) \geq z \right\} \leq K_{1.9} z^2 \bar{\Phi}(z), \qquad \forall z \geq 1. \tag{1.9}$$

For a refinement see Pickands (1967), and also Qualls and Watanabe (1971).

The apparent similarity between Theorem 1.4 and (1.9) is based on more than mere analogy. Indeed, Theorems 1.1 and 1.4 together imply (1.9) as a corollary. This can be readily checked; cf. the last line of §4.1.

As a third sample from our present work, we show a pathwise implication of Theorem 1.4. This is the dynamical analogue of the celebrated "integral test" of Erdős (1942). Define the map $\mathfrak{I}(H)$, for all nonnegative measurable functions $H$, by

$$\mathfrak{I}(H) := \int_1^\infty \frac{H^4(t)\bar{\Phi}(H(t))}{t}\, dt. \tag{1.10}$$

**Theorem 1.5.** *Suppose that $H$ is a nonnegative nondecreasing function. Then:*

(i) *If $\mathfrak{I}(H) < +\infty$, then with probability one,*

$$\sup_{t\in[0,1]} S_n(t) < H(n)\sqrt{n}, \quad \textit{for all but a finite number of } n\textit{'s}. \tag{1.11}$$

(ii) *Conversely, if $\mathfrak{I}(H) = +\infty$, then with probability one there exists a $t \in [0,1]$, such that*

$$S_n(t) \geq H(n)\sqrt{n}, \quad \textit{for an infinite number of } n\textit{'s}. \tag{1.12}$$

*Remark* 1.6. Owing to (1.17) below, we have

$$\mathfrak{I}(H) < +\infty \iff \int_1^\infty H^3(t) e^{-\frac{1}{2}H^2(t)} \frac{dt}{t} < +\infty. \tag{1.13}$$

We recall that the Erdős integral test asserts that $S_n(0) > H(n)\sqrt{n}$ for infinitely many $n$ (a.s.) if and only if $\int_1^\infty H(t)e^{-\frac{1}{2}H^2(t)}t^{-1}\, dt < +\infty$. Combining the preceding remark with Theorem 1.5 immediately leads us to the following result whose elementary proof is omitted.

**Corollary 1.7.** *Given $\tau \in [0,1]$,*

$$\limsup_{n\to\infty} \frac{[S_n(\tau)]^2 - 2n\ln\ln n}{n\ln\ln\ln n} = 3, \qquad a.s. \tag{1.14}$$

*On the other hand, there exists a (random) $T \in [0,1]$, such that*

$$\limsup_{n\to\infty} \frac{[S_n(T)]^2 - 2n\ln\ln n}{n\ln\ln\ln n} = 5, \qquad a.s. \tag{1.15}$$

*Remark* 1.8. In the terminology of Benjamini et al. (2003), our Theorem 1.5 has the consequence that the Erdős characterization of the upper class of a Gaussian random walk is "dynamically sensitive." This is in contrast to the fact that the LIL itself is "dynamically stable." In plain terms, the latter means that with probability one,

$$\limsup_{n\to\infty} \frac{S_n(t)}{\sqrt{2n\ln\ln n}} = 1, \qquad \forall t \in [0,1]. \tag{1.16}$$

See Benjamini et al. (2003, Theorem 1.2).

The organization of this paper is as follows: In §2 we state and prove a theorem on the Poisson clocks that, informally speaking, asserts that with overwhelming probability the typical clock is at mean-field all the time, and this happens simultaneously "over a variety of scales." This material may be of independent technical interest to the reader.

In §3, we make a few computations with Gaussian random variables. These calculations are simple consequences of classical regression analysis of mathematical statistics, but since we need the exact forms of the ensuing estimates, we include some of the details.

After a brief discussion of the space $D([0,1]^2)$, Theorem 1.1 is then proved in §4. Our proof relies heavily on the general machinery of Bickel and Wichura (1971).

Theorem 1.4 is more difficult to prove; its proof is split across §5, §6, and §7. The key idea here is that estimates, similar to those in Theorem 1.4, hold in the quenched setting, where the implied conditioning is made with respect to the clocks.

Finally, we derive Theorem 1.5 in §8. Our proof combines Theorem 1.4, a localization trick, and the combinatorial method of Erdős (1942).

Throughout, we frequently use the elementary facts that for all $y > 0$,

$$\bar{\Phi}(y) \le e^{-y^2/2}, \text{ and } \bar{\Phi}(z) = \frac{1+o(1)}{z\sqrt{2\pi}} e^{-z^2/2} \qquad (z \to \infty). \tag{1.17}$$

We have used Bachmann's "little-$o$/big-$O$" notation to simplify the exposition.

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## 2. Regularity of the Clocks

Consider the random field $\{N^n_{s\to t};\ 0 \le s \le t,\, n \ge 1\}$ that is defined as follows: Given $s \le t$ and $n \ge 1$, $N^n_{s\to t}$ denotes the Poisson-based number of changes made from time $s$ to time $t$; i.e.,

$$N^n_{s\to t} := \sum_{j=1}^n \mathbf{1}_{\{X_j(t)\neq X_j(s)\}}. \tag{2.1}$$

It is clear that $N^n_{s\to t}$ is a sum of $n$ i.i.d. $\{0,1\}$-valued random variables. Because we know also that $\mathsf{P}\{X_1(s) = X_1(t)\} = e^{-|t-s|}$, we can deduce from the strong law for such binomials that for $n$ large, $N^n_{s\to t} \simeq n(1 - e^{-|t-s|})$. The following is an estimate that ensures that, in the mentioned approximation, a good amount of uniformity in $s$ and $t$ is preserved.

**Theorem 2.1.** *If $\{\Delta_j\}_{j=1}^\infty$ is a sequence in $[0,1]$ such that $\lim_{n\to\infty} \Delta_n = 0$, then for all $n \ge 1$ and $\alpha \in (0,1)$,*

$$\mathsf{P}\left\{ \sup_{\substack{0\le s\le t\le 1:\\ t-s\ge\Delta_n}} \left| \frac{N^n_{s\to t}}{\mathsf{E}N^n_{s\to t}} - 1 \right| \ge \alpha \right\} \le \frac{512}{\alpha^2\Delta_n^2} \exp\left( -\frac{3\alpha^3 n\Delta_n}{2304} \right), \tag{2.2}$$

*where* $\sup \varnothing := 0$.

This, and the Borel–Cantelli lemma, together imply the following result that we shall need later on. In rough terms, it states that as long as the "window size" is not too small, then the Poisson clocks are mean-field.

**Corollary 2.2.** *If $\Delta_n \to 0$ in $[0,1]$ satisfies $\lim_{n\to\infty} n(\log n)^{-1}\Delta_n = +\infty$, then with probability one,*

$$\lim_{n\to\infty} \sup_{\substack{0\le s\le t\le 1:\\ t-s\ge\Delta_n}} \left| \frac{N^n_{s\to t}}{\mathsf{E}N^n_{s\to t}} - 1 \right| = 0. \tag{2.3}$$

It is not hard to convince oneself that the preceding fails if the "window size" $\Delta_n$ decays too rapidly.

*Proof of Theorem 2.1.* Throughout this proof, $\alpha \in (0,1)$ is held fixed.

We first try to explain the significance of the condition $t - s \ge \Delta_n$ by obtaining a simple lower bound on $\mathsf{E}N^n_{s\to t}$ in this case.

Observe the following simple bound:

$$\frac{x}{2} \le 1 - e^{-x} \le x, \qquad \forall x \in [0,1]. \tag{2.4}$$

This shows that

$$\inf_{\substack{0\le s\le t\le 1:\\ t-s\ge\Delta_n}} \mathsf{E}N^n_{s\to t} \ge \frac{n\Delta_n}{2}. \tag{2.5}$$

Next we recall an elementary large deviations bound for Binomials. According to Bernstein's inequality (cf. Bennett (1962); also see the elegant inequalities of Hoeffding (1963)), if $\{B_j\}_{j=1}^\infty$ are i.i.d. Bernoulli random variables with $\mathsf{P}\{B_1 = 1\} := p$, then

$$\mathsf{P}\left\{|B_1 + \cdots + B_n - np| \ge n\lambda\right\} \le 2\exp\left(-\frac{n\lambda^2}{2p + \frac{2}{3}\lambda}\right). \tag{2.6}$$

Apply this with $B_j := \mathbf{1}_{\{X_j(s)\neq X_j(t)\}}$, for arbitrary $s \le t$ and $\lambda := \alpha[1 - e^{-(t-s)}]$, to deduce that for all $\alpha \in (0,1)$ and $n \ge 1$,

$$\begin{aligned} &\mathsf{P}\left\{|N^n_{s\to t} - \mathsf{E}N^n_{s\to t}| \ge \alpha \mathsf{E}N^n_{s\to t}\right\} \\ &\quad\le 2\exp\left(-\frac{\alpha^2 n\left[1 - e^{-(t-s)}\right]}{2 + \frac{2}{3}\alpha}\right) \\ &\quad\le 2\exp\left(-\frac{3\alpha^2 n\left[1 - e^{-(t-s)}\right]}{8}\right). \end{aligned} \tag{2.7}$$

From (2.4) we can deduce that for all $\alpha \in (0,1)$ and $n \ge 1$,

$$\sup_{\substack{0\le s\le t\le 1:\\ |s-t|\ge\Delta_n}} \mathsf{P}\left\{|N^n_{s\to t} - \mathsf{E}N^n_{s\to t}| \ge \alpha\mathsf{E}N^n_{s\to t}\right\} \le 2\exp\left(-\frac{3\alpha^2 n\Delta_n}{16}\right). \tag{2.8}$$

Next, we choose and fix integers $k_1 < k_2 < \cdots \to \infty$ as follows:

$$k_n := \left\lfloor 1 + \frac{8}{\alpha\Delta_n} \right\rfloor \qquad \text{so that} \quad \frac{\alpha\Delta_n}{9} \le k_n^{-1} \le \frac{\alpha\Delta_n}{8}. \tag{2.9}$$

Based on these, we define

$$\Gamma_n := \left\{ \frac{j}{k_n};\ 0 \le j \le k_n \right\}. \tag{2.10}$$

Then it follows immediately from (2.8) and (2.9) that

$$\mathsf{P}\left\{\sup_{\substack{0\le s\le t\le 1:\\ s,t\in\Gamma_n}}\left|\frac{N^n_{s\to t}}{\mathsf{E}N^n_{s\to t}}-1\right|\ge\alpha\right\}\le(k_n+1)^2\exp\left(-\frac{3\alpha^3 n\Delta_n}{144}\right). \tag{2.11}$$

Given any point $u\in[0,1]$, define

$$\begin{aligned}\underline{u}_n &:= \max\left\{r\in[0,u]:\ r\in\Gamma_n\right\}\\ \overline{u}_n &:= \min\left\{r\in[u,1]:\ r\in\Gamma_n\right\}.\end{aligned} \tag{2.12}$$

These are the closest points to $u$ in $\Gamma_n$ from below and above respectively. We note, in passing, that $0\le\overline{u}_n-\underline{u}_n\le k_n^{-1}$. Moreover, thanks to (2.9), whenever $0\le s\le t\le 1$ satisfy $t-s\ge\Delta_n$, it follows that $\overline{s}_n<\underline{t}_n$ with room to spare. We will use this fact without further mention. Moreover, for such a pair $(s,t)$,

$$N^n_{\overline{s}_n\to\underline{t}_n}\le N^n_{s\to t}\le N^n_{\underline{s}_n\to\overline{t}_n}. \tag{2.13}$$

This follows from the fact that with $\mathsf{P}$-probability one, once one of the $X_j(u)$'s is updated, then from that point on it will never be replaced back to its original state. (This is so because the chances are zero that two independent normal variates are equal to one another.) The preceding display motivates the following bound: For all $0\le s\le t\le 1$,

$$\begin{aligned}\mathsf{E}\left\{\left|N^n_{\underline{s}_n\to\overline{t}_n}-N^n_{\overline{s}_n\to\underline{t}_n}\right|\right\}&=ne^{-(\underline{t}_n-\overline{s}_n)}\left[1-e^{-(\overline{t}_n-\underline{t}_n)-(\overline{s}_n-\underline{s}_n)}\right]\\ &\le\frac{2n}{k_n},\end{aligned} \tag{2.14}$$

where the last inequality follows from (2.4). Owing to (2.5) and (2.9), we have the crucial estimate,

$$\sup_{\substack{0\le s\le t\le 1:\\ t-s\ge\Delta_n}}\mathsf{E}\left\{\left|N^n_{\underline{s}_n\to\overline{t}_n}-N^n_{\overline{s}_n\to\underline{t}_n}\right|\right\}\le\frac{\alpha}{2}\inf_{\substack{0\le u\le v\le 1:\\ v-u\ge\Delta_n}}\mathsf{E}N^n_{u\to v}. \tag{2.15}$$

This and (2.13) together imply the following bound uniformly for all $0\le s\le t\le 1$ that satisfy $t-s\ge\Delta_n$:

$$\left|\widetilde{N}^n_{s\to t}\right|\le\frac{\alpha}{2}\inf_{\substack{0\le u\le v\le 1\\ v-u\ge\Delta_n}}\mathsf{E}N^n_{u\to v}+\max\left(\left|\widetilde{N}^n_{\overline{s}_n\to\underline{t}_n}\right|\ ,\ \left|\widetilde{N}^n_{\underline{s}_n\to\overline{t}_n}\right|\right), \tag{2.16}$$

where $\widetilde{Z}:=Z-\mathsf{E}Z$ for any integrable random variable $Z$. Therefore,

$$\begin{aligned}&\mathsf{P}\left\{{}^\exists t-s\ge\Delta_n:\ \left|\widetilde{N}^n_{s\to t}\right|\ge\alpha\mathsf{E}N^n_{s\to t}\right\}\\ &\le\mathsf{P}\left\{{}^\exists t-s\ge\Delta_n:\ \max\left(\left|\widetilde{N}^n_{\overline{s}_n\to\underline{t}_n}\right|\ ,\ \left|\widetilde{N}^n_{\underline{s}_n\to\overline{t}_n}\right|\right)\ge\frac{\alpha}{2}\mathsf{E}N^n_{s\to t}\right\}.\end{aligned} \tag{2.17}$$

Another application of (2.15) yields

$$
\begin{aligned}
&\mathsf{P}\left\{ {}^{\exists} t-s\geq\Delta_n :\ \left|\widetilde{N}^n_{s\to t}\right| \geq \alpha \mathsf{E}N^n_{s\to t}\right\}\\
&\leq \mathsf{P}\left\{ {}^{\exists} t-s\geq\Delta_n :\ \left|\widetilde{N}^n_{\bar{s}_n\to \underline{t}_n}\right| \geq \frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)\mathsf{E}N^n_{\bar{s}_n\to \underline{t}_n}\right\}\\
&\quad + \mathsf{P}\left\{ {}^{\exists} t-s\geq\Delta_n :\ \left|\widetilde{N}^n_{\underline{s}_n\to \bar{t}_n}\right| \geq \frac{\alpha}{2}\left(1-\frac{\alpha}{2}\right)\mathsf{E}N^n_{\underline{s}_n\to \bar{t}_n}\right\}\\
&\leq 2\mathsf{P}\left\{ \max_{\substack{0\leq u\leq v\leq 1:\\ u,v\in\Gamma_n}} \left|\frac{N^n_{u\to v}}{\mathsf{E}N^n_{u\to v}}-1\right| \geq \frac{\alpha}{4}\right\}\\
&\leq 2\left(k_n^2+1\right)\exp\left(-\frac{3\alpha^3 n\Delta_n}{2304}\right),
\end{aligned}
\tag{2.18}
$$

owing to (2.11). Because $k_n+1\leq 16(\alpha\Delta_n)^{-1}$, this proves the theorem. ■

## 3. A Little Regression Analysis

Define $\mathfrak{F}^n_t$ to be the augmented right-continuous $\sigma$-algebra generated by the variables $\{S_n(v);\, v\leq t\}$ and $\mathfrak{N}$, where the latter is the $\sigma$-algebra generated by all of the Poisson clocks. For convenience, we write $\mathsf{P}_{\mathfrak{N}}\{\cdots\}$ and $\mathsf{E}_{\mathfrak{N}}\{\cdots\}$ in place of $\mathsf{P}\{\cdots\,|\,\mathfrak{N}\}$ and $\mathsf{E}\{\cdots\,|\,\mathfrak{N}\}$, respectively. We refer to $\mathsf{P}_{\mathfrak{N}}$ as a random "quenched" measure, and $\mathsf{E}_{\mathfrak{N}}$ is its corresponding expectation operator. We will also write $\mathrm{Var}_{\mathfrak{N}}$ for the corresponding conditional variance.

**Lemma 3.1.** *If $0\leq u\leq v$, then the following hold $\mathsf{P}$-almost surely: For all $x\in\mathbb{R}$,*

$$
\begin{aligned}
\mathsf{E}_{\mathfrak{N}}\left\{S_n(v)\,\middle|\, S_n(u)=x\right\} &= \left(1-\frac{N^n_{u\to v}}{n}\right)x,\\
\mathrm{Var}_{\mathfrak{N}}\left(S_n(v)\,\middle|\, S_n(u)=x\right) &= N^n_{u\to v}\left[2-\frac{N^n_{u\to v}}{n}\right].
\end{aligned}
\tag{3.1}
$$

*Proof.* From time $u$ to time $v$, $N^n_{u\to v}$-many of the increments are changed; the remaining $(n-N^n_{u\to v})$ increments are left unchanged. Therefore, we can write

$$
\begin{aligned}
S_n(u) &= V_1+V_2\\
S_n(v) &= V_1+V_3,
\end{aligned}
\tag{3.2}
$$

where: (i) $V_1$, $V_2$, and $V_3$ are independent; (ii) the distribution of $V_1$ is the same as that of $S_{n-N^n_{u\to v}}(0)$; and (iii) $V_2$ and $V_3$ are identically distributed and their common distribution is that of $S_{N^n_{u\to v}}(0)$. The result follows from standard calculations from classical regression analysis. ■

This immediately yields the following.

**Lemma 3.2.** *For all $x,y\geq 0$, all times $0\leq u\leq v$, and all integers $n\geq 1$,*

$$
\begin{aligned}
\mathsf{P}_{\mathfrak{N}}\left\{S_n(v)\geq y\,\middle|\,\mathfrak{F}^n_u\right\} &= \mathsf{P}_{\mathfrak{N}}\left\{S_n(v)\geq y\,\middle|\, S_n(u)\right\}\\
&= \bar{\Phi}\left(\frac{y-\left(1-\frac{1}{n}N^n_{u\to v}\right)S_n(u)}{\sqrt{N^n_{u\to v}\left(2-\frac{1}{n}N^n_{u\to v}\right)}}\right), \qquad \mathsf{P}\text{-}a.s.
\end{aligned}
\tag{3.3}
$$

We will also have need for the following whose elementary proof we omit.

**Lemma 3.3.** *For all $z\geq 1$ and $\varepsilon>0$, we have $\bar{\Phi}(z+\varepsilon z)\leq e^{-z^2\varepsilon}\bar{\Phi}(z)$.*

Next is a "converse" inequality. Unlike the latter lemma, however, this one merits a brief derivation.

**Lemma 3.4.** *If* $\gamma > 0$*, then*

$$\bar\Phi\left(z - \frac{\gamma}{z}\right) \le \left(1 + e^{2\gamma}\right)\bar\Phi(z), \qquad \forall z \ge \sqrt{\gamma}. \tag{3.4}$$

*Proof.* We make a direct computation:

$$\begin{aligned}\bar\Phi\left(z - \frac{\gamma}{z}\right) &= \frac{1}{\sqrt{2\pi}}\int_z^\infty \exp\left\{-\frac{1}{2}\left(y - \frac{\gamma}{z}\right)^2\right\} dy \\ &\le \frac{1}{\sqrt{2\pi}}\int_z^{2z} e^{-\frac{1}{2}y^2} e^{\gamma y/z}\, dy + \bar\Phi\left(2z - \frac{\gamma}{z}\right) \\ &\le e^{2\gamma}\bar\Phi(z) + \bar\Phi\left(2z - \frac{\gamma}{z}\right).\end{aligned} \tag{3.5}$$

On the other hand, if $z \ge \gamma/z$, then $2z - \gamma/z \ge z$, and so $\bar\Phi(2z - \gamma z^{-1}) \le \bar\Phi(z)$. This completes the proof. ■

## 4. Weak Convergence

4.1. **The Space** $D([0,1]^2)$**.** Let us first recall some facts about the Skorohod space $D([0,1]^2)$ which was introduced and studied in Neuhaus (1971), Straf (1972), and Bickel and Wichura (1971). Bass and Pyke (1987) provide a theory of weak convergence in $D(A)$ which subsumes that in $D([0,1]^2)$.

In a nutshell, $D([0,1]^2)$ is the collection of all bounded functions $f : [0,1]^2 \to \mathbb{R}$ such that $f$ is càdlàg with respect to the partial order $\prec$, where

$$(s,t) \prec (s',t') \iff s \le s', \text{ and } t \le t'. \tag{4.1}$$

Of course, $f$ is càdlàg with respect to $\prec$ if and only if: (i) As $(s,t) \downarrow (u,v)$ (with respect to $\prec$), $f(s,t) \to f(u,v)$; and (ii) if $(s,t) \uparrow (u,v)$, then $f((u,v)^-) := \lim f(s,t)$ exists.

Once it is endowed with a Skorohod-type metric, the space $D([0,1]^2)$ becomes a complete separable metric space (Bickel and Wichura, 1971, p. 1662).

If $X, X_1, X_2, \ldots$ are random elements of $D([0,1]^2)$, then $X_n$ is said to converge weakly to $X$ (written $X_n \Rightarrow X$) if for all bounded continuous functions $\phi : D([0,1]^2) \to \mathbb{R}$, $\lim_{n\to\infty} \mathsf{E}[\phi(X_n)] = \mathsf{E}[\phi(X)]$. Since the identity map from $C([0,1]^2)$ onto itself is a topological embedding of $C([0,1]^2)$ in $D([0,1]^2)$, if $\phi$ is a continuous functional on $C([0,1]^2)$, then it is also a continuous functional on $D([0,1]^2)$.

An important example of such a continuous functional is

$$\phi(x) := \sup_{t\in[0,1]} x(t), \qquad \forall x \in D([0,1]^2). \tag{4.2}$$

This example should provide ample details for deriving Mountford's theorem (1.9) from Theorems 1.1 and 1.4 of the present article.

4.2. **Proof of Theorem 1.1.** The proof, as is usual in weak convergence, involves two parts. First, we prove the convergence of all finite-dimensional distributions. This portion is done in the quenched setting, for then all processes involved are Gaussian and we need to compute a covariance or two only. The more interesting portion is the second part and amounts to proving tightness. Here we use, in a crucial way, a theorem of Bickel and Wichura (1971).

*Proof of Theorem 1.1.* (Finite-Dimensional Distributions) Given any four (fixed) values of $s, t, s', t' \in [0,1]$,

$$\begin{aligned} \mathsf{E}_{\mathfrak{N}}\left\{U_t^n(s)U_{t'}^n(s')\right\} &= \frac{1}{n}\mathsf{E}_{\mathfrak{N}}\left\{S_{\lfloor nt \rfloor}(s)S_{\lfloor nt' \rfloor}(s')\right\} \\ &= \frac{1}{n}\mathsf{E}_{\mathfrak{N}}\left\{S_{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor}(s)S_{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor}(s')\right\}. \end{aligned} \tag{4.3}$$

Thanks to Lemma 3.1, $\mathsf{P}$-almost surely,

$$\begin{aligned} &\mathsf{E}_{\mathfrak{N}}\left\{U_t^n(s)U_{t'}^n(s')\right\} \\ &\quad = \frac{1}{n}\left(1 - \frac{N^{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor}_{(s\wedge s') \to (s \vee s')}}{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor}\right)\left(\lfloor nt \rfloor \wedge \lfloor nt' \rfloor\right). \end{aligned} \tag{4.4}$$

On the other hand, by the strong law of large numbers, as $n \to \infty$,

$$\begin{aligned} \frac{N^{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor}_{(s\wedge s') \to (s \vee s')}}{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor} &= (1+o(1))\frac{\mathsf{E}N^{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor}_{(s\wedge s') \to (s \vee s')}}{\lfloor nt \rfloor \wedge \lfloor nt' \rfloor} \\ &\to 1 - e^{-|s'-s|}, \qquad \text{a.s. } [\mathsf{P}]. \end{aligned} \tag{4.5}$$

Therefore, $\mathsf{P}$-almost surely, $\lim_{n\to\infty} \mathsf{E}_{\mathfrak{N}}\{U_t^n(s)U_{t'}^n(s')\} = \mathsf{E}\{U_t(s)U_{t'}(s')\}$. This readily implies that $\mathsf{P}$-almost surely, the finite-dimensional distributions of $U^n$ converge weakly $[\mathsf{P}_{\mathfrak{N}}]$ to those of $U$. By the dominated convergence theorem, this implies the weak convergence, under $\mathsf{P}$, of the finite-dimensional distributions of $U^n$ to those of $U$. ■

In order to prove tightness, we appeal to a refinement to the Bickel–Wichura Theorem 3; cf. Bickel and Wichura (1971, p. 1665). To do so, we need to first recall some of the notation of Bickel and Wichura (1971).

A *block* is a two-dimensional half-open rectangle whose sides are parallel to the axes; i.e., $I$ is a block if and only if it has the form $(s,t] \times (u,v] \subseteq (0,1]^2$. Two blocks $I$ and $I'$ are *neighboring* if either: (i) $I = (s,t] \times (u,v]$ and $I' = (s',t'] \times (u,v]$ (horizontal neighboring); or (ii) $I = (s,t] \times (u,v]$ and $I' = (s,t] \times (u',v']$ (vertical neighboring).

Given any two-parameter stochastic process $Y := \{Y(s,t);\ s,t \in [0,1]\}$, and any block $I := (s,t] \times (u,v]$, the *increment of* $Y$ *over* $I$ [written as $\mathcal{Y}(I)$] is defined as

$$\mathcal{Y}(I) := Y(t,v) - Y(t,u) - Y(s,v) + Y(s,u). \tag{4.6}$$

We are ready to recall the following important result of Bickel and Wichura (1971). We have stated it in a way that best suits our later needs.

**Lemma 4.1** (Refinement to Bickel and Wichura (1971, Theorem 3))**.** *Denote by* $\{Y_n\}_{n\ge1}$ *a sequence of random fields in* $D([0,1]^2)$ *such that for all* $n \ge 1$, $Y_n(s,t) = 0$ *if* $st = 0$. *Suppose that there exist constants* $K_{4.1} > 1$, $\theta_1, \theta_2, \gamma_1, \gamma_2 > 0$ *such that they are all independent of* $n$, *and whenever* $I := (s,t] \times (u,v]$ *and* $J := (s',t'] \times (u',v']$ *are neighboring blocks, and if* $s,t,s',t' \in n^{-1}\mathbb{Z} \cap [0,1]$, *then*

$$\mathsf{E}\left\{|\mathcal{Y}_n(I)|^{\theta_1}\,|\mathcal{Y}_n(J)|^{\theta_2}\right\} \le K_{4.1}\,|I|^{\gamma_1}\,|J|^{\gamma_2}, \tag{4.7}$$

*where* $|I|$ *and* $|J|$ *denote respectively the planar Lebesgue measures of* $I$ *and* $J$. *If, in addition,* $\gamma_1 + \gamma_2 > 1$, *then* $\{Y_n\}_{n\ge1}$ *is a tight sequence.*

This is the motivation behind our next lemma which is the second, and final, step in the proof of Theorem 1.1.

**Lemma 4.2.** *The process $Y_n(t,s) := U^n_t(s)$ satisfies (4.7) with the values $K_{4.1} := 10$, $\theta_1 = \theta_2 = 2$, and $\gamma_1 = \gamma_2 = 1$. In particular, $\{U^n\}_{n\geq 1}$ is a tight sequence in $D([0,1]^2)$.*

*Proof.* We begin by proving that (4.7) indeed holds with the stated constants. This is a laborious, but otherwise uninspiring, computation which we include for the sake of completeness. This computation is divided into two successive steps, one for each possible configuration of the neighboring blocks $I$ and $J$.

*Step 1.* (Horizontal Neighboring) By stationarity, it suffices to consider only the case $I := (0,s] \times (0,u]$ and $J := (s,t] \times (0,u]$ where $s,t \in n^{-1}\mathbb{Z}$. In this case,

$$\begin{aligned} \mathcal{Y}_n(I) &= \frac{S_{ns}(u) - S_{ns}(0)}{\sqrt{n}}, \\ \mathcal{Y}_n(J) &= \frac{S_{nt}(u) - S_{nt}(0) - S_{ns}(u) + S_{ns}(0)}{\sqrt{n}}, \end{aligned} \tag{4.8}$$

which implies the independence of the two [under $\mathsf{P}_{\mathfrak{N}}$ and/or $\mathsf{P}$], since $k \mapsto S_k$ is a random walk on $D([0,1])$. Now, with $\mathsf{P}$-probability one,

$$\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\right\} = \frac{2ns - 2\mathsf{E}_{\mathfrak{N}}\left\{S_{ns}(u)S_{ns}(0)\right\}}{n} = \frac{2N^{ns}_{0\to u}}{n}. \tag{4.9}$$

See Lemma 3.1. Therefore, $\mathsf{E}\{|\mathcal{Y}_n(I)|^2\} = 2s[1-e^{-u}] \leq 2su = 2|I|$. By this and the stationarity of the infinite-dimensional random walk $k \mapsto S_k$, $\mathsf{E}\{|\mathcal{Y}_n(J)|^2\} \leq 2|J|$. In summary, in this first case of Step 1, we have shown that $\mathsf{E}\{|\mathcal{Y}_n(I)\mathcal{Y}_n(J)|^2\} \leq 4|I| \times |J|$, which is certainly less than $10|I| \times |J|$.

*Step 2.* (Vertical Neighboring) By stationarity, we need to consider only the case where $I = (0,s] \times (0,u]$ and $J = (0,s] \times (u,v]$, where $s \in n^{-1}\mathbb{Z}$. In this case,

$$\mathcal{Y}_n(I) = \frac{S_{ns}(u) - S_{ns}(0)}{\sqrt{n}}, \text{ and } \mathcal{Y}_n(J) = \frac{S_{ns}(v) - S_{ns}(u)}{\sqrt{n}}. \tag{4.10}$$

These are not independent random variables, and consequently the calculations are slightly lengthier in this case.

Using the Markov property and Lemma 3.1, we $\mathsf{P}$-almost surely have the following:

$$\begin{aligned} &\mathsf{E}_{\mathfrak{N}}\left\{ |\mathcal{Y}_n(J)|^2 \,\middle|\, \mathfrak{F}^n_u \right\} \\ &= \mathrm{Var}_{\mathfrak{N}}\left( \frac{S_{ns}(v)}{\sqrt{n}} \,\middle|\, S_{ns}(u) \right) + \left[ \mathsf{E}_{\mathfrak{N}}\left\{ \frac{S_{ns}(v) - S_{ns}(u)}{\sqrt{n}} \,\middle|\, S_{ns}(u) \right\} \right]^2 \\ &= \frac{N^{ns}_{u\to v}}{n}\left(2 - \frac{N^{ns}_{u\to v}}{ns}\right) + \left(\frac{N^{ns}_{u\to v}}{ns}\right)^2 \frac{[S_{ns}(u)]^2}{n} \\ &\leq \frac{N^{ns}_{u\to v}}{n}\left[2 + \frac{[S_n(u)]^2}{ns}\right]. \end{aligned} \tag{4.11}$$

In particular, $\mathsf{P}$-almost surely,

$$\begin{aligned}\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\,|\mathcal{Y}_n(J)|^2\right\} &= \mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\,\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(J)|^2\,\middle|\,\mathfrak{F}_u^N\right\}\right\}\\ &\le \frac{N^{ns}_{u\to v}}{n}\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\left[2+\frac{[S_{ns}(u)]^2}{ns}\right]\right\}\\ &= \frac{N^{ns}_{u\to v}}{n}\left[\frac{4N^{ns}_{0\to u}}{n}+\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\,\frac{[S_{ns}(u)]^2}{ns}\right\}\right].\end{aligned}\tag{4.12}$$

See (4.9) for the last line. Applying the Cauchy–Bunyakovsky–Schwarz inequality, we obtain

$$\begin{aligned}\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\,\frac{[S_{ns}(u)]^2}{ns}\right\} &\le \sqrt{\mathsf{E}_{\mathfrak{N}}\,|\mathcal{Y}_n(I)|^4\times\mathsf{E}_{\mathfrak{N}}\left\{\frac{[S_{ns}(u)]^4}{n^2s^2}\right\}}\\ &= \sqrt{3\mathsf{E}_{\mathfrak{N}}\,|\mathcal{Y}_n(I)|^4},\end{aligned}\tag{4.13}$$

since whenever $G$ is a centered Gaussian variate, $\mathsf{E}G^4 = 3(\mathsf{E}G^2)^2$. By applying this identity once more in conjunction with (4.9), we have

$$3\mathsf{E}_{\mathfrak{N}}\,|\mathcal{Y}_n(I)|^4 \le 9\left[\mathsf{E}_{\mathfrak{N}}\,|\mathcal{Y}_n(I)|^2\right]^2 = 36\left[\frac{N^{ns}_{0\to u}}{n}\right]^2.\tag{4.14}$$

Plugging (4.14) into (4.13) yields the following $\mathsf{P}$-almost sure inequality:

$$\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\,\frac{[S_{ns}(u)]^2}{ns}\right\} \le 6\frac{N^{ns}_{0\to u}}{n}.\tag{4.15}$$

We can plug this into (4.12) to deduce that $\mathsf{P}$-a.s.,

$$\mathsf{E}_{\mathfrak{N}}\left\{|\mathcal{Y}_n(I)|^2\,|\mathcal{Y}_n(J)|^2\right\} \le 10\frac{N^{ns}_{u\to v}}{n}\,\frac{N^{ns}_{0\to u}}{n}.\tag{4.16}$$

On the other hand, $N^{ns}_{u\to v}$ and $N^{ns}_{0\to u}$ are independent. Therefore,

$$\begin{aligned}\mathsf{E}\left\{|\mathcal{Y}_n(I)|^2\,|\mathcal{Y}_n(J)|^2\right\} &\le 10\mathsf{E}\left[\frac{N^{ns}_{u\to v}}{n}\right]\mathsf{E}\left[\frac{N^{ns}_{0\to u}}{n}\right]\\ &= 10s^2\left[1-e^{-(v-u)}\right]\left[1-e^{-u}\right]\\ &\le 10su\times s(v-u)\\ &= 10|I|\times|J|.\end{aligned}\tag{4.17}$$

We have verified (4.7) with $K_{4.7}=10$, $\theta_1=\theta_2=2$, $\gamma_1=\gamma_2=1$. Now if it were the case that $Y_n(s,t)=0$ whenever $st=0$, we would be done. However, this is not so. To get around this small difficulty, note that what we have shown thus far reveals that the random fields $(s,t)\mapsto Y_n(s,t)-n^{1/2}S_{ns}(0)$ $(n=1,2,\ldots)$ are tight. On the other hand, by Donsker's invariance principle, the processes $s\mapsto n^{-1/2}S_{ns}(0)$ $(n=1,2,\ldots)$ are tight, and the lemma follows from this and the triangle inequality. ■

## 5. A Quenched Upper Bound

Without further ado, next is the main result of this section. Note that it gives quenched tail estimates for $\sup_{t\in[r,r+1]} S_n(t)$ since the latter has the same distribution as $\sup_{t\in[0,1]} S_n(t)$.

**Theorem 5.1.** *Suppose $\{z_j\}_{j=1}^\infty$ is a nonrandom sequence that satisfies property (1.7). Then with $\mathsf{P}$-probability one, for all $\varepsilon > 0$, there exists an integer $n_0 \geq 1$ such that for all $n \geq n_0$,*

$$\mathsf{P}_{\mathfrak{N}}\left\{\sup_{t\in[0,1]} S_n(t) \geq z_n\sqrt{n}\right\} \leq (2+\varepsilon)z_n^2\bar{\Phi}(z_n). \tag{5.1}$$

In the remainder of this section we prove Theorem 5.1. Throughout, we choose and fix a sequence $z_n$ that satisfies (1.7). Based on these $z_n$'s, we define the "window size,"

$$\Delta_n := \frac{1}{16z_n^2}, \qquad \forall n \geq 1. \tag{5.2}$$

According to (1.7), the sequence $\{\Delta_j\}_{j=1}^\infty$ satisfies the conditions of Theorem 2.1. Next, define for all $n \geq 1$,

$$J_n := \int_0^1 \mathbf{1}_{\{S_n(v)\geq z_n\sqrt{n}\}}\, dv. \tag{5.3}$$

Thanks to Lemma 3.2, for any $u \geq 0$, $n \geq 1$,

$$\mathsf{E}_{\mathfrak{N}}\left\{J_n \,\middle|\, \mathfrak{F}_u^n\right\} \geq \int_u^1 \bar{\Phi}\left(\frac{z_n\sqrt{n} - \left(1-\frac{1}{n}N_{u\to v}^n\right)S_n(u)}{\sqrt{N_{u\to v}^n\left(2-\frac{1}{n}N_{u\to v}^n\right)}}\right) dv. \tag{5.4}$$

Now consider the following "good" events, where $n \geq 1$ is an integer, and $\alpha \in (0,1)$ is an arbitrarily small parameter:

$$\begin{aligned} A_{n,\alpha} &:= \left\{\sup_{\substack{0\leq s\leq t\leq 1:\\ t-s\geq\Delta_n}} \left|\frac{N_{s\to t}^n}{\mathsf{E}N_{s\to t}^n} - 1\right| \leq \alpha\right\}, \\ B_n(u) &:= \left\{S_n(u) \geq z_n\sqrt{n}\right\}. \end{aligned} \tag{5.5}$$

Next is a key technical estimate.

**Lemma 5.2.** *Choose and fix integers $n, m \geq 1$, $u \in [0, 1-\frac{1}{m}]$, and $\alpha \in (0,1)$. Then, $\mathsf{P}$-a.s.,*

$$\mathsf{E}_{\mathfrak{N}}\left\{J_n \,\middle|\, \mathfrak{F}_u^n\right\} \geq \frac{1}{(1+\alpha)z_n^2}\int_0^{z_n^2/m} \bar{\Phi}\left(\sqrt{t}\right) dt \cdot \mathbf{1}_{A_{n,\alpha}\cap B_n(u)}. \tag{5.6}$$

*Proof.* Thanks to (5.4), for any $u \geq 0$,

$$\begin{aligned} &\mathsf{E}_{\mathfrak{N}}\left\{J_n \,\middle|\, \mathfrak{F}_u^n\right\} \\ &\geq \int_u^1 \bar{\Phi}\left(\frac{z_n\sqrt{n} - \left(1-\frac{1}{n}N_{u\to v}^n\right)S_n(u)}{\sqrt{N_{u\to v}^n\left(2-\frac{1}{n}N_{u\to v}^n\right)}}\right) dv \cdot \mathbf{1}_{A_{n,\alpha}\cap B_n(u)}. \end{aligned} \tag{5.7}$$

We will estimate the terms inside $\bar\Phi$. On $B_n(u)$, we have

$$\begin{aligned}\frac{z_n\sqrt{n}-\left(1-\frac{1}{n}N^n_{u\to v}\right)S_n(u)}{\sqrt{N^n_{u\to v}\left(2-\frac{1}{n}N^n_{u\to v}\right)}} &\le \frac{z_n\sqrt{n}-\left(1-\frac{1}{n}N^n_{u\to v}\right)z_n\sqrt{n}}{\sqrt{N^n_{u\to v}}}\\ &= z_n\sqrt{\frac{N^n_{u\to v}}{n}}.\end{aligned}\tag{5.8}$$

On the other hand, on $A_{n,\alpha}$,

$$N^n_{u\to v}\le(1+\alpha)n\left(1-e^{-|v-u|}\right)\le(1+\alpha)(v-u)n.\tag{5.9}$$

Consequently, on $A_{n,\alpha}\cap B_n(u)$, the preceding two displays combine to yield the following:

$$\frac{z_n\sqrt{n}-\left(1-\frac{1}{n}N^n_{u\to v}\right)S_n(u)}{\sqrt{N^n_{u\to v}\left(2-\frac{1}{n}N^n_{u\to v}\right)}}\le z_n\sqrt{(1+\alpha)(v-u)}.\tag{5.10}$$

Because $\bar\Phi$ is decreasing, the above can be plugged into (5.7) to yield:

$$\begin{aligned}\mathsf{E}_{\mathfrak{N}}\left\{J_n\,\middle|\,\mathfrak{F}^n_u\right\} &\ge \int_u^1\bar\Phi\left(z_n\sqrt{(1+\alpha)(v-u)}\right)\,dv\cdot\mathbf{1}_{A_{n,\alpha}\cap B_n(u)}\\ &=\frac{1}{(1+\alpha)z_n^2}\int_0^{(1-u)(1+\alpha)z_n^2}\bar\Phi\left(\sqrt{t}\right)\,dt\cdot\mathbf{1}_{A_{n,\alpha}\cap B_n(u)}.\end{aligned}\tag{5.11}$$

The result follows readily from this. ■

*Proof of Theorem 5.1.* Clearly, the following holds $\mathsf{P}$-a.s. on $A_{n,\alpha}$:

$$\begin{aligned}&\mathsf{P}_{\mathfrak{N}}\left\{^\exists u\in\left[0,1-\tfrac{1}{m}\right]:\ S_n(u)\ge z_n\sqrt{n}\right\}\\ &=\mathsf{P}_{\mathfrak{N}}\left\{\sup_{u\in\left[0,1-\frac{1}{m}\right]\cap\mathbb{Q}}\mathbf{1}_{A_{n,\alpha}\cap B_n(u)}=1\right\}.\end{aligned}\tag{5.12}$$

Therefore, we can appeal to Lemma 5.2 to deduce that $\mathsf{P}$-almost surely,

$$\begin{aligned}&\mathbf{1}_{A_{\alpha,n}}\times\mathsf{P}_{\mathfrak{N}}\left\{^\exists u\in\left[0,1-\tfrac{1}{m}\right]:\ S_n(u)\ge z_n\sqrt{n}\right\}\\ &\le\mathsf{P}_{\mathfrak{N}}\left\{\sup_{u\in\left[0,1-\frac{1}{m}\right]\cap\mathbb{Q}}\mathsf{E}_{\mathfrak{N}}\left\{J_n\,\middle|\,\mathfrak{F}^n_u\right\}\ge\frac{1}{(1+\alpha)z_n^2}\int_0^{z_n^2/m}\bar\Phi\left(\sqrt{t}\right)\,dt.\right\}\\ &\le\frac{(1+\alpha)z_n^2}{\int_0^{z_n^2/m}\bar\Phi\left(\sqrt{t}\right)\,dt}\mathsf{E}_{\mathfrak{N}}\left\{J_n\right\}=\frac{(1+\alpha)z_n^2}{\int_0^{z_n^2/m}\bar\Phi\left(\sqrt{t}\right)\,dt}\bar\Phi(z_n).\end{aligned}\tag{5.13}$$

The final line uses Doob's inequality (under $\mathsf{P}_{\mathfrak{N}}$), and the stationarity of $S_n(u)$. According to Corollary 2.2, with $\mathsf{P}_{\mathfrak{N}}$-probability one, for all but finitely-many of the $n$'s, $\mathbf{1}_{A_{\alpha,n}}=1$. To finish, we note that

$$\int_0^\infty\bar\Phi\left(\sqrt{t}\right)\,dt=\frac{1}{2}.\tag{5.14}$$

Theorem 5.1 follows after letting $m\to\infty$ and $\alpha\to0$. ■

## 6. A Quenched Lower Bound

**Theorem 6.1.** *Suppose $\{z_j\}_{j=1}^\infty$ is a sequence of real numbers that satisfies (1.7). Then, there exists a random variable $n_1$ such that* P*-almost surely the following holds:*

$$\mathsf{P}_{\mathfrak{N}}\left\{\sup_{t\in[0,1]} S_n(t) \geq z_n\sqrt{n}\right\} \geq \frac{1}{9} z_n^2 \bar\Phi(z_n), \qquad \forall n \geq n_1. \tag{6.1}$$

We begin by proving Theorem 6.1.

**Lemma 6.2.** *There is some $\alpha_0 > 0$ so that for any fixed $\alpha < \alpha_0$, there exists a random variable $n_2$ such that with* P*-probability one, the following holds: For all $n \geq n_2$,*

$$\begin{aligned}&\mathsf{P}_{\mathfrak{N}}\left\{S_n(u) \geq z_n\sqrt{n}\ ,\ S_n(v) \geq z_n\sqrt{n}\right\}\\ &\qquad \leq 2\exp\left(-\frac{z_n^2(1-\alpha)(v-u)}{4}\right)\bar\Phi(z_n),\end{aligned} \tag{6.2}$$

*for all $0 \leq u \leq v \leq 1$ such that $v - u \geq \Delta_n$, where $\Delta_n$ is defined in (5.2).*

*Proof.* In the course of our proof of Theorem 5.1 we observed that for any $\alpha \in (0,1)$, $\mathbf{1}_{A_{n,\alpha}} = 1$ for all but a finite number of $n$'s. Thus, it suffices to derive the inequality of this lemma on the set $A_{n,\alpha}$. Recall that the latter event was defined in (5.5).

By Lemma 3.2,

$$\begin{aligned}&\mathsf{P}_{\mathfrak{N}}\left\{S_n(v) \geq z_n\sqrt{n}\ ,\ S_n(u) \geq z_n\sqrt{n}\right\}\\ &\quad = \int_{z_n}^\infty \bar\Phi\left(\frac{z_n\sqrt{n} - x\sqrt{n}\left(1 - \frac{1}{n}N^n_{u\to v}\right)}{\sqrt{N^n_{u\to v}\left[2 - \frac{1}{n}N^n_{u\to v}\right]}}\right)\Phi(dx).\end{aligned} \tag{6.3}$$

A computation shows that if $x \geq z_n$, then the function

$$\frac{z_n - x(1-u)}{\sqrt{u(2-u)}}, \tag{6.4}$$

is increasing for $u \in [0,1]$. On the other hand, on $A_{n,\alpha}$, we have

$$N^n_{u\to v} \geq n(1-\alpha)(1 - e^{-(v-u)}) \geq n\frac{1}{2}(1-\alpha)(v-u); \tag{6.5}$$

cf. (2.4). Therefore,

$$\begin{aligned}&\mathsf{P}_{\mathfrak{N}}\left\{S_n(v) \geq z_n\sqrt{n}\ ,\ S_n(u) \geq z_n\sqrt{n}\right\}\\ &\leq \int_{z_n}^\infty \bar\Phi\left(\frac{z_n - x\left(1 - \frac{1}{2}(1-\alpha)(v-u)\right)}{\sqrt{\frac{1}{2}(1-\alpha)(v-u)\left[2 - \frac{1}{2}(1-\alpha)(v-u)\right]}}\right)\Phi(dx)\\ &= \int_{z_n}^\infty \bar\Phi\left(\frac{\frac{1}{2}x(1-\alpha)(v-u) - (x - z_n)}{\sqrt{\frac{1}{2}(1-\alpha)(v-u)}}\right)\Phi(dx)\\ &:= \mathrm{I}_1 + \mathrm{I}_2,\end{aligned} \tag{6.6}$$

where $\mathrm{I}_1 := \int_{z_n}^{(1+\eta)z_n} \bar\Phi(\cdots)\,\Phi(dx)$, $\mathrm{I}_2 := \int_{z_n(1+\eta)}^\infty \bar\Phi(\cdots)\,\Phi(dx)$, and

$$\eta := \frac{\gamma}{2}(1-\alpha)(v-u). \tag{6.7}$$

$\gamma \in (0,1)$ is a parameter to be determined. For the estimation of $\mathrm{I}_1$, we note that if $x \in [z_n, z_n(1+\eta)]$, then $\frac{1}{2}x\sqrt{n}(1-\alpha)(v-u)-(x-z_n)\sqrt{n} \geq z_n\frac{1}{2}(1-\alpha)(v-u)\,(1-\gamma)$, and we obtain the following:

$$\begin{aligned} \mathrm{I}_1 &\leq \int_{z_n}^{\infty} \bar{\Phi}\left(z_n(1-\gamma)\sqrt{\frac{1}{2}(1-\alpha)(v-u)}\right)\Phi(dx) \\ &\leq \exp\left(-\frac{z_n^2(1-\gamma)^2(1-\alpha)(v-u)}{4}\right)\bar{\Phi}(z_n), \end{aligned} \tag{6.8}$$

where the last line follows from (1.17). The integral $\mathrm{I}_2$ is also easily estimated: Since $\bar{\Phi}(t) \leq 1$, we have

$$\mathrm{I}_2 \leq \bar{\Phi}\left(z_n(1+\eta)\right) \leq \exp\left(-\eta z_n^2\right)\bar{\Phi}(z_n) \leq e^{-\eta z_n^2}\bar{\Phi}(z_n). \tag{6.9}$$

We have appealed to Lemma 3.3 in the penultimate inequality. Now replace $\eta$ by its value defined in (6.7) in order to obtain

$$\mathrm{I}_2 \leq \exp\left(-z_n^2\frac{\gamma}{2}(1-\alpha)(v-u)\right)\bar{\Phi}(z_n)\,. \tag{6.10}$$

Taking $\gamma$ to be the solution of $\gamma = \frac{(1-\gamma)^2}{2}$ in $[0,1]$ we have that

$$\mathrm{I}_1 + \mathrm{I}_2 \leq 2\exp\left(-\left(2-\sqrt{3}\right)(1-\alpha)(v-u)\right)\bar{\Phi}(z_n)\,, \tag{6.11}$$

the result follows from the fact that $(2-\sqrt{3}) \leq \frac{1}{4}$. ■

*Proof of Theorem 6.1.* We recall (5.3) and appeal to Lemma 6.2 to see that $\mathsf{P}$-a.s., for all $n \geq n_3$,

$$\begin{aligned} \mathsf{E}_{\mathfrak{N}}\left\{J_n^2\right\} &= 2\int_0^1\int_u^1 \mathsf{P}_{\mathfrak{N}}\left\{S_n(v) \geq z_n\sqrt{n}\ ,\ S_n(u) \geq z_n\sqrt{n}\right\}dv\,du \\ &\leq 2\bar{\Phi}(z_n)\int_0^{1-\Delta_n}\int_{u+\Delta_n}^1 \exp\left(-\frac{z_n^2(1-\alpha)(v-u)}{4}\right)dv\,du \\ &\quad + 2\Delta_n\bar{\Phi}(z_n) \\ &\leq z_n^{-2}\bar{\Phi}(z_n)\left[\frac{8}{(1-\alpha)} + \frac{2}{16}\right]. \end{aligned} \tag{6.12}$$

We have used the definition (5.2) of $\Delta_n$ in the last line. Let us choose $\alpha$ small enough so that $8/(1-\alpha)+1/8 < 9$. Then, we obtain:

$$\mathsf{E}_{\mathfrak{N}}\left\{J_n^2\right\} \leq 9z_n^{-2}\bar{\Phi}(z_n), \text{ a.s. on } A_{\alpha,n}. \tag{6.13}$$

Thus, by the Paley–Zygmund inequality, almost surely on $A_{\alpha,n}$,

$$\mathsf{P}_{\mathfrak{N}}\left\{J_n > 0\right\} \geq \frac{\left(\mathsf{E}_{\mathfrak{N}}J_n\right)^2}{\mathsf{E}_{\mathfrak{N}}J_n^2} \geq \frac{1}{9}z_n^2\bar{\Phi}(z_n). \tag{6.14}$$

The theorem follows readily from this and the obvious fact that $\{J_n(z_n) > 0\} \subseteq \{{}^\exists u \leq 1 : S_n(u) \geq z_n\sqrt{n}\}$. ■

## 7. Proof of Theorem 1.4

We start by proving the simpler lower bound. Fix $\alpha \in (0,1)$, let $W_n$ denote the $\mathsf{P}_{\mathfrak{N}}$-probability that $\sup_{t\in[0,1]} S_n(t) \geq z_n\sqrt{n}$, and define $f_n := z_n^2\bar{\Phi}(z_n)$. [We will use this notation throughout the proof.] Then, according to (6.14), $9W_n \geq f_n$, $\mathsf{P}$-almost surely on $A_{\alpha,n}$. Theorem 2.1 implies that $\mathsf{P}(A_{\alpha,n}^{\complement}) \to 1$, as $n \to \infty$. In particular, as $n \to \infty$, $\mathsf{P}\{9W_n \geq f_n\} = 1 + o(1)$. This, and Chebyshev's inequality, together imply that $9\mathsf{E}W_n \geq (1 + o(1))f_n$, which is the desired lower bound in scrambled form. We now prove the corresponding probability upper bound of Theorem 1.4.

Let $\Pi_n$ denote the total number of replacements to the incremental processes $\{X_k(\cdot)\}_{k=1}^n$ during the time-interval $[0,1]$. That is,

$$\Pi_n := \sum_{s\in(0,1]} \Delta\Pi_n(s), \qquad \text{where } \Delta\Pi_n(s) := \sum_{k=1}^n \mathbf{1}_{\{X_k(s)-X_k(s-)\neq 0\}} \tag{7.1}$$

Because $\Pi_n$ is a Poisson random variable with mean $n$, $\mathsf{E}\{e^{t\Pi_n}\} = \exp(-n + e^t n)$ for all $t > 0$. This readily yields the following well-known Chernoff-type bound: For all $x > 0$,

$$\mathsf{P}\left\{\Pi_n \geq x\right\} \leq \inf_{t>0} \exp\left(-n + e^t n - tx\right) = \exp\left\{-n - x\ln\left(\frac{x}{en}\right)\right\}. \tag{7.2}$$

Consequently, by (1.7),

$$\mathsf{P}\left(G_n^{\complement}\right) \leq e^{-n} = o\left(f_n\right), \text{ where } G_n := \left\{\Pi_n \leq 3n\right\}, \qquad \forall n \geq 1. \tag{7.3}$$

A significant feature of the event $G_n$ is that $\mathsf{P}$-almost surely,

$$\mathbf{1}_{G_n} W_n \leq 3n\mathsf{P}\left\{S_n(0) \geq z_n\sqrt{n}\right\} = 3n\bar{\Phi}(z_n). \tag{7.4}$$

(Indeed, if $G_n$ holds, then $W_n$ is the chance that the maximum of at most $3n$ dependent Gaussian random walks exceeds $z_n\sqrt{n}$.) Thus, we can write the almost sure [$\mathsf{P}$] bound,

$$\mathbf{1}_{A_{\alpha,n}^{\complement}} W_n \leq \mathbf{1}_{G_n^{\complement}} + 3n\bar{\Phi}(z_n)\mathbf{1}_{A_{\alpha,n}^{\complement}}. \tag{7.5}$$

Combined with (5.13) and (6.2) (for suitable small $\alpha$), this yields

$$W_n \leq (2 + o(1))f_n + \mathbf{1}_{G_n^{\complement}} + 3n\bar{\Phi}(z_n)\mathbf{1}_{A_{\alpha,n}^{\complement}}. \tag{7.6}$$

In this formula, $o(1)$ denotes a nonrandom term that goes to zero as $n$ tends to infinity. We take expectations and appeal to Theorem 2.1 with $\Delta_n := (16z_n^2)^{-1}$ (cf. 5.2), as well as (7.3), to deduce the following:

$$\mathsf{E}\left\{W_n\right\} \leq (2 + o(1)\,)f_n + \frac{8192}{\alpha^2} n z_n^2 f_n \exp\left(-\frac{3\alpha^3 n}{36864 z_n^2}\right). \tag{7.7}$$

Condition (1.7) guarantees that the right-hand side is asymptotically equal to $(2 + o(1))f_n$, as $n \to \infty$. This proves the theorem. □

## 8. Proof of Theorem 1.5

Throughout, $\log(x) := \log x := \ln(e \vee x)$, and consider the *Erdős sequence*:

$$\mathsf{e}_n := \mathsf{e}(n) := \left\lfloor \exp\left(\frac{n}{\log(n)}\right)\right\rfloor, \qquad \forall n \geq 1. \tag{8.1}$$

Note that the sequence $\{\mathsf{e}_j\}_{j=1}^{\infty}$ satisfies the following *gap property*:

$$\mathsf{e}_{n+1} - \mathsf{e}_n = \frac{\mathsf{e}_n}{\log(n)}(1+o(1)) = \frac{\mathsf{e}_n}{\log\log(\mathsf{e}_n)}(1+o(1)), \quad (n \to \infty). \tag{8.2}$$

[This was noted in Erdős (1942, eq. (0.11))] Furthermore, we can combine the truncation argument of Erdős (1942) [eq.'s (1.2) and (3.4)] with our equation (1.16) to deduce the following: Without loss of generality,

$$\sqrt{\log\log(t)} \le H(t) \le 2\sqrt{\log\log(t)} \quad {}^{\forall} t \ge 1. \tag{8.3}$$

The following is a standard consequence.

**Lemma 8.1.** *If $H$ is a nonnegative nondecreasing measurable function that satisfies (8.3), then*

$$\mathfrak{J}(H) < +\infty \iff \sum_n H^2(\mathsf{e}_n)\bar{\Phi}(H(\mathsf{e}_n)) < +\infty, \tag{8.4}$$

*where $\mathfrak{J}(H)$ is defined in (1.10).*

We are ready to prove (the easier) part (i) of Theorem 1.5.

*Proof of Theorem 1.5 (First Half).* In the first portion of our proof, we assume that $\mathfrak{J}(H) < +\infty$, and recall that without loss of generality, (8.3) is assumed to hold.

It is easy to see that $\{X_j\}_{j=1}^{\infty}$ are i.i.d. elements of $D([0,1])$—the space of càdlàg real paths on $[0,1]$—which implies that $n \mapsto S_n$ is a symmetric random walk on $D([0,1])$. In particular, an infinite-dimensional reflection argument implies that for all $n \ge 1$ and $\lambda > 0$,

$$\mathsf{P}\left\{ \max_{1\le k\le n} \sup_{t\in[0,1]} S_k(t) \ge \lambda \right\} \le 2\mathsf{P}\left\{ \sup_{t\in[0,1]} S_n(t) \ge \lambda \right\}. \tag{8.5}$$

See Khoshnevisan (2003, Lemma 3.5) for the details of this argument. Consequently, as $n \to \infty$,

$$\begin{aligned}
&\mathsf{P}\left\{ \max_{1\le k\le \mathsf{e}(n+1)} \sup_{t\in[0,1]} S_k(t) \ge H(\mathsf{e}_n)\sqrt{\mathsf{e}_n} \right\} \\
&\le 2\mathsf{P}\left\{ \sup_{t\in[0,1]} S_{\mathsf{e}(n+1)}(t) \ge H(\mathsf{e}_n)\sqrt{\mathsf{e}_n} \right\} \\
&\le 2\mathsf{P}\left\{ \sup_{t\in[0,1]} S_{\mathsf{e}(n+1)}(t) \ge H(\mathsf{e}_n)\sqrt{\mathsf{e}_{n+1}} \left[1 - \frac{2+o(1)}{H(\mathsf{e}_n)}\right] \right\}.
\end{aligned} \tag{8.6}$$

We have appealed to (8.2) in the last line. At this point, (8.3) and Theorem 1.4 together imply that as $n \to \infty$,

$$\begin{aligned}
&\mathsf{P}\left\{ \max_{1\le k\le \mathsf{e}(n+1)} \sup_{t\in[0,1]} S_k(t) \ge H(\mathsf{e}_n)\sqrt{\mathsf{e}_n} \right\} \\
&\le [4+o(1)]H^2(\mathsf{e}_n)\bar{\Phi}\left( H(\mathsf{e}_n)\left[1 - \frac{2+o(1)}{H(\mathsf{e}_n)}\right]\right) \\
&\le (e^4 4 + o(1))H^2(\mathsf{e}_n)\bar{\Phi}\left(H(\mathsf{e}_n)\right),
\end{aligned} \tag{8.7}$$

the last line follows from Lemma 3.4. Lemma 8.1 and the finiteness assumption on $\mathfrak{J}(H)$ together yield the summability of the left-most probability in the preceding display. By the Borel–Cantelli lemma, almost surely for all but a finite number of $n$'s,

$$\max_{1\le k\le \mathsf{e}(n+1)} \sup_{t\in[0,1]} S_k(t) < H(\mathsf{e}_n)\sqrt{\mathsf{e}_n}. \tag{8.8}$$

Now any $m$ can be sandwiched between $\mathsf{e}_n$ and $\mathsf{e}_{n+1}$ for some $n := n(m)$. Hence, a.s. for all but a finite number of $m$'s,

$$\sup_{t\in[0,1]} S_m(t) \le \max_{1\le k\le \mathsf{e}(n+1)} \sup_{t\in[0,1]} S_k(t) < H(\mathsf{e}_n)\sqrt{\mathsf{e}_n} \le H(m)\sqrt{m}. \tag{8.9}$$

This completes our proof of part (i). ■

The remainder of this section is concerned with proving the more difficult second part of Theorem 1.5. We will continue to use the Erdős sequence $\{\mathsf{e}_j\}_{j=1}^\infty$ as defined in (8.1). We will also assume—still without loss of generality—that (8.3) holds, although now $\mathcal{J}(H) = +\infty$.

We introduce the following notation in order to simplify the exposition:

$$\begin{aligned}
S_n^* &:= \sup_{t\in[0,1]} S_{\mathsf{e}(n)}(t)\\
H_n &:= H(\mathsf{e}_n)\\
\mathcal{I}_n &:= \left[H_n\sqrt{\mathsf{e}_n}, \left(H_n + \frac{14}{H_n}\right)\sqrt{\mathsf{e}_n}\right]\\
L_n &:= \sum_{j=1}^n \mathbf{1}_{\{S_j^*\in\mathcal{I}_j\}}\\
f(z) &:= z^2\bar\Phi(z), \quad \forall z>0.
\end{aligned} \tag{8.10}$$

Here is a little localization lemma that states that $\mathcal{I}_n$ and $[H_n\sqrt{\mathsf{e}_n}, +\infty]$ have, more or less, the same dynamical-walk-measure.

**Lemma 8.2.** *As $n\to\infty$,*

$$\left(10^{-2} + o(1)\right) \le \frac{\mathsf{P}\left\{S_n^*\in\mathcal{I}_n\right\}}{\mathsf{P}\left\{S_n^* \ge H_n\sqrt{\mathsf{e}_n}\right\}} \le 1. \tag{8.11}$$

*Proof.* Because $9^{-1} \ge 0.1$, Theorem 1.4 implies that as $n\to\infty$,

$$\begin{aligned}
\mathsf{P}\left\{S_n^*\in\mathcal{I}_n\right\} &\ge (0.1+o(1))\, f(H_n) - (2+o(1))\, H_n^2\bar\Phi\left(H_n + \frac{14}{H_n}\right)\\
&\ge (0.1+o(1))\, f(H_n) - (2+o(1))\, e^{-14} f(H_n).
\end{aligned} \tag{8.12}$$

(The second line holds because of Lemma 3.3.) Since $0.1 - 2e^{-14} \le 0.09$, the lemma follows Theorem 1.4 and a few lines of arithmetic. ■

Since we are assuming that $\mathcal{J}(H) = +\infty$, Lemmas 8.1 and 8.2 together imply that as $n\to\infty$, $\mathsf{E}L_n \to +\infty$. We intend to show that

$$\limsup_{n\to\infty} \frac{\mathsf{E}\left\{L_n^2\right\}}{\left(\mathsf{E}L_n\right)^2} < +\infty. \tag{8.13}$$

If so, then the Chebyshev inequality shows that $\limsup_{n\to\infty} L_n/\mathsf{E}L_n > 0$ with positive probability. This implies that with positive probability, $L_\infty = +\infty$, so that the following would then conclude the proof.

**Lemma 8.3.** *If $\rho := \mathsf{P}\{L_\infty = +\infty\} > 0$, then $\rho = 1$, and part (ii) of Theorem 1.5 holds.*

*Proof.* We have already observed that $n \mapsto S_n$ is a random walk in $D([0,1])$. Therefore, by the Hewitt–Savage 0–1 law, $L_\infty = +\infty$, a.s.

Now consider

$$\mathcal{W}_n := \left\{ t \geq 0 : S_{\mathsf{e}(n)}(t) \vee S_{\mathsf{e}(n)}(t-) \geq H_n \sqrt{\mathsf{e}_n} \right\}, \qquad \forall n \geq 1. \tag{8.14}$$

This is a random open set, and

$$\{L_\infty = +\infty\} \subseteq \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty \{\mathcal{W}_m \cap [0,1] \neq \varnothing\}. \tag{8.15}$$

More generally still, for any $0 \leq a < b$,

$$\{L_\infty(a,b) = +\infty\} \subseteq \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty \{\mathcal{W}_m \cap [a,b] \neq \varnothing\}, \tag{8.16}$$

where $L_n(a,b) := \sum_{j=1}^n \mathbf{1}_{\{\sup_{t\in[a,b]} S_{\mathsf{e}(j)}(t) \in \mathcal{I}_j\}}$. But by the stationarity of the $\mathbb{R}^\infty$-valued process $t \mapsto S_\bullet(t)$, $L_\infty(a,b)$ has the same distribution as $L_\infty(0, b-a)$, and this means that with probability one, $L_\infty(a,b) = +\infty$ for all rational $0 \leq a < b$. Therefore, according to (8.16),

$$\mathsf{P}\left\{ \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty \{\mathcal{W}_m \cap [a,b] \neq \varnothing\} \right\} = 1. \tag{8.17}$$

This development shows that for any $n$, $\mathcal{W}^n := \cup_{m \geq n} \mathcal{W}_m$ is a random open set that is a.s. everywhere dense. Thanks to the Baire category theorem, $\mathcal{W} := \cap_n \mathcal{W}^n \cap [0,1]$ is [a.s.] uncountable. Now any $t \in \mathcal{W} \cap [0,1]$ satisfies the following:

$$S_\ell(t) \vee S_\ell(t-) \geq H(\ell)\sqrt{\ell}, \text{ for infinitely many } \ell\text{'s}. \tag{8.18}$$

On the other hand, the jump structure of the Poisson clocks tells us that $\mathcal{J} := \cup_{\ell \geq 1}\{t \geq 0 : S_\ell(t) \neq S_\ell(t-)\}$ is [a.s.] denumerable. Because $\mathcal{W}$ is uncountable [a.s.], any $t \in \mathcal{W} \cap \mathcal{J}^{\mathsf{C}}$ satisfies assertion (ii) of Theorem 1.5. ∎

We now begin working toward our proof of (8.13). We write

$$\mathsf{E}\left\{L_n^2\right\} = \mathsf{E}L_n + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathcal{P}_{i,j}, \tag{8.19}$$

where

$$\mathcal{P}_{i,j} = \mathsf{P}\left\{S_i^* \in \mathcal{I}_i\ ,\ S_j^* \in \mathcal{I}_j\right\}, \qquad \forall i > j \geq 1. \tag{8.20}$$

In estimating $\mathcal{P}_{i,j}$, our first observation is the following.

**Lemma 8.4.** *There exists a finite and positive universal constant $K_{8.4}$ such that for all $j > i \geq 1$,*

$$\mathcal{P}_{i,j} \leq K_{8.4} \mathsf{P}\left\{S_i^* \in \mathcal{I}_i\right\} \mathcal{Q}_{i,j} \tag{8.21}$$

*where*

$$\mathcal{Q}_{i,j} := f\left( H_j \sqrt{\frac{\mathsf{e}_j}{\mathsf{e}_j - \mathsf{e}_i}} - H_i \sqrt{\frac{\mathsf{e}_i}{\mathsf{e}_j - \mathsf{e}_i}} - \frac{14}{H_i}\sqrt{\frac{\mathsf{e}_i}{\mathsf{e}_j - \mathsf{e}_i}} \right). \tag{8.22}$$

*Proof.* Recall that $n \mapsto S_n$ is a random walk on $D([0,1])$. Therefore,

$$\begin{aligned} \mathcal{P}_{i,j} &\le \mathsf{P}\left\{S_i^* \in \mathcal{I}_j\right\} \\ &\quad \times \mathsf{P}\left\{ \sup_{t\in[0,1]} \left(S_{\mathsf{e}_j}(t) - S_{\mathsf{e}_i}(t)\right) \ge H_j\sqrt{\mathsf{e}_j} - \sqrt{\mathsf{e}_i}\left[H_i + \frac{14}{H_i}\right]\right\} \\ &= \mathsf{P}\left\{S_i^* \in \mathcal{I}_i\right\} \mathsf{P}\left\{ \sup_{t\in[0,1]} S_{\mathsf{e}_j - \mathsf{e}_i}(t) \ge H_j\sqrt{\mathsf{e}_j} - \sqrt{\mathsf{e}_i}\left[H_i + \frac{14}{H_i}\right]\right\}. \end{aligned} \tag{8.23}$$

Therefore, Theorem 1.4 will do the rest, once we check that uniformly for all $j > i$,

$$\frac{H_j\sqrt{\mathsf{e}_j}}{\sqrt{\mathsf{e}_j - \mathsf{e}_i}} = o\left(\sqrt{\frac{\mathsf{e}_j - \mathsf{e}_i}{\log(\mathsf{e}_j - \mathsf{e}_i)}}\right) \qquad (i \to \infty). \tag{8.24}$$

Equivalently, we wish to prove that uniformly for all $j > i$,

$$H_j\sqrt{\mathsf{e}_j} = o\left(\frac{\mathsf{e}_j - \mathsf{e}_i}{\sqrt{\log(\mathsf{e}_j - \mathsf{e}_i)}}\right) \qquad (i \to \infty). \tag{8.25}$$

By (8.3), the left-hand side is bounded above as follows:

$$H_j\sqrt{\mathsf{e}_j} \le (2 + o(1))\sqrt{\mathsf{e}_j \log\log \mathsf{e}_j} = O\left(\sqrt{\mathsf{e}_j \log j}\right), \quad (j \to \infty). \tag{8.26}$$

On the other hand,

$$\frac{\mathsf{e}_j - \mathsf{e}_i}{\sqrt{\log(\mathsf{e}_j - \mathsf{e}_i)}} \ge \frac{\mathsf{e}_j - \mathsf{e}_i}{\sqrt{\log \mathsf{e}_j}} = (\mathsf{e}_j - \mathsf{e}_i)\sqrt{\frac{\log j}{j}}. \tag{8.27}$$

In light of (8.26) and (8.27), (8.24)—and hence the lemma—is proved once we verify that as $i \to \infty$, $\sqrt{j\mathsf{e}_j} = o(\mathsf{e}_j - \mathsf{e}_i)$ uniformly for all $j > i$. But this follows from the gap condition of the sequence $\mathsf{e}_1, \mathsf{e}_2, \ldots$. Indeed, (8.2) implies that uniformly for all $j > i$,

$$\mathsf{e}_j - \mathsf{e}_i \ge \mathsf{e}_j - \mathsf{e}_{j-1} = (1 + o(1))\frac{\mathsf{e}_j}{\log j} \quad (i \to \infty). \tag{8.28}$$

So it suffices to check that as $j \to \infty$, $\sqrt{j\mathsf{e}_j} = o(\mathsf{e}_j/\log j)$, which is a trivial matter. ■

Motivated by the ideas of Pál Erdős (1942), we consider the size of $\mathcal{Q}_{i,j}$ on three different scales, where $\mathcal{Q}_{i,j}$ is defined in (8.22). The mentioned scales are based on the size of the "correlation gap," $(j - i)$. Our next three lemmas reflect this viewpoint.

**Lemma 8.5.** *There exists a finite and positive universal constant $K_{8.5}$ such that for all integers $i$ and $j > i + [\log i]^{10}$,*

$$\mathcal{Q}_{i,j} \le K_{8.5}\mathsf{P}\left\{S_j^* \in \mathcal{I}_j\right\}. \tag{8.29}$$

*Proof.* We will require the following consequence of (8.2): Uniformly for all integers $j > i$,

$$\mathsf{e}_j - \mathsf{e}_i = \sum_{l=i}^{j-1}(\mathsf{e}_{l+1} - \mathsf{e}_l) \ge \frac{(j-i)\mathsf{e}_i}{\log i}(1 + o(1)) \qquad (i \to \infty). \tag{8.30}$$

Now we proceed with the proof.

Since $\mathsf{e}_j/(\mathsf{e}_j - \mathsf{e}_i) \geq 1$, (8.22) implies that

$$\mathcal{Q}_{i,j} \leq f\left(H_j - \sqrt{\frac{\mathsf{e}_i}{\mathsf{e}_j - \mathsf{e}_i}}\left[H_i + \frac{14}{H_i}\right]\right). \tag{8.31}$$

We intend to prove that uniformly for every integer $j \geq i + [\log i]^{10}$,

$$\sqrt{\frac{\mathsf{e}_i}{\mathsf{e}_j - \mathsf{e}_i}}\left[H_i + \frac{14}{H_i}\right] = O\left(H_j^{-1}\right) \qquad (i \to \infty). \tag{8.32}$$

Given this for the time being, we finish the proof as follows: Note that the preceding display and (3.4) together prove that uniformly for every integer $j \geq i + [\log i]^{10}$, $\mathcal{Q}_{i,j} = O(f(H_j))$ as $i \to \infty$. According to Theorem 1.4, for this range of $(i, j)$, $\mathcal{Q}_{i,j} = O(\mathsf{P}\{S_j^* \geq H_j\sqrt{\mathsf{e}_j}\})$. Thanks to Lemma 8.2, this is $O(\mathsf{P}\{S_j^* \in \mathcal{I}_j\})$. The result follows easily from this, therefore it is enough to derive (8.32).

Because of (8.3), equation (8.32) is equivalent to the following: Uniformly for every integer $j \geq i + [\log i]^{10}$,

$$\frac{\mathsf{e}_i(\log i)(\log j)}{\mathsf{e}_j - \mathsf{e}_i} = O(1) \qquad (i \to \infty). \tag{8.33}$$

But thanks to (8.30), uniformly for all integers $j > i + [\log i]^{10}$, the left-hand side is at most

$$(1 + o(1))\frac{[\log i]^2 \log\left(i + [\log i]^{10}\right)}{[\log i]^{10}} = o(1) \qquad (i \to \infty). \tag{8.34}$$

This completes our proof. ■

**Lemma 8.6.** *Uniformly for all integers $j \in [i + \log i, i + [\log i]^{10}]$,*

$$\mathcal{Q}_{i,j} \leq i^{-\frac{1}{4}+o(1)} \qquad (i \to \infty). \tag{8.35}$$

*Proof.* Whenever $j > i$, we have $H_j \geq H_i$. Thus, the (eventual) monotonicity of $f$ implies that as $i \to \infty$, the following holds uniformly for all $j > i$:

$$\begin{aligned}
\mathcal{Q}_{i,j} &\leq f\left(H_i\left[\sqrt{\frac{\mathsf{e}_j}{\mathsf{e}_j - \mathsf{e}_i}} - \sqrt{\frac{\mathsf{e}_i}{\mathsf{e}_j - \mathsf{e}_i}} - \frac{14}{H_i^2}\sqrt{\frac{\mathsf{e}_i}{\mathsf{e}_j - \mathsf{e}_i}}\right]\right) \\
&= f\left(H_i\left[\frac{\sqrt{\mathsf{e}_j - \mathsf{e}_i}}{\sqrt{\mathsf{e}_j} + \sqrt{\mathsf{e}_i}} - \frac{14}{H_i^2}\sqrt{\frac{\mathsf{e}_i}{\mathsf{e}_j - \mathsf{e}_i}}\right]\right) \\
&\leq f\left(H_i\left[\frac{\sqrt{\mathsf{e}_j - \mathsf{e}_i}}{\sqrt{\mathsf{e}_j} + \sqrt{\mathsf{e}_i}} - \frac{14 + o(1)}{H_i^2}\sqrt{\frac{\mathsf{e}_i \log j}{\mathsf{e}_j}}\right]\right).
\end{aligned} \tag{8.36}$$

[The last line relies on (8.28).] According to (8.3), and after appealing to the trivial inequality that $\mathsf{e}_j \geq \mathsf{e}_i$, we arrive at the following: As $i \to \infty$, then uniformly for all integers $j \in [i + \log i, i + [\log i]^{10}]$,

$$\begin{aligned}
\mathcal{Q}_{i,j} &\leq f\left(\frac{1 + o(1)}{2}\sqrt{\log i}\left[\sqrt{\frac{\mathsf{e}_j - \mathsf{e}_i}{\mathsf{e}_j}} - O\left(\frac{\sqrt{\log j}}{\log i}\right)\right]\right) \\
&\leq f\left(\frac{1 + o(1)}{2}\left[\sqrt{\log i}\sqrt{\frac{\mathsf{e}_j - \mathsf{e}_i}{\mathsf{e}_j}} - O(1)\right]\right) \\
&\leq \exp\left\{-\frac{1 + o(1)}{4}\left[\frac{\mathsf{e}_j - \mathsf{e}_i}{\mathsf{e}_j}\right]\log i\right\}.
\end{aligned} \tag{8.37}$$

[The last line holds because of the first inequality in (1.17).] On the other hand, uniformly for all $j \geq i + \log i$,

$$\begin{aligned}\frac{\mathsf{e}_j}{\mathsf{e}_i} &= \exp\left(\frac{j}{\log j} - \frac{i}{\log i}\right)\\ &\geq \exp\left(\frac{i+\log i}{\log(i+\log i)} - \frac{i}{\log i}\right)\\ &\geq 2 + o(1) \qquad (i \to \infty).\end{aligned} \tag{8.38}$$

Consequently, $\mathsf{e}_j - \mathsf{e}_i \geq (1+o(1))\mathsf{e}_j$. This and (8.37) together yield the lemma. ■

**Lemma 8.7.** *Uniformly for all integers $j \in (i, i+\log i]$,*

$$\mathcal{Q}_{i,j} \leq \exp\left\{-\frac{1+o(1)}{4e}(j-i)\right\} \qquad (i \to \infty). \tag{8.39}$$

*Proof.* Equation (8.30) tell us that uniformly for all integers $j > i$, and as $i \to \infty$, $\mathsf{e}_j - \mathsf{e}_i \geq (1+o(1))\mathsf{e}_i(j-i)/\log i$. On the other hand, for $j \in (i, i+\log i]$,

$$\frac{\mathsf{e}_j}{\mathsf{e}_i} = \exp\left(\frac{j}{\log j} - \frac{i}{\log i}\right) \leq \exp\left(\frac{j-i}{\log i}\right) \leq e. \tag{8.40}$$

The preceding two displays together yield that uniformly for all integers $j \in (i, i+\log i]$, $\mathsf{e}_j^{-1}(\mathsf{e}_j - \mathsf{e}_i) \geq (1+o(1))(j-i)/(e\log i)$ $(i \to \infty)$. The lemma follows from this and (8.37). ■

We are ready to commence with the following.

*Proof of Theorem 1.5.* Recall that $\mathsf{E}L_n \to \infty$, and our goal is to verify (8.13). According to Lemma 8.4, given any two positive integers $n > k$,

$$\begin{aligned}\mathsf{E}\left\{(L_n - L_k)^2\right\} &= \mathsf{E}\left\{L_n - L_k\right\} + 2\sum_{i=k}^{n-1}\sum_{j=i+1}^{n} \mathcal{P}_{i,j}\\ &\leq \mathsf{E}L_n + 2K_{8.4}\sum_{i=k}^{n-1}\sum_{j=i+1}^{n} \mathsf{P}\left\{S_i^* \in \mathcal{I}_i\right\}\mathcal{Q}_{i,j}.\end{aligned} \tag{8.41}$$

We split the double-sum according to whether $j > i + [\log i]^{10}$, $j \in (i+\log i, i+[\log i]^{10}]$, or $j \in (i, i+\log i]$ and respectively apply Lemmas 8.5, 8.6, and 8.7 to deduce the existence of an integer $\nu \geq 1$ such that for all $n > \nu$,

$$\begin{aligned}&\mathsf{E}\left\{(L_n - L_\nu)^2\right\}\\ &\leq \mathsf{E}L_n + 2K_{8.4}K_{8.5}\sum_{\substack{\nu \leq i \leq n\\ n \geq j > i+[\log i]^{10}}}\sum \mathsf{P}\left\{S_i^* \in \mathcal{I}_i\right\}\mathsf{P}\left\{S_j^* \in \mathcal{I}_j\right\}\\ &\quad + 2K_{8.4}\sum_{\substack{\nu \leq i \leq n\\ j \in \left(i+\log i, i+[\log i]^{10}\right]}}\sum i^{-1/8}\mathsf{P}\left\{S_i^* \in \mathcal{I}_i\right\}\\ &\quad + 2K_{8.4}\sum_{\substack{\nu \leq i \leq n\\ j \in (i, i+\log i]}}\sum e^{-(j-i)/12}\mathsf{P}\left\{S_i^* \in \mathcal{I}_i\right\}.\end{aligned} \tag{8.42}$$

Since $\mathsf{E}L_n \to \infty$, the above is at most $2K_{8.4}K_{8.5}(1+o(1))(\mathsf{E}L_n)^2$ as $n \to \infty$. This proves our claim (8.13). ■

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Department of Mathematics, The University of Utah, 155 S. 1400 E., Salt Lake City, UT 84112–0090

*E-mail address*: `davar@math.utah.edu`

*URL*: `http://www.math.utah.edu/~davar`

Department of Mathematics, The University of Utah, 155 S. 1400 E., Salt Lake City, UT 84112–0090

*E-mail address*: `levin@math.utah.edu`

*URL*: `http://www.math.utah.edu/~levin`

Department of Mathematics, The University of Utah, 155 S. 1400 E., Salt Lake City, UT 84112–0090

*E-mail address*: `mendez@math.utah.edu`

*URL*: `http://www.math.utah.edu/~mendez`