Intermittency and chaos for a non-linear stochastic wave equation in dimension 1

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This paper is dedicated to Professor David Nualart, whose scientific innovations have influenced us greatly.

Abstract

Consider a non-linear stochastic wave equation driven by space-time white noise in dimension one. We discuss the intermittency of the solution, and then use those intermittency result in order to demonstrate that in many cases the solution is chaotic. For the most part, the novel portion of our work is about the two cases where: (1) The initial conditions have compact support, where the global maximum of the solution remains bounded; and (2) The initial conditions are positive constants, where the global maximum is almost surely infinite. Bounds are also provided on the behavior of the global maximum of the solution in Case (2).

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1 Introduction

Let us consider the following hyperbolic stochastic PDE of the wave type

$$(\square u)(t\,,x) = \sigma(u(t\,,x))\dot{W}(t\,,x) \qquad (t>0,\, x\in\mathbf{R}). \tag{1.1}$$

Here, $\square$ denotes the [massless] wave operator

$$\square := \frac{\partial^2}{\partial t^2} - \kappa^2 \frac{\partial^2}{\partial x^2},$$

$\sigma : \mathbf{R} \to \mathbf{R}$ is a globally Lipschitz function with Lipschitz constant

$$\mathrm{Lip}_\sigma := \sup_{-\infty<x<y<\infty} \frac{|\sigma(y)-\sigma(x)|}{y-x},$$

$\dot{W}$ denotes space-time white noise, and $\kappa > 0$ is a fixed constant. The initial function and the initial velocity are denoted respectively by $u_0 : \mathbf{R} \to \mathbf{R}$ and $v_0 : \mathbf{R} \to \mathbf{R}$, and we might refer to the pair $(u_0\,, v_0)$ as the "initial conditions" of the stochastic wave equation (1.1). [The terminology is standard in PDEs, and so we use it freely.] When the initial value $x \mapsto u_0(x)$ is assumed to be a constant, we write the constant as u_0; similar remarks apply to v_0. In those cases, we state quite clearly that u_0 and v_0 are constants in order to avoid ambiguities.

The stochastic wave equation (1.1) has been studied extensively by Carmona and Nualart [9] and Walsh [31]. Among other things, these references contain the theorem that the random wave equation (1.1) has a unique continuous solution u as long as

$$u_0 \text{ and } v_0 \text{ are bounded and measurable functions,}$$

an assumption that is made tacitly throughout this paper. All of this is about the wave equation in dimension $1+1$ (that is one-dimensional time and one-dimensional space). There are also some existence theorems in the more delicate dimensions $1+d$ (that is one-dimensional time and d-dimensional space), where $d > 1$ and the 1-D wave operator $\square$ is replaced by the d-dimensional wave operator $\partial^2_{tt} - \kappa^2\Delta$, where Δ denotes the Laplacian on $\mathbf{R}^d$; see Conus and Dalang [11], Dalang [16], Dalang and Frangos [17], and Dalang and Mueller [19].

Parabolic counterparts to the random hyperbolic equation (1.1) are well-studied stochastic PDEs. For example, when $\sigma(u) = u$ and the wave operator $\square$ is replaced by the heat operator $\partial_t - \kappa^2\partial^2_{xx}$, the resulting stochastic PDE becomes a continuous *parabolic Anderson model* [8] and has connections to the study of random polymer measures and the KPZ equation [1, 2, 22, 25, 26, 28], and numerous other problems of mathematical physics and theoretical chemistry [8, Introduction]. The mentioned references contain a great deal of further information about these sorts of parabolic SPDEs.

From a purely-mathematical point of view, (1.1) is the hyperbolic counterpart to the stochastic heat equation, and in particular $\sigma(u) = \text{const} \cdot u$ ought to be a hyperbolic counterpart to the parabolic Anderson model. From a more pragmatic point of view, we believe that the analysis of the present hyperbolic equations might one day also lead to a better understanding of numerical-analysis problems that arise when trying to solve families of chaotic hyperbolic stochastic PDEs.

It is well-known, and easy to verify directly, that the Green function for the wave operator $\square$ is

$$\Gamma_t(x) := \tfrac{1}{2}\mathbf{1}_{[-\kappa t,\kappa t]}(x) \qquad \text{for } t>0 \text{ and } x\in\mathbf{R}. \tag{1.2}$$

According to general theory [9, 16, 31], the stochastic wave equation (1.1) has an a.s.-unique continuous solution $\{u(t\,,x)\}_{t>0,x\in\mathbf{R}}$ which has the following mild formulation:

$$u(t\,,x) = U_0(t\,,x) + V_0(t\,,x) + \int_{(0,t)\times\mathbf{R}} \Gamma_{t-s}(y-x)\sigma(u(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y). \tag{1.3}$$

The integral is understood to be a stochastic integral in the sense of Walsh [31, Chapter 2], and:

$$U_0(t\,,x) := \frac{u_0(x+\kappa t)+u_0(x-\kappa t)}{2}; \qquad V_0(t\,,x) := \frac{1}{2}\int_{x-\kappa t}^{x+\kappa t} v_0(y)\,\mathrm{d}y. \tag{1.4}$$

In the special case that u_0 and v_0 are constants, the preceding simplifies to

$$u(t\,,x) = u_0 + v_0\kappa t + \tfrac{1}{2}\int_{(0,t)\times(x-\kappa t,x+\kappa t)} \sigma(u(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y). \tag{1.5}$$

Recall [8, 22] that the process $\{u(t\,,x)\}_{t>0,x\in\mathbf{R}}$ is said to be *weakly intermittent* if the upper moment Lyapunov exponents,

$$\bar\gamma(p) := \limsup_{t\to\infty}\frac{1}{t}\sup_{x\in\mathbf{R}}\log \mathrm{E}\left(|u(t\,,x)|^p\right) \qquad (1\leqslant p<\infty), \tag{1.6}$$

have the property that

$$\bar\gamma(2)>0 \quad \text{and} \quad \bar\gamma(p)<\infty \quad \text{for every } p\in[2\,,\infty). \tag{1.7}$$

Various questions from theoretical physics [28] have motivated the study of intermittency for the stochastic heat equation. A paper [22] by Foondun and Khoshnevisan introduces methods for the intermittency analysis of fully-nonlinear parabolic stochastic PDEs. That paper also contains an extensive bibliography, with pointers to the large literature on the subject.

As far as we know, far less is known about the intermittent structure of the stochastic wave equation. In fact, we are aware only of two bodies of research: There is the recent work of Dalang and Mueller [20] that establishes intermittency for (1.1) in dimension $1+3$ (1 for time and 3 for space), where:

(1) $\sigma(u) = \lambda u$ (the hyperbolic Anderson model) for some $\lambda > 0$; (2) $\dot{W}$ is replaced by a generalized Gaussian field that is white in its time and has correlations in its space variable; and (iii) The 1-D wave operator in dimension is replaced by the 3-D wave operator. We are aware also of a recent paper by two of the present authors [15], where the solution to (1.1) is shown to be intermittent in the case that the initial function u_0 and the initial velocity v_0 are both sufficiently-smooth functions of compact support and $\dot{W}$ is a space-time white noise. The latter paper contains also detailed results on the geometry of the peaks of the solution.

The purpose of the present paper is to study intermittency and chaotic properties of the fully-nonlinear stochastic wave equation (1.1). We follow mainly the exposition style of Foondun and Khoshnevisan [22] for our results on weak intermittency: We will show that (1.7) holds provided that σ is a function of truly-linear growth (Theorems 3.1 and 3.3). We will also illustrate that this condition is somehow necessary by proving that weak intermittency fails to hold when σ is bounded (Theorem 3.4).

Regarding the chaotic properties of the solution u to (1.1), we follow mainly the exposition style of Conus, Joseph, and Khoshnevisan [12] who establish precise estimates on the asymptotic behavior of $\sup_{|x|\leqslant R} u(t\,,x)$, as $R\to\infty$ for fixed $t>0$, for the parabolic counterpart to (1.1). In the present hyperbolic case, we first prove that the solution to (1.1) satisfies $\sup_{x\in\mathbf{R}} |u(t\,,x)| < \infty$ a.s. for all $t\geqslant 0$, if that the initial function and the initial velocity are functions of compact support (Theorem 4.1). Then we return to the case of central importance to this paper, and prove that $\sup_{x\in\mathbf{R}} |u(t\,,x)| = \infty$ a.s. for all $t > 0$ when u_0 and v_0 are positive constants. Also, we obtain some quantitative estimates on the behavior of the supremum under varying assumptions on the nonlinearity σ (Theorems 7.1 and 7.2).

When consider in conjunction, the results of this paper imply that the solution to (1.1) is chaotic in the sense that slightly-different initial conditions can lead to drastically-different qualitative behaviors for the solution. This phenomenon is entirely due to the presence of noise in the system (1.1), and does not arise in typical deterministic wave equations.

This paper might be of interest for two main reasons: First of all, we obtain estimates on the supremum of the solution to hyperbolic stochastic PDEs, and use them to show that the solution can be chaotic. We believe that these estimates might have other uses and are worthy of record in their own right. Secondly, we shall see that the analysis of the 1-D wave equation is simplified by the fact that fundamental solution Γ of the wave operator $\Box$—see (1.2)—is a bounded function of compact support. As such, one can also view the present paper, in part, as a gentle introduction to the methods of the more-or-less companion paper [12].

Let us conclude the Introduction with an outline of the paper. Section 3 below mainly recalls intermittency results for (1.1). These facts are mostly known in the folklore, but we document them here, in a systematic manner, for what appears to be the first time. The reader who is familiar with [22] will undoubtedly recognize some of the arguments of §3.

Section 4 is devoted to the study of the case where the initial value and velocity have compact support [and hence are *not* constants]. We will show that in such cases, $\sup_{x\in\mathbf{R}}|u(t\,,x)| < \infty$ a.s. for all $t > 0$. Sections 5 and 6 contain novel tail-probability estimates that depend on various forms of the nonlinearity σ. These estimates are of independent interest. Here, we use them in order to establish various localization properties. Finally, in Section 7, we combine our earlier estimates and use them to state and prove the main results of the present paper about the asymptotic behavior of $\sup_{|x|\leqslant R}|u(t\,,x)|$ as $R\to\infty$. More specifically, we prove that if u_0 is a positive constant, v_0 is a non-negative constant, and $\inf_{z\in\mathbf{R}}|\sigma(z)| > 0$, then the peaks of the solution in $x\in[-R\,,R]$ grow at least as $(\kappa\log R)^{1/3}$. More precisely, we prove that there exists an almost-surely finite random random variable $R_0 > 0$ and a positive and finite constant a such that

$$\sup_{|x|\leqslant R}|u(t\,,x)|^3 \geqslant a\kappa\log R \qquad \text{for all } R > R_0.$$

Furthermore, we will prove that a does not depend on κ, as long as κ is sufficiently small; this assertion measures the effect of the noise on the intermittency properties of u. If $0 < \inf\sigma \leqslant \sup\sigma < \infty$, then we prove that the preceding can be improved to the existence of an a.s.-finite R_1 together with positive and finite constants b and c such that

$$b\kappa\log R \leqslant \sup_{|x|\leqslant R}|u(t\,,x)|^2 \leqslant c\kappa\log R \qquad \text{for all } R > R_1.$$

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2 Preliminaries

In this section we introduce some notation and preliminary results that are used throughout the paper. For a random variable Z, we denote by $\|Z\|_p := \{\mathrm{E}(|Z|^p)\}^{1/p}$ the standard norm on $L^p(\Omega)$ $(1\leqslant p<\infty)$.

On several occasions we apply the following form of the Burkholder–Davis–Gundy inequality [4, 5, 6] for continuous $L^2(\Omega)$ martingales: If $\{X_t\}_{t\geqslant 0}$ is a continuous $L^2(\Omega)$ martingale with running maximum $X_t^* := \sup_{s\in[0,t]}|X_s|$ and quadratic variation process $\langle X\rangle$, then for all $p\in[2\,,\infty)$ and $t\in(0\,,\infty)$,

$$\|X_t^*\|_p \leqslant (4p)^{1/2}\cdot\|\langle X\rangle_t\|_{p/2}^{1/2}. \tag{2.1}$$

The multiplicative prefactor $4p$ is the asymptotically-optimal bound, due to Carlen and Kree [7], for the sharp constant in the Burkholder–Davis–Gundy inequality that was discovered by Davis [21].

Given numbers $p \in [1\,,\infty)$ and $\beta \in (0\,,\infty)$, and given a space-time random field $\{Z(t\,,x)\}_{t>0,x\in\mathbf{R}}$, let us recall the following norm [22]:

$$\|Z\|_{p,\beta} := \left\{\sup_{t\geqslant 0}\sup_{x\in\mathbf{R}} \mathrm{e}^{-\beta t}\mathrm{E}\left(|Z(t\,,x)|^p\right)\right\}^{1/p}. \tag{2.2}$$

We also use the following norm [12]:

$$\mathcal{N}_{p,\beta}(Z) := \left(\sup_{t\geqslant 0}\sup_{x\in\mathbf{R}} \mathrm{e}^{-\beta t}\|Z\|_p^2\right)^{1/2}. \tag{2.3}$$

Clearly, the two norms are related via the elementary relations,

$$\mathcal{N}_{p,\beta}(Z) = \|Z\|_{p,p\beta/2} \quad \text{and} \quad \|Z\|_{p,\beta} = \mathcal{N}_{p,2\beta/p}(Z). \tag{2.4}$$

However, the difference between the norms becomes relevant to us when we need to keep track of some constants.

Finally, we mention the following elementary formulas about the fundamental solution Γ to the wave operator $\Box$: For all $t, \beta > 0$;

$$\|\Gamma_t\|_{L^2(\mathbf{R})}^2 = \frac{\kappa t}{2}, \int_0^t \|\Gamma_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s = \frac{\kappa t^2}{4}, \int_0^\infty \mathrm{e}^{-\beta s}\|\Gamma_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s = \frac{\kappa}{2\beta^2}. \tag{2.5}$$

3 Intermittency

We are ready to state and prove the intermittency of the solution to (1.1). Our methods follow closely those of Foondun and Khoshnevisan [22], for the heat equation, and Conus and Khoshnevisan [15], for the wave equation.

In order to establish weak intermittency for the solution to (1.1) we need to obtain two different results: (1) We need to derive a finite upper bound for $\bar{\gamma}(p)$ for every $p \geqslant 2$; and (2) We need to establish a positive lower bound for $\bar{\gamma}(2)$. It might help to recall that the Lyapunov exponents $\bar{\gamma}(p)$ were defined in (1.6).

Theorem 3.1. *If u_0 and v_0 are both bounded and measurable functions, then*

$$\bar{\gamma}(p) \leqslant p^{3/2}\mathrm{Lip}_\sigma\sqrt{\kappa/2} \qquad \textit{for all } p \in [2\,,\infty).$$

Remark 3.2. Since the optimal constant in the Burkholder-Davis-Gundy L^2 inequality is 1, an inspection of the proof of Theorem 3.1 yields the improved bound $\bar{\gamma}(2) \leqslant \mathrm{Lip}_\sigma\sqrt{\kappa/2}$ in the case that $p = 2$. $\Box$

For our next result we define

$$\mathrm{L}_\sigma := \inf_{x\neq 0} |\sigma(x)/x|\,. \tag{3.1}$$

Theorem 3.3. *If u_0 and v_0 are bounded and measurable,* $\inf_{x\in\mathbf{R}} u_0(x) > 0$, *$v_0 \geqslant 0$ pointwise, and $\mathrm{L}_\sigma > 0$, then $\bar{\gamma}(2) \geqslant \mathrm{L}_\sigma\sqrt{\kappa/2}$.*

Theorems 3.1 and 3.3 are similar to Theorems 2.1 and 2.7 of [22] for the heat equation. Together, they prove that the solution u is weakly intermittent provided that u_0 is bounded away from 0, $v_0 \geqslant 0$ and σ has linear growth. Intermittency in the case where u_0 and v_0 have compact support has been proved in [15] (see also Section 4). Theorems 3.1 and 3.3 illustrate that the wave equation exhibits a similar qualitative behavior as the heat equation. However, the quantitative behavior is different: Here, $\bar{\gamma}(p)$ is of order $p^{3/2}$, whereas it is of order p^3 for the stochastic heat equation.

The linear growth of σ is somehow necessary for intermittency as the following result suggests.

Theorem 3.4. *If u_0, v_0, and σ are all bounded and measurable functions, then*

$$\mathrm{E}\left(|u(t\,,x)|^p\right) = O(t^p) \qquad \text{as } t \to \infty, \qquad \text{for all } p \in [2\,,\infty).$$

This estimate is sharp when $u_0(x) \geqslant 0$ for all $x \in \mathbf{R}$, and $\inf_{z\in\mathbf{R}} v_0(z) > 0$.

The preceding should be compared to Theorem 2.3 of [22]. There it was shown that if u were replaced by the solution to the stochastic heat equation, then there is the much smaller bound $\mathrm{E}(|u(t\,,x)|^p) = o(t^{p/2})$, valid under boundedness assumptions on u_0 and σ.

Analogues of the preceding three theorems above are known in the parabolic setting [22, 15]. Therefore, we will describe only outlines of their proof.

We will use a stochastic Young-type inequality for stochastic convolutions (Proposition 3.5 below), which is a ready consequence of [15, Proposition 2.5].

For a random-field $\{Z(t\,,x)\}_{t>0,x\in\mathbf{R}}$, we denote by $\Gamma * Z\dot{W}$ the random-field defined by

$$(\Gamma * Z\dot{W})(t\,,x) = \int_{(0,t)\times\mathbf{R}} \Gamma_{t-s}(y-x) Z(s\,,y)\, W(\mathrm{d}s\,\mathrm{d}y),$$

provided that the stochastic integral is well-defined in the sense of Walsh [31].

Proposition 3.5. *For all $\beta > 0$ and $p \in (2\,,\infty)$,*

$$\|\Gamma * Z\dot{W}\|_{2,\beta} \leqslant \frac{\kappa^{1/2}}{\beta\sqrt{2}}\|Z\|_{2,\beta} \quad \text{and} \quad \|\Gamma * Z\dot{W}\|_{p,\beta} \leqslant \frac{p^{3/2}\kappa^{1/2}}{\beta\sqrt{2}}\|Z\|_{p,\beta}.$$

Proof. We appeal to (2.1) in order to deduce that

$$\begin{aligned}
&\mathrm{E}\left(|(\Gamma * Z\dot{W})(t\,,x)|^p\right) \qquad (3.2)\\
&\leqslant (4p)^{p/2}\mathrm{E}\left[\left(\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \Gamma^2_{t-s}(y-x)|Z(s\,,y)|^2\right)^{p/2}\right]\\
&\leqslant (4p)^{p/2}\left(\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \Gamma^2_{t-s}(y-x)\left\{\mathrm{E}\left(|Z(s\,,y)|^p\right)\right\}^{2/p}\right)^{p/2};
\end{aligned}$$

the last inequality is justified by Minkowski's inequality. Next we raise both sides of the preceding inequality to the power $2/p$, and then multiply both sides by $\mathrm{e}^{-\beta t}$ in order to obtain

$$\begin{aligned}\left[\mathcal{N}_{p,\beta}(\Gamma * Z\dot{W})\right]^2 &\leqslant 4p\int_0^\infty \mathrm{d}s\int_{-\infty}^\infty \mathrm{d}y\ \mathrm{e}^{-\beta(t-s)}\Gamma_{t-s}^2(y-x)\mathrm{e}^{-\beta s}\left\{\mathrm{E}\left(|Z(s\,,y)|^p\right)\right\}^{2/p}\\ &\leqslant 4p\left[\mathcal{N}_{p,\beta}(Z)\right]^2\int_0^\infty \mathrm{d}s\int_{-\infty}^\infty \mathrm{d}y\ \mathrm{e}^{-\beta s}[\Gamma_s(y)]^2\\ &= \frac{2p\kappa}{\beta^2}\left[\mathcal{N}_{p,\beta}(Z)\right]^2,\end{aligned}$$

thanks to (2.5). The relation (2.4) concludes the proof in the case that $p > 2$. When $p = 2$ is handled the same way, but the prefactor $(4p)^{p/2} = 8$ of (3.2) can be improved to one, owing to the $L^2(\Omega)$ isometry of Walsh integrals. □

We are now ready to prove the main results of this section.

Proof of Theorem 3.1. Since u_0 and v_0 are bounded, we clearly have

$$\sup_{x\in\mathbf{R}}|U_0(t\,,x)+V_0(t\,,x)| \leqslant \text{const}\cdot(1+t) \qquad (t\geqslant 0),$$

whence

$$\|U_0+V_0\|_{p,\beta} = \left(\sup_{t\geqslant 0}\mathrm{e}^{-\beta t}\sup_{x\in\mathbf{R}}|U_0(t\,,x)+V_0(t\,,x)|^p\right)^{1/p} \leqslant K, \tag{3.3}$$

where $K := K_{p,\beta}$ is a positive and finite constant that depends only on p and β.

We apply (1.3), (3.3), and Proposition 3.5, together with the fact that $|\sigma(u)| \leqslant |\sigma(0)| + \mathrm{Lip}_\sigma|u|$, in order to conclude that for all $\beta \in (0\,,\infty)$ and $p \in [2\,,\infty)$,

$$\|u\|_{p,\beta} \leqslant K + \frac{p^{3/2}\kappa^{1/2}}{\beta\sqrt{2}}\left(|\sigma(0)| + \mathrm{Lip}_\sigma\|u\|_{p,\beta}\right). \tag{3.4}$$

This inequality implies that $\|u\|_{p,\beta} < \infty$, provided that $\beta > p^{3/2}\mathrm{Lip}_\sigma\sqrt{\kappa/2}$, and Theorem 3.1 follows. □

Proof of Theorem 3.3. We need to follow the proof of Theorem 2.7 of [22] closely, and so merely recall the necessary steps. It suffices to prove that

$$\int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}\left(|u(t\,,x)|^2\right)\,\mathrm{d}t = \infty \qquad \text{when } \beta \leqslant \mathrm{L}_\sigma\sqrt{\kappa/2}. \tag{3.5}$$

Theorem 3.3 will follow from this. This can be seen as follows: By the very definition of $\bar{\gamma}(2)$, we know that for all fixed $\epsilon > 0$ there exists a finite constant $t_\epsilon > 1$ such that $\mathrm{E}(|u(t\,,x)|^2) \leqslant t_\epsilon \exp((\bar{\gamma}(2)+\epsilon)t)$ whenever $t > t_\epsilon$. Consequently,

$$\int_{t_\epsilon}^\infty \mathrm{e}^{-\beta t}\mathrm{E}(|u(t\,,x)|^2)\,\mathrm{d}t \leqslant t_\epsilon\int_{t_\epsilon}^\infty \mathrm{e}^{-(\beta-\bar{\gamma}(2)-\epsilon)t}\,\mathrm{d}t.$$

We may conclude from this and (3.5) that $\bar{\gamma}(2) \geqslant \mathrm{L}_\sigma\sqrt{\kappa/2} - \epsilon$, and this completes the proof because $\epsilon > 0$ were arbitrary. It remains to verify (3.5).

A direct computation, using the L^2 isometry that defines Walsh's stochastic integrals, shows us that

$$\begin{aligned}
&\mathrm{E}\left(|u(t\,,x)|^2\right)\\
&= |U_0(t\,,x) + V_0(t\,,x)|^2 + \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \Gamma_{t-s}^2(y-x)\mathrm{E}\left(|\sigma(u(s\,,y)|^2\right)\\
&\geqslant \frac{C}{\beta} + \mathrm{L}_\sigma^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \Gamma_{t-s}^2(y-x)\mathrm{E}\left(|u(s\,,y)|^2\right),
\end{aligned} \tag{3.6}$$

with $C := \inf_{z\in\mathbf{R}}[u_0(z)]^2$. Define

$$M_\beta(x) := \int_0^\infty \mathrm{e}^{-\beta t}\mathrm{E}\left(|u(t\,,x)|^2\right)\,\mathrm{d}t, \quad H_\beta(x) := \int_0^\infty \mathrm{e}^{-\beta t}[\Gamma_t(x)]^2\,\mathrm{d}t.$$

We can rewrite (3.6) in terms of M_β and H_β as follows:

$$M_\beta(t) \geqslant \frac{C}{\beta} + \mathrm{L}_\sigma^2(M_\beta * H_\beta)(x);$$

where $*$ denotes spatial convolution. The preceding is a renewal inequation, and can be solved directly: We set

$$(\mathcal{H}f)(x) := \mathrm{L}_\sigma^2(H_\beta * f)(x) \qquad (x \in \mathbf{R}),$$

for every non-negative measurable function $f : \mathbf{R} \to \mathbf{R}_+$, and deduce the functional recursion $M_\beta \geqslant C/\beta + (\mathcal{H} * M_\beta)$, whence

$$M_\beta(x) \geqslant \beta^{-1}\sum_{n=0}^\infty (\mathcal{H}^n C)(x),$$

where we have identified the constant C with the function $C(x) := C$, as usual. Now $(\mathcal{H}C)(x) = C\mathrm{L}_\sigma^2\int_0^\infty \mathrm{e}^{-\beta t}\|\Gamma_t\|_{L^2(\mathbf{R})}^2\,\mathrm{d}t = C[\mathrm{L}_\sigma^2\kappa/(2\beta^2)]$; see (2.5). We can iterate this computation to see that $(\mathcal{H}^n C)(x) = C[\mathrm{L}_\sigma^2\kappa/(2\beta^2)]^n$ for all $n \geqslant 0$, and hence

$$M_\beta(x) \geqslant C\beta^{-1}\sum_{n=0}^\infty \left(\frac{\mathrm{L}_\sigma^2\kappa}{2\beta^2}\right)^n.$$

The preceding infinite series is equal to $+\infty$ if and only if $\beta \leqslant \mathrm{L}_\sigma\sqrt{\kappa/2}$. This establishes (3.5), and concludes the proof of Theorem 3.3. □

Proof of Theorem 3.4. Because u_0 and v_0 are bounded, $|U_0(t\,,x) + V_0(t\,,x)| = O(t)$ as $t \to \infty$, uniformly in $x \in \mathbf{R}$. Therefore, the boundedness of σ, (1.3), (2.1), (2.5), and (3.3) together imply that

$$\begin{aligned}
\|u(t\,,x)\|_p &\leqslant O(t) + \sup_{x\in\mathbf{R}}|\sigma(x)|\left(4p\int_0^t \|\Gamma_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s\right)^{1/2}\\
&\leqslant O(t) + \sqrt{p\kappa}\sup_{x\in\mathbf{R}}|\sigma(x)|t = O(t) \qquad (t \to \infty).
\end{aligned} \tag{3.7}$$

The main assertion of Theorem 3.4 follows. In order to establish the remaining claim about the sharpness of the estimator, suppose $u_0(x) \geqslant 0$ and $\inf_{z\in\mathbf{R}} v_0(y) > 0$, and consider $p = 2$. Thanks to (3.6), $\|u(t\,,x)\|_2 \geqslant V_0(t\,,x) \geqslant \inf_{z\in\mathbf{R}} v_0(z)\cdot \kappa t$. The claim follows from this and Jensen's inequality. □

There are many variations of the sharpness portion of Theorem 3.4. Let us conclude this section with one such variation.

Lemma 3.6. *If $\sigma(u) = \lambda$ is a constant, and u_0 and v_0 are both constants, then*

$$\lim_{t\to\infty} \frac{1}{t^2} \mathrm{E}\left(|u(t\,,x)|^2\right) = (v_0\kappa)^2 + \frac{\lambda^2\kappa}{4} \quad \textit{for all } x \in \mathbf{R}.$$

Proof. In accord with (2.5), the second moment of $\int_{(0,t)\times\mathbf{R}} \Gamma_{t-s}(y-x)\, W(\mathrm{d}s\,\mathrm{d}y)$ is $\kappa t^2/4$. Therefore, (1.5) implies that

$$\mathrm{E}\left(|u(t\,,x)|^2\right) = (u_0 + v_0\kappa t)^2 + \frac{\lambda^2\kappa t^2}{4} = \left\{(v_0\kappa)^2 + \frac{\lambda^2\kappa}{4} + o(1)\right\} t^2,$$

as $t \to \infty$. □

4 Compact-support initial data

This section is devoted to the study of the behavior of the [spatial] supremum $\sup_{x\in\mathbf{R}} |u(t\,,x)|$ of the solution to (1.1) when t is fixed. Throughout this section we assume the following:

The initial function u_0 and initial velocity v_0 have compact support. (4.1)

We follow the ideas of Foondun and Khoshnevisan [23]. However, the present hyperbolic setting lends itself to significant simplifications that arise mainly because the Green's function has the property that Γ_t has compact support at every fixed time $t > 0$.

Throughout this section, we assume also that

$$\sigma(0) = 0 \qquad \text{and} \qquad \mathrm{L}_\sigma > 0, \tag{4.2}$$

where L_σ was defined in (3.1). Since (1.1) has a unique solution, the preceding conditions imply that if $u_0(x) \equiv 0$, then $u_t(x) \equiv 0$ for all $t > 0$.

The idea, borrowed from [23], is to compare $\sup_{x\in\mathbf{R}} |u(t\,,x)|$ with the $L^2(\mathbf{R})$-norm of the infinite-dimensional stochastic process $\{u(t\,,\cdot)\}_{t\geqslant 0}$. This comparison will lead to the result, since it turns out that the compact-support property of u_0 and v_0 will lead us to show that $u(t\,,\cdot)$ also has compact support. This compact-support property does *not* hold for parabolic variants of (1.1); see Mueller [29].

Next is the main result of this section.

Theorem 4.1. *Suppose* $\mathrm{L}_\sigma > 0$*,* $\sigma(0) = 0$*, and* u_0 *is Hölder-continuous with Hölder index* $\geqslant 1/2$*. Suppose also that* u_0 *and* v_0 *are non-negative functions, both supported compactly in* $[-K\,,K]$ *for some* $K > 0$*. Then,* $u(t\,,\cdot) \in L^2(\mathbf{R})$ *a.s. for all* $t \geqslant 0$ *and*

$$\begin{aligned}\mathrm{L}_\sigma\sqrt{\frac{\kappa}{2}} &\leqslant \limsup_{t\to\infty} \frac{1}{t} \sup_{x\in\mathbf{R}} \log \mathrm{E}\left(|u(t\,,x)|^2\right) \\ &\leqslant \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}\left(\sup_{x\in\mathbf{R}} |u(t\,,x)|^2\right) \leqslant \mathrm{Lip}_\sigma\sqrt{\frac{\kappa}{2}}.\end{aligned} \tag{4.3}$$

Remark 4.2. Theorem 4.1 implies that $\sup_{x\in\mathbf{R}} |u(t\,,x)| < \infty$ a.s. for all $t \geqslant 0$ provided that the initial function and the initial velocity both have compact support [and are mildly smooth]. We are going to show in §7 that $\sup_{x\in\mathbf{R}} |u(t\,,x)| = \infty$ a.s. if the initial function and velocity are non-zero constants, even if those constants are quite close to zero. This discrepancy suggests strongly that the stochastic wave equation (1.1) is chaotic [two mildly-different initial conditions can lead to a drastically-different solutions]. This form of chaos is due entirely to the presence of the noise $\dot{W}$ in (1.1). □

Before we turn to the proof of Theorem 4.1, we will need a few intermediary results.

Proposition 4.3. *Suppose that* $\mathrm{L}_\sigma > 0$*,* $\sigma(0) = 0$*, and that* $u_0 \not\equiv 0$ *and* v_0 *are non-negative functions in* $L^2(\mathbf{R})$*. Then,* $u(t\,,\cdot) \in L^2(\mathbf{R})$ *a.s for all* $t > 0$*, and*

$$\mathrm{L}_\sigma\sqrt{\frac{\kappa}{2}} \leqslant \limsup_{t\to\infty} \frac{1}{t} \log \mathrm{E}\left(\|u(t\,,\cdot)\|_{L^2(\mathbf{R})}^2\right) \leqslant \mathrm{Lip}_\sigma\sqrt{\frac{\kappa}{2}} \tag{4.4}$$

Proof. The proof resembles that of Theorem 2.1 of [22]. The latter is valid for parabolic equations; therefore, we show how one can adapt that argument to the present hyperbolic setting.

Since $u_0 \geqslant 0$, it follows that

$$\frac{1}{2}\|u_0\|_{L^2(\mathbf{R})}^2 \leqslant \|U_0(t\,,\cdot)\|_{L^2(\mathbf{R})}^2 \leqslant \|u_0\|_{L^2(\mathbf{R})}^2.$$

Moreover, since $v_0 \geqslant 0$, we have

$$0 \leqslant \|V_0(t\,,\cdot)\|_{L^2(\mathbf{R})}^2 = \int_{-\infty}^{\infty} \mathrm{d}x \left(\int_{-\kappa t}^{\kappa t} \mathrm{d}y\; v_0(y+x)\right)^2 \leqslant 4\kappa^2 t^2 \, \|v_0\|_{L^2(\mathbf{R})}^2,$$

thanks to the Cauchy–Schwarz inequality.

Now, we deduce from (1.3) that

$$\begin{aligned}&\mathrm{E}\left(\|u(t\,,\cdot)\|_{L^2(\mathbf{R})}^2\right) \\ &\geqslant \|U_0(t\,,\cdot)\|_{L^2(\mathbf{R})}^2 + \|V_0(t\,,\cdot)\|_{L^2(\mathbf{R})}^2 + \mathrm{L}_\sigma^2 \int_0^t \mathrm{d}s \; \mathrm{E}\left(\|u(s\,,\cdot)\|_{L^2(\mathbf{R})}^2\right) \|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2 \\ &\geqslant \frac{1}{2}\|u_0\|_{L^2(\mathbf{R})}^2 + \mathrm{L}_\sigma^2 \int_0^t \mathrm{d}s \; \mathrm{E}\left(\|u(s\,,\cdot)\|_{L^2(\mathbf{R})}^2\right) \|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2.\end{aligned} \tag{4.5}$$

Define

$$U(\lambda) := \int_0^\infty \mathrm{e}^{-\lambda t}\mathrm{E}\left(\|u(t\,,\cdot)\|_{L^2(\mathbf{R})}^2\right)\mathrm{d}t. \tag{4.6}$$

In this way, we can conclude from (2.5) and (4.5) that the non-negative function U that was just defined satisfies the recursive inequality,

$$U(\lambda) \geqslant \frac{\|u_0\|_{L^2(\mathbf{R})}^2}{2\lambda} + \frac{\kappa \mathrm{L}_\sigma^2}{2\lambda^2}\,U(\lambda). \tag{4.7}$$

Since $u_0 \not\equiv 0$, the first term on the right-hand side of (4.7) is strictly positive, whence it follows that whenever $\lambda \leqslant \mathrm{L}_\sigma\sqrt{\kappa/2}$, we have $U(\lambda)=\infty$. This proves the first asserted inequality in Proposition 4.3.

As regards the other bound, we consider the Picard iteration scheme that defines u from (1.3). Namely, we set $u_0(t\,,x) := 0$ and then define iteratively

$$\begin{aligned} &u_{n+1}(t\,,x)\\ &\quad= U_0(t\,,x) + V_0(t\,,x) + \int_{(0,t)\times\mathbf{R}} \Gamma_{t-s}(y-x)\sigma(u_n(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y). \end{aligned}\tag{4.8}$$

Next we may proceed, as we did for (4.5) but develop upper bounds in place of lower bounds, in order to deduce the following:

$$\begin{aligned} &\mathrm{E}\left(\|u_{n+1}(t\,,\cdot)\|_{L^2(\mathbf{R})}^2\right) \\ &\leqslant 2\|U_0(t\,,\cdot)\|_{L^2(\mathbf{R})}^2 + 2\|V_0(t\,,\cdot)\|_{L^2(\mathbf{R})}^2 + \mathrm{Lip}_\sigma^2\int_0^t \mathrm{d}s\ \mathrm{E}\left(\|u_n(s\,,\cdot)\|_{L^2(\mathbf{R})}^2\right)\|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2\\ &\leqslant 2\|u_0\|_{L^2(\mathbf{R})}^2 + 8\kappa^2t^2\,\|v_0\|_{L^2(\mathbf{R})}^2 + \mathrm{Lip}_\sigma^2\int_0^t \mathrm{d}s\ \mathrm{E}\left(\|u_n(s\,,\cdot)\|_{L^2(\mathbf{R})}^2\right)\|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2. \end{aligned}\tag{4.9}$$

In order to analyze this inequality let us define

$$M_n(\lambda) := \sup_{t\geqslant 0}\left[\mathrm{e}^{-\lambda t}\mathrm{E}\left(\|u_n(t\,,\cdot)\|_{L^2(\mathbf{R})}^2\right)\right] \qquad (\lambda>0,\ n=1,1,\ldots).$$

In accord with (4.9) and (2.5), the $M_j(\lambda)$'s satisfy the recursive inequality

$$M_{n+1}(\lambda) \leqslant 2\|u_0\|_{L^2(\mathbf{R})}^2 + \frac{8\kappa^2}{\lambda^2}\,\|v_0\|_{L^2(\mathbf{R})}^2 + \frac{\kappa\mathrm{Lip}_\sigma^2}{2\lambda^2}\,M_n(\lambda).$$

It follows readily from this recursion that if $\lambda > \mathrm{Lip}_\sigma\sqrt{\kappa/2}$, then $\sup_{n\geqslant 0} M_n(\lambda) < \infty$. Finally, we take the limit as $n\to\infty$ in order to deduce the lower bound in Proposition 4.3. □

Among other things, Proposition 4.3 proves the first claim, made in Theorem 4.1, that $u(t\,,\cdot)\in L^2(\mathbf{R})$ almost surely for every $t\geqslant 0$.

We plan to deduce Theorem 4.1 from Proposition 4.3 by showing that $\|u(t\,,\cdot)\|_{L^2(\mathbf{R})}$ and $\sup_{x\in\mathbf{R}}|u(t\,,x)|$ are "comparable."

We start by a "compact-support property" of the solution u, which is associated strictly to the hyperbolicity of the wave operator. As such, our next result should be contrasted with Lemma 3.3 of [23], valid for parabolic stochastic partial differential equations.

Proposition 4.4. *Under the assumptions of Theorem 4.1, the random function $x \mapsto u(t\,,x)$ is a.s. supported in $[-K-\kappa t\,,K+\kappa t]$ for every $t>0$.*

Proof. Let $u_0(t\,,x) := 0$ and define iteratively u_{n+1}, in terms of u_n, as Picard iterates; see (4.8). Note that Γ_{t-s} is supported in $[-\kappa(t-s)\,,\kappa(t-s)]$ for all $s \in (0\,,t)$. Because $U_0(t\,,\cdot)$ and $V_0(t\,,\cdot)$ are both supported in the interval $[-K-\kappa t\,,K+\kappa t]$, it follows from (4.8), the fact that $\sigma(0)=0$, and induction [on $n \geqslant 0$] that $u_n(s\,,\cdot)$ is a.s. supported in $[-K-\kappa s\,,K+\kappa s]$ for all $s>0$ and $n \geqslant 0$. Now we know from Dalang's theory [16] that $\lim_{n\to\infty} u_n(t\,,x) = u(t\,,x)$ in probability. Therefore, the result follows. ☐

Remark 4.5. Proposition 4.4 improves some of the estimates that were obtained previously in [15]. Namely that, $u(t\,,\cdot)$ does not have large peaks more than a distance $\kappa t + o(t)$ away from the origin as $t \to \infty$. ☐

In order to be able to prove Theorem 4.1, we need some continuity estimates for the solution u. The continuity of the solution itself has been known for a long time; see [16, 31] for instance. We merely state the results in the form that we need.

Lemma 4.6. *If u_0 is Hölder-continuous of order $\geqslant 1/2$, then for all integers $p \geqslant 1$ and for every $\beta > \bar\gamma(2p)$, there exists a constant $C_{p,\beta} \in (0,\infty)$ such that, for all $t \geqslant 0$,*

$$\sup_{j\in\mathbf{Z}} \sup_{j\leqslant x<x'\leqslant j+1} \left\| \frac{u(t\,,x)-u(t\,,x')}{|x-x'|^{1/2}} \right\|_{2p} \leqslant C_{p,\beta} \mathrm{e}^{\beta t/2p}. \tag{4.10}$$

Proof. We may observe that $|U_0(t\,,x) - U_0(t\,,x')| \leqslant \text{const}\cdot |x-x'|^{1/2}$ and $|V_0(t\,,x) - V_0(t\,,x')| \leqslant 2\sup_{z\in\mathbf{R}} |v_0(z)| \cdot |x-x'| \leqslant 2\sup_{z\in\mathbf{R}} |v_0(z)|\cdot|x-x'|^{1/2}$, as long as $|x-x'| \leqslant 1$. Therefore, we apply (1.3) and (2.1) to deduce that uniformly for all $x,x' \in \mathbf{R}$ such that $|x-x'| \leqslant 1$,

$$\begin{aligned}\|u(t\,,x)-u(t\,,x')\|_{2p} &\leqslant \text{const}\cdot|x-x'|^{1/2} \\ &\quad + \mathrm{Lip}_\sigma \left(4p\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \|u(s\,,y)\|_{2p}^2 |\Gamma_{t-s}(y-x) - \Gamma_{t-s}(y-x')|^2 \right)^{1/2}.\end{aligned} \tag{4.11}$$

Theorem 3.1 shows that $\|u\|_{2p,\beta} < \infty$ provided $\beta > \bar\gamma(2p)$, and a direct calculation shows that

$$\int_{-\infty}^{\infty} \mathrm{d}y\ |\Gamma_s(y-x) - \Gamma_s(y-x')|^2 \leqslant 2|x-x'| \tag{4.12}$$

for all $s > 0$. As a consequence,

$$
\begin{aligned}
&\int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \|u(s\,,y)\|_{2p}^2 |\Gamma_{t-s}(y-x) - \Gamma_{t-s}(y-x')|^2 \qquad (4.13)\\
&\leqslant \mathrm{e}^{\beta t/p} \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ \mathrm{e}^{-\beta s/p} \|u(s,y)\|_{2p}^2\ \mathrm{e}^{-\beta(t-s)/p} |\Gamma_{t-s}(y-x) - \Gamma_{t-s}(y-x')|^2\\
&\leqslant \|u\|_{2p,\beta}^2\, \mathrm{e}^{\beta t/p} \int_0^{\infty} \mathrm{d}s\ \mathrm{e}^{-\beta s/p} \int_{-\infty}^{\infty} \mathrm{d}y\ |\Gamma_s(y-x) - \Gamma_s(y-x')|^2\\
&\leqslant \|u\|_{2p,\beta}^2 \frac{2p}{\beta}\, \mathrm{e}^{\beta t/p} |x - x'|, \qquad (4.14)
\end{aligned}
$$

by (4.12). The theorem follows from (4.11) and (4.14). □

By analogy with Lemmas 3.5 and 3.6 of [23], we can extend the preceding result to all real numbers $p \in (1\,,2)$ and to a uniform modulus of continuity estimate.

Lemma 4.7. *Suppose the conditions of Lemma 4.6 are satisfied. Then, for all $p \in (1\,,2)$ and $\epsilon, \delta \in (0\,,1)$, there exists a constant $C_{p,\epsilon,\delta} \in (0\,,\infty)$ such that for all $t \geqslant 0$,*

$$
\sup_{j \in \mathbf{Z}} \left\| \sup_{j \leqslant x < x' \leqslant j+1} \frac{|u(t\,,x) - u(t\,,x')|^2}{|x - x'|^{1-\epsilon}} \right\|_p \leqslant C_{p,\epsilon,\delta}\, \mathrm{e}^{(1+\delta)\lambda_p t}, \tag{4.15}
$$

where $\lambda_p := (2-p)\bar{\gamma}(2) + (p-1)\bar{\gamma}(4)$.

Proof. The proof works exactly as in [23, Lemmas 3.5 and 3.6]. First, one proves that

$$
\mathrm{E}\left(|u(t\,,x) - u(t\,,x')|^{2p}\right) \leqslant C_{p,\delta} |x - x'|^p \exp((1+\delta)\lambda(p)), \tag{4.16}
$$

for all $\delta \in (0\,,1)$, $|x - x'| \leqslant 1$ and $p \in [1\,,2]$. This is a direct application of convexity of L^p norms and Lemma 4.6. We refer to [23, Lemma 3.5] for a detailed argument. As a second step, it is possible to use a suitable form of the Kolomogorov continuity theorem in order to obtain an estimate that holds uniformly for $j \leqslant x < x' \leqslant j+1$, as stated. We refer to [18] for a detailed proof; see in particular, the proof of Theorem 4.3 therein. □

We are ready to prove Theorem 4.1. This is similar to the proof of Theorem 1.1 in [23], but because of Proposition 4.4, some of the technical issues of [23] do not arise.

Proof of Theorem 4.1. We have already proved that $u(t\,,\cdot) \in L^2(\mathbf{R})$ for every $t > 0$; see Proposition 4.3. Therefore, it remains to prove (4.3).

The lower bound is a direct consequence of Propositions 4.3 and 4.4. Indeed, Proposition 4.3 implies that

$$\begin{aligned}\exp\left(\left[\mathrm{L}_\sigma\sqrt{\frac{\kappa}{2}}+o(1)\right]t\right) &\leqslant \mathrm{E}\left(\int_{-\infty}^{\infty}|u(t\,,x)|^2\,\mathrm{d}x\right)\\ &= \mathrm{E}\left(\int_{-K-\kappa t}^{K+\kappa t}|u(t\,,x)|^2\,\mathrm{d}x\right)\\ &\leqslant 2(K+\kappa t)\sup_{x\in\mathbf{R}}\mathrm{E}\left(|u(t\,,x)|^2\right).\end{aligned}\tag{4.17}$$

The first inequality in (4.3) follows.

As regards the second inequality in (4.3), we may observe that for all $p\in(1\,,2)$, $\epsilon\in(0\,,1)$, $j\in\mathbf{Z}$, and $t\geqslant 0$,

$$\begin{aligned}\sup_{j\leqslant x\leqslant j+1}|u(t\,,x)|^{2p} &\leqslant 2^{2p-1}\left(|u(t\,,j)|^{2p}+\sup_{j\leqslant x\leqslant j+1}|u(t\,,x)-u(t\,,j)|^{2p}\right)\\ &\leqslant 2^{2p-1}\left(|u(t\,,j)|^{2p}+\Omega_j^p\right),\end{aligned}$$

where

$$\Omega_j^p := \sup_{j\leqslant x\leqslant x'\leqslant j+1}\frac{|u(t\,,x)-u(t\,,x')|^2}{|x-x'|^{1-\epsilon}}.\tag{4.18}$$

Consequently,

$$\mathrm{E}\left(\sup_{j\leqslant x\leqslant j+1}|u(t\,,x)|^{2p}\right)\leqslant 2^{2p-1}\left\{\mathrm{E}\left(|u(t\,,j)|^{2p}\right)+\mathrm{E}\left(\Omega_j^p\right)\right\}.\tag{4.19}$$

Lemma 4.7 implies that $\mathrm{E}(\Omega_j^p)\leqslant C_{p,\epsilon,\delta}\,\mathrm{e}^{p(1+\delta)\lambda_p t}$. Moreover, $u(t\,,j)=0$ a.s. for $|j|>K+\kappa t$ [Proposition 4.4], and $\mathrm{E}(|u(t\,,j)|^{2p})\leqslant \mathrm{const}\cdot\mathrm{e}^{(\bar\gamma(2p)+o(1))t}$ whenever $|j|\leqslant K+\kappa t$ [Theorem 3.1]. It follows that for all large t,

$$\begin{aligned}\mathrm{E}\left(\sup_{x\in\mathbf{R}}|u(t\,,x)|^{2p}\right) &= \mathrm{E}\left(\sup_{|x|\leqslant\lceil K+\kappa t\rceil}|u(t\,,x)|^{2p}\right)\\ &\leqslant \mathrm{const}\cdot\lceil K+\kappa t\rceil\left(\mathrm{e}^{(\bar\gamma(2p)+o(1))t}+C_{p,\epsilon,\delta}\mathrm{e}^{p(1+\delta)\lambda_p t}\right),\end{aligned}\tag{4.20}$$

whence

$$\limsup_{t\to\infty}\frac{1}{t}\,\log\mathrm{E}\left(\sup_{x\in\mathbf{R}}|u(t\,,x)|^{2p}\right)\leqslant\max\{p(1+\delta)\lambda_p\ ;\ \bar\gamma(2p)\}.\tag{4.21}$$

We let $\delta\to 0$, then use Jensen's inequality and finally take $p\to 1$. Since, $\lambda_p\to\overline{\gamma}(2)$ as $p\to 1$, this will lead us to the bounds

$$\limsup_{t\to\infty}\frac{1}{t}\,\log\mathrm{E}\left(\sup_{x\in\mathbf{R}}|u(t\,,x)|^{2}\right)\leqslant\bar\gamma(2)\leqslant\mathrm{Lip}_\sigma\sqrt{\frac{\kappa}{2}},\tag{4.22}$$

by Theorem 3.1 and Remark 3.2. The last inequality in (4.3) follows. This completes our proof. $\square$

5 Moment and tail probability estimates

In this section, we will first present technical estimates on the L^p moments of the solution u, and then use those estimates in order to establish estimates on tail probabilities of the solution. We will use the efforts of this section later in §7 in order to deduce the main results of this paper. This section contains the hyperbolic analogues of the results of [12], valid for parabolic equations. Some of the arguments of [12] can be simplified greatly, because we are in a hyperbolic setting. But in several cases, one uses arguments similar to those in [12]. Therefore, we skip some of the details.

Convention. Throughout §5, we will consider only the case that u_0 and v_0 are constants.

Without loss of much generality, we will assume that $u_0 \equiv 1$. The general case follows from this by scaling. However, we will have to keep track of the numerical value of v_0. Hence, (1.3) becomes

$$u(t\,,x) = 1 + v_0\kappa t + \int_{(0,t)\times\mathbf{R}} \Gamma_{t-s}(y-x)\sigma(u(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y), \tag{5.1}$$

for $t \geqslant 0$, $x \in \mathbf{R}$. In accord with the results of Dalang [16], the law of $u_t(x)$ is independent of x, since the initial velocity v_0 and position $u_0 \equiv 1$ are constants.

We start our presentation by stating a general upper bound for the moments of the solution.

Proposition 5.1. *Suppose $u_0 \equiv 1$ and v_0 is a constant. Choose and fix $T > 0$ and define $a := T\mathrm{Lip}_\sigma\sqrt{\kappa}$. Then there exists a finite constant $C > 0$ such that*

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}} \mathrm{E}\left(|u(t\,,x)|^p\right) \leqslant C^p \exp\left(ap^{3/2}\right) \qquad \textit{for all } p \in [1\,,\infty). \tag{5.2}$$

The preceding is a direct consequence of our proof of Theorem 3.1. Indeed, we proved there that $\|u\|_{p,\beta} < \infty$ provided that $\beta > p^{3/2}\mathrm{Lip}_\sigma\sqrt{\kappa/2}$. Proposition 5.1 follows upon unscrambling this assertion.

Let us recall the following "stretched-exponential" bound for $\log X$:

Lemma 5.2 (Lemma 3.4 of [12])**.** *Suppose X is a non-negative random variable that satisfies the following: There exists finite numbers $a, C > 0$ and $b > 1$ such that $\mathrm{E}(X^p) \leqslant C^p \exp(ap^b)$ for all $p \in [1\,,\infty)$. Then,*

$$\mathrm{E}\exp\left(\alpha\left[\log_+ X\right]^{b/(b-1)}\right) < \infty,$$

where $\log_+ u := \log(u \vee \mathrm{e})$, provided that $0 < \alpha < (1-b^{-1})/(ab)^{1/(b-1)}$.

Thanks to the preceding lemma and Chebyshev's inequality, Proposition 5.1 implies readily the following upper bound on the tail of the distribution of $|u(t\,,x)|$.

Corollary 5.3. *For all $T \in (0\,,\infty)$ and $\alpha \in (0\,,\frac{4}{27}(T^2(\mathrm{Lip}_\sigma \vee 1)^2\kappa)^{-1})$,*

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}} \mathrm{E}\left[\exp\left\{\alpha\left(\log_+|u(t\,,x)|\right)^3\right\}\right] < \infty. \tag{5.3}$$

Consequently,

$$\limsup_{\lambda\to\infty}\frac{1}{(\log\lambda)^3}\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}}\log \mathrm{P}\{|u(t\,,x)|>\lambda\} \leqslant -\frac{4}{27\,T^2(\mathrm{Lip}_\sigma\vee 1)^2\kappa}. \tag{5.4}$$

In plainer terms, Corollary 5.3 asserts that there is a finite constant $A := A_T > 1$ such that for all λ sufficiently large,

$$\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbf{R}} \mathrm{P}\left\{|u(t\,,x)|\geqslant\lambda\right\} \leqslant A\exp\left(-A^{-1}|\log\lambda|^3\right).$$

In order to bound lower bounds on tail probabilities we need to have more specific information on the non-linearity σ. Let us start with the case that σ is bounded uniformly away from zero.

Proposition 5.4. *If $\epsilon_0 := \inf_{z\in\mathbf{R}}\sigma(z) > 0$, then for all $t\in(0\,,\infty)$,*

$$\inf_{x\in\mathbf{R}} \mathrm{E}\left(|u(t\,,x)|^{2p}\right) \geqslant \left(\sqrt{2}+o(1)\right)(\mu_t p)^p \qquad \text{as } p\to\infty, \tag{5.5}$$

where the $o(1)$ term only depends on p and

$$\mu_t := \epsilon_0^2\kappa t^2/(2\mathrm{e}). \tag{5.6}$$

Proof. We follow the proof of Lemma 3.6 of [12] closely.

Since the law of $u(t\,,x)$ does not depend on x, the inf in (5.5) is redundant. From now on, we will consider only the case that $x=0$.

Choose and fixed a finite $t>0$, and notice that $u(t\,,0) = 1+v_0\kappa t + M_t$, where $(M_\tau)_{0\leqslant\tau\leqslant t}$ is the continuous mean-zero martingale that is defined by

$$M_\tau := \int_{(0,\tau)\times\mathbf{R}} \Gamma_{t-s}(y)\sigma(u(s\,,y))\,W(\mathrm{d}s\,\mathrm{d}y). \tag{5.7}$$

The quadratic variation of M is given by

$$\langle M\rangle_\tau = \int_0^\tau \mathrm{d}s\int_{-\infty}^\infty \mathrm{d}y\ \Gamma_{t-s}^2(y)\sigma^2(u(s\,,y)). \tag{5.8}$$

According to Itô's formula, if $p\in[2\,,\infty)$ then

$$M_t^{2p} = 2p\int_0^t M_s^{2p-1}\,\mathrm{d}M_s + p(2p-1)\int_0^t M_s^{2p-2}\mathrm{d}\langle M\rangle_s. \tag{5.9}$$

We expectations of both sides and replace $\langle M\rangle$ using (5.8), in order to obtain the following:

$$\begin{aligned}\mathrm{E}\left(M_t^{2p}\right) &= p(2p-1)\int_0^t\mathrm{d}s\int_{-\infty}^\infty \mathrm{d}y\ \mathrm{E}\left(M_s^{2(p-1)}\sigma^2(u(s\,,y))\right)\Gamma_{t-s}^2(y)\\ &\geqslant p(2p-1)\epsilon_0^2\cdot\int_0^t\mathrm{d}s\int_{-\infty}^\infty\mathrm{d}y\ \mathrm{E}\left(M_s^{2(p-1)}\right)\Gamma_{t-s}^2(y).\end{aligned}$$

We iterate this process, using (5.8), to obtain the following lower bound for the moments of M:

$$\mathrm{E}\left(M_t^{2p}\right) \geqslant \sum_{k=0}^{p-1} C_k(p) \int_0^t \nu(t\,,\mathrm{d}s_1) \int_0^{s_1} \nu(s_1\,,\mathrm{d}s_2) \cdots \int_0^{s_k} \nu(s_k\,,\mathrm{d}s_{k+1}), \quad (5.10)$$

where

$$\nu(t\,,\mathrm{d}s) := \mathbf{1}_{[0,t]}(s)\, \|\Gamma_{t-s}\|_{L^2(\mathbf{R})}^2\, \mathrm{d}s = \tfrac{1}{2}\kappa(t-s)\mathbf{1}_{[0,t]}(s)\,\mathrm{d}s \qquad \text{[see (2.5)]},$$

and

$$C_k(p) := \epsilon_0^{2(k+1)} \cdot \prod_{j=0}^{k} \binom{2p-2j}{2}.$$

For similar moment computations, also valid for hyperbolic equations, see [11]. The right-hand side of (5.10) is the exact expression for the p^{th} moment of u if σ were identically ϵ_0. Therefore,

$$\mathrm{E}\left(|u(t\,,0)|^{2p}\right) \geqslant \mathrm{E}\left(M_t^{2p}\right) \geqslant \mathrm{E}\left(N_t^{2p}\right), \quad (5.11)$$

where $N_t := \epsilon_0 \cdot \int_{(0,t)\times\mathbf{R}} \Gamma_{t-s}(y)\, W(\mathrm{d}s\,\mathrm{d}y)$ is a Gaussian random variable with mean zero and variance $\mathrm{E}(N_t^2) = \epsilon_0^2 \cdot \int_0^t \|\Gamma_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s = \epsilon_0^2\kappa t^2/4$. Therefore, for every integer $p \geqslant 2$,

$$\mathrm{E}\left(N_t^{2p}\right) = \frac{(2p)!}{2^p\, p!} \left\{\mathrm{E}\left(N_t^2\right)\right\}^p = \frac{(2p)!}{2^p\, p!} \left(\frac{\epsilon_0^2\kappa t^2}{4}\right)^p. \quad (5.12)$$

Stirling's formula, (5.11) and (5.12) together prove the result if $p \to \infty$ along integers. For other values of p, we use the integer case for $\lceil p \rceil$, and apply Jensen's inequality to bound the $\|u(t\,,0)\|_p$ by $\|u(t\,,0)\|_{\lceil p \rceil}$. □

The preceding moment bound yields the next probability estimate.

Proposition 5.5. *If $\inf_{z\in\mathbf{R}} \sigma(z) = \epsilon_0 > 0$, then there exists a constant $C \in (0\,,\infty)$ such that for all $t > 0$,*

$$\liminf_{\lambda\to\infty} \frac{1}{\lambda^3} \inf_{x\in\mathbf{R}} \log \mathrm{P}\{|u(t\,,x)| \geqslant \lambda\} \geqslant -C\, \frac{(\mathrm{Lip}_\sigma \vee 1)}{\epsilon_0^3 t^2 \kappa}. \quad (5.13)$$

Proof. We follow the proof of [12, Proposition 3.7].

The classical Paley–Zygmund inequality implies that

$$\begin{aligned} \mathrm{P}\left\{|u(t\,,x)| \geqslant \tfrac{1}{2}\|u(t\,,x)\|_{2p}\right\} &\geqslant \frac{\left\{\mathrm{E}\left(|u(t\,,x)|^{2p}\right)\right\}^2}{4\mathrm{E}\left(|u(t\,,x)|^{4p}\right)} \\ &\geqslant \exp\left(-8t(\mathrm{Lip}_\sigma \vee 1)\kappa^{1/2}\, p^{3/2}\right), \end{aligned} \quad (5.14)$$

owing to Propositions 5.1 and 5.4. Proposition 5.4 tells us that $\|u(t\,,x)\|_{2p}$ is bounded below by $(1+o(1))$ times $(\mu_t p)^{1/2}$ as $p\to\infty$, where μ_t is given by (5.6). Therefore,

$$\mathrm{P}\left\{|u(t\,,x)|\geqslant\frac{1}{2}(\mu_t p)^{1/2}\right\}\geqslant\exp\left(-8t(\mathrm{Lip}_\sigma\vee 1)\kappa^{1/2}p^{3/2}\right),\tag{5.15}$$

for all sufficiently-large p. Set $\lambda:=\frac{1}{2}(\mu_t p)^{1/2}$ to complete the proof. □

Let us write "$f(x)\gtrsim g(x)$ as $x\to\infty$" instead of "there exists a constant $C\in(0\,,\infty)$ such that $\liminf_{x\to\infty}f(x)/g(x)\geqslant C$". In this way, we may summarize the findings of this section, so far, as follows:

Corollary 5.6. *Suppose $u_0\equiv 1$ and $v_0\equiv$ a constant. If $\inf_{z\in\mathbf{R}}\sigma(z)=\epsilon_0>0$, then for all $t>0$,*

$$-\frac{\lambda^3}{\kappa}\lesssim\log\mathrm{P}\{|u(t\,,x)|\geqslant\lambda\}\lesssim-\frac{(\log\lambda)^3}{\kappa}\qquad\text{as }\lambda\to\infty.\tag{5.16}$$

The implied constants do not depend on $(x\,,\kappa)$.

Proposition 5.1 and Corollary 5.3 assumed that σ was a Lipschitz function. If we assume, in addition, that σ is bounded above (as well as bounded away from 0), then we can obtain a nearly-optimal improvement to Corollary 5.6. In fact, the following shows that the lower bound of Proposition 5.4 is sharp in such cases.

Proposition 5.7. *If $S_0:=\sup_{z\in\mathbf{R}}\sigma(z)<\infty$, then for all $t>0$ and all integers $p\geqslant 1$,*

$$\sup_{x\in\mathbf{R}}\mathrm{E}\left(|u(t\,,x)|^{2p}\right)\leqslant\left(2\sqrt{2}+o(1)\right)(\tilde{\mu}_t p)^p\qquad\text{as }p\to\infty,\tag{5.17}$$

where the $o(1)$ term only depends on p and

$$\tilde{\mu}_t:=2S_0^2\kappa t^2/\mathrm{e}.\tag{5.18}$$

Proof. We apply an argument that is similar to the one used in the proof of Proposition 5.4. Namely, we consider the same martingale $\{M_\tau\}_{0\leqslant\tau\leqslant t}$, as we did for the proof of Proposition 5.4. We apply exactly the same argument as we did there, but reverse the inequalities using the bound $\sigma(z)\leqslant S_0$ for all $z\in\mathbf{R}$, in order to deduce the following:

$$\begin{aligned}\mathrm{E}\left(|u(t\,,0)|^{2p}\right)&\leqslant 2^{2p}(1+v_0\kappa t)^{2p}+2^{2p}\mathrm{E}\left(M_t^{2p}\right)\\&\leqslant 2^{2p}(1+v_0\kappa t)^{2p}+2^{2p}\mathrm{E}\left(N_t^{2p}\right),\end{aligned}$$

where $N_t:=S_0\cdot\int_{(0,t)\times\mathbf{R}}\Gamma_{t-s}(y)\,W(\mathrm{d}s\,\mathrm{d}y)$. Similar computations as in Proposition 5.4 prove the result. □

We can now turn this bound into Gaussian tail-probability estimates.

Proposition 5.8. *If $0 < \epsilon_0 := \inf_{z\in\mathbf{R}} \sigma(z) \leqslant \sup_{z\in\mathbf{R}} \sigma(z) := S_0 < \infty$, then for all $t > 0$ there exists finite constants $C > c > 0$ such that*

$$c \exp\left(-C\frac{\lambda^2}{\kappa}\right) \leqslant \mathrm{P}\{|u(t\,,x)| \geqslant \lambda\} \leqslant C \exp\left(-c\frac{\lambda^2}{\kappa}\right), \tag{5.19}$$

simultaneously for all λ large enough and $x \in \mathbf{R}$.

Proof. The lower bound is obtained in the exact same manner as in the proof of Proposition 5.5: We use the Paley–Zygmund inequality, though we now appeal to Proposition 5.7 instead of Proposition 5.1.

We establish the upper bound by first applying Proposition 5.7 in order to see that $\sup_{x\in\mathbf{R}} \mathrm{E}(|u(t\,,x)|^{2m}) \leqslant (A\kappa)^m\, m!$ for all integers $m \geqslant 1$, for some constant $A \in (0,\infty)$. This inequality implies that for al $0 < \xi < (Ak)^{-1}$,

$$\sup_{x\in\mathbf{R}} \mathrm{E}\left(\mathrm{e}^{\xi|u(t\,,x)|^2}\right) \leqslant \sum_{m=0}^{\infty} (\xi A\kappa)^m = \frac{1}{1-\xi A\kappa} < \infty. \tag{5.20}$$

Therefore, Chebyshev's inequality implies that if $0 < \xi < (A\kappa)^{-1}$, then

$$\sup_{x\in\mathbf{R}} \mathrm{P}\{|u(t\,,x)| > \lambda\} \leqslant \frac{\exp(-\xi\lambda^2)}{1-\xi A\kappa} \qquad (\lambda > 0). \tag{5.21}$$

We choose $\xi = \mathrm{const}\cdot\kappa^{-1}$, for a suitably-large constant to finish. $\square$

6 Localization

In §7 below, we will establish the chaotic behavior of the solution u to (1.1). The analysis of §7 will rest on a series of observations; one of the central ones is that the random function u is highly "localized." We will make this more clear in this section. In the mean time, let us say sketch, using only a few words, what localization means in the present context.

Essentially, localization is the property that if x_1 and x_2 are chosen "sufficiently" far apart, then $u(t\,,x_1)$ and $u(t\,,x_2)$ are "approximately independent."

As we did in Section 5, we will assume throughout this section that the initial conditions are identically constant, and that $u_0 \equiv 1$. [Recall that the latter assumption is made without incurring any real loss in generality.] Note, in particular, that the solution u can be written in the mild form (1.5). Equivalently,

$$u(t\,,x) = 1 + v_0\kappa t + \tfrac{1}{2}\int_{(0,t)\times(x-\kappa t,x+\kappa t)} \sigma(u(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y), \tag{6.1}$$

for all $t > 0$, $x \in \mathbf{R}$.

For all integers $n \geqslant 0$, let $\{u_n(t\,,x)\}_{t\geqslant 0,x\in\mathbf{R}}$ denote the n-th step Picard approximation to u. Namely, we have $u_0 \equiv 0$ and, for $n \geqslant 1$, $t \geqslant 0$ and $x \in \mathbf{R}$,

$$u_n(t\,,x) = 1 + v_0\kappa t + \tfrac{1}{2}\int_{(0,t)\times(x-\kappa t,x+\kappa t)} \sigma(u_{n-1}(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y). \tag{6.2}$$

Our next result estimates the order of convergence of the Picard iteration.

Proposition 6.1. *Let u denote the solution to* (1.1) *with constant initial velocity v_0, and constant initial function $u_0 \equiv 1$. Let u_n be defined as above. Then, for all $n \geqslant 0$, $t \geqslant 0$, and $p \in [2\,,\infty)$,*

$$\sup_{x\in\mathbf{R}} \mathrm{E}\left(|u(t\,,x) - u_n(t\,,x)|^p\right) \leqslant C^p \exp\left(ap^{3/2}t - np\right), \tag{6.3}$$

where the constants $C, a \in (0\,,\infty)$ do not depend on $(n\,,t\,,p)$

Proof. Recall the norms $\|\cdots\|_{p,\beta}$ from (2.2). In accord with Proposition 3.5 and (6.2),

$$\|u - u_n\|_{p,\beta} \leqslant \text{const}\cdot \frac{p^{3/2}\kappa^{1/2}\mathrm{Lip}_\sigma^2}{4\beta\sqrt{2}}\,\|u - u_{n-1}\|_{p,\beta}.$$

We apply (2.5) with $\beta := \mathrm{e}2^{-5/2}\kappa^{1/2}\mathrm{Lip}_\sigma^2 p^{3/2}$ in order to deduce, for this choice of β, the inequality $\|u - u_n\|_{p,\beta} \leqslant \mathrm{e}^{-1}\|u - u_{n-1}\|_{p,\beta}$, whence $\|u - u_n\|_{p,\beta} \leqslant \mathrm{e}^{-n}\|u - u_0\|_{p,\beta}$ by iteration. In other words, we have proved that

$$\mathrm{E}\left(|u(t\,,x) - u_n(t\,,x)|^p\right) \leqslant \mathrm{e}^{-np+\beta t}\|u\|_{p,\beta}^p. \tag{6.4}$$

An appeal to Proposition 5.1 concludes the proof. □

We plan to use the Picard iterates $\{u_n\}_{n=0}^\infty$ in order to establish the localization of u. The following is the next natural step in this direction.

Proposition 6.2. *Let $t > 0$ and choose and fix a positive integer n. Let $\{x_i\}_{i\geqslant 0}$ denote a sequence of real numbers such that $|x_i - x_j| > 2n\kappa t$ whenever $i \neq j$. Then $\{u_n(t\,,x_i)\}_{i\geqslant 0}$ is a collection of i.i.d. random variables.*

Proof. It is easy to verify, via induction, that the random variable $u_n(t\,,x)$ depends only on the value of the noise $\dot{W}$ evaluated on $[0\,,t]\times[x - n\kappa t\,,x + n\kappa t]$. Indeed, it follows from (6.2) that $u_1(t\,,x) = 1 + v_0\kappa t$ is deterministic, and (6.2) does the rest by induction.

With this property in mind, we now choose and fix a sequence $\{x_i\}_{i\geqslant 0}$ as in the statement of the proposition. Without loss of too much generality, let us consider x_1 and x_2. By the property that was proved above, $u_n(t\,,x_1)$ only depends only on the noise on $I_1 := [0\,,t]\times[x_1 - n\kappa t\,,x_1 + n\kappa t]$, whereas $u_n(t\,,x_2)$ depends only on the noise on $I_2 := [0\,,t]\times[x_2 - n\kappa t\,,x_2 + n\kappa t]$. According to the defining property of the x_i's, $|x_1 - x_2| > 2n\kappa t$, and hence I_1 and I_2 are disjoint. Therefore, it follows from the independence properties of white noise that $u(t\,,x_1)$ and $u(t\,,x_2)$ are independent. Moreover, the stationarity properties of stochastic integrals imply that $u(t\,,x_1)$ and $u(t\,,x_2)$ are identically distributed as well [here we use also the assumption of constant initial data]. This proves the result for $n = 2$. The general case is proved by expanding on this case a little bit more. We omit the remaining details.

Let us conclude by mentioning that the preceding is the sketch of a complete argument. A fully-rigorous proof would require us to address a few technical issues about Walsh stochastic integral. They are handled as in the proof of Lemma 4.4 in [12], and the arguments are not particularly revealing; therefore, we omit the details here as well. □

7 Chaotic behavior

We are now ready to state and prove the two main results of this paper. The first one addresses the case that σ is bounded away uniformly from zero, and shows a universal blow-up rate of $(\log R)^{1/3}$.

Theorem 7.1. *If $u_0 > 0$, $v_0 \geqslant 0$, and $\inf_{z\in\mathbf{R}} \sigma(z) = \epsilon_0 > 0$, then for all $t > 0$ there exists a constant $c := c_t \in (0\,,\infty)$—independent of κ—such that*

$$\liminf_{R\to\infty} \frac{1}{(\log R)^{1/3}} \sup_{x\in[-R,R]} |u(t\,,x)| \geqslant c\kappa^{1/3}.$$

Proof. The basic idea is the following: Consider a sequence of spatial points $\{x_i\}_{i\geqslant 0}$, as we did in Proposition 6.2, in order to obtain an i.i.d. sequence $\{u_n(t\,,x_i)\}_{i\geqslant 0}$. The tail probability estimates of §5 will imply that every random variable $u_n(t\,,x_i)$ has a positive probability of being "very large." Therefore, a Borel-Cantelli argument will imply that if we have enough spatial points, then eventually one of the $u_n(t\,,x_i)$'s will have a "very large" value a.s. A careful quantitative analysis of this outline leads to the estimates of Theorem 7.1. Now let us add a few more details.

Fix integers $n, N > 0$ and let $\{x_i\}_{i=1}^N$ be a sequence of points as in Proposition 6.2. According to Proposition 6.2, $\{u_n(t\,,x_i)\}_{i=1}^N$ is a sequence of independent random variables. For every $\lambda > 0$,

$$\begin{aligned} &\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u(t\,,x_j)| < \lambda\right\} \\ &\qquad \leqslant \mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_n(t\,,x_j)| < 2\lambda\right\} + \mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u(t\,,x_j) - u_n(t\,,x_j)| > \lambda\right\}. \end{aligned}$$

An inspection of the proof of Proposition 5.5 shows us that the proposition continues to hold after u is replaced by u_n. Therefore,

$$\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_n(t\,,x_j)| < 2\lambda\right\} \leqslant \left(1 - c_1 \mathrm{e}^{-c_2(2\lambda)^3}\right)^N, \tag{7.1}$$

for some constants c_1 and c_2. Moreover, Chebyshev's inequality and Proposition 6.1 together yield

$$\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u(t\,,x_j) - u_n(t\,,x_j)| > \lambda\right\} \leqslant NC^p \mathrm{e}^{ap^{3/2}t - np}\lambda^{-p}, \tag{7.2}$$

and hence

$$\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u(t\,,x_j)| < \lambda\right\} \leqslant \left(1 - c_1 \mathrm{e}^{-c_2(2\lambda)^3}\right)^N + NC^p \mathrm{e}^{ap^{3/2}t - np}\lambda^{-p}. \tag{7.3}$$

Now, we select the various parameters with some care. Namely, we set $\lambda := p$, $N := p\exp(c_2 p^3)$, and $n = \varrho p^2$ for some constant $\varrho > 8c_2$. With these parameter choices, (7.3) reduces to the following:

$$\begin{aligned}
&\mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u(t\,,x_j)|<p\right\}\\
&\leqslant \mathrm{e}^{-c_1p}+\exp\Big(c_2(2p)^3+\log p+atp^{3/2}-\varrho p^3-p\log p\Big)\\
&\leqslant 2\mathrm{e}^{-c_1p}.
\end{aligned}\tag{7.4}$$

We may consider the special case $x_i=\pm 2i\kappa tn$ in order to deduce the following:

$$\mathrm{P}\left\{\sup_{|x|\leqslant 2N\kappa tn}|u(t\,,x)|<p\right\}\leqslant 2\mathrm{e}^{-c_1p}.\tag{7.5}$$

Note that $2N\kappa tn=O(\mathrm{e}^{c_2p^3})$ as $p\to\infty$, Let us choose $R:=\exp(c_2p^3)$, equivalently $p:=(\log R/c_2)^{1/3}$. Then by the Borel-Cantelli lemma,

$$\sup_{|x|<R}|u(t\,,x)|\geqslant \text{const}\cdot\left(\frac{\log R}{c_2}\right)^{1/3}.\tag{7.6}$$

A monotonicity argument shows that the preceding inequality continues to hold for non-integer R [for a slightly smaller constant, possibly]. A careful examination of the content of Proposition 5.5 shows that we can at best choose $c_2=\text{const}\cdot\kappa^{-1}$. The result follows. □

The second result of this section [and the second main result of the present paper] contains an analysis of the case that σ is bounded both uniformly above 0 and below ∞. In that case, we will obtain an exact order of growth for $\sup_{|x|<R}|u(t\,,x)|$, as $R\to\infty$. We can deduce by examining that growth order that the behavior of the solution u is similar to the case where σ is identically a constant. [In the latter case, u is a Gaussian process.]

Theorem 7.2. *Assume constant initial data with $u_0>0$ and $v_0\geqslant 0$. Suppose also that $0<\inf_{z\in\mathbf{R}}\sigma(z)\leqslant\sup_{z\in\mathbf{R}}\sigma(z)<\infty$. Then, for all $t>0$ there exists finite constants $C:=C_t>c:=c_t>0$ such that a.s.,*

$$c\kappa^{1/2}\leqslant\liminf_{R\to\infty}\frac{\sup_{x\in[-R,R]}|u(t\,,x)|}{(\log R)^{1/2}}\leqslant\limsup_{R\to\infty}\frac{\sup_{x\in[-R,R]}|u(t\,,x)|}{(\log R)^{1/2}}\leqslant C\kappa^{1/2}.$$

Moreover, there exists a finite constant $\kappa_0=\kappa_0(t)>0$ such that c and C do not depend on κ whenever $\kappa\in(0\,,\kappa_0)$.

We first need an estimate for the quality of the spatial continuity of the solution u.

Lemma 7.3. *Suppose $0<\epsilon_0:=\inf_{z\in\mathbf{R}}\sigma(z)\leqslant\sup_{z\in\mathbf{R}}\sigma(z):=S_0<\infty$. Then, for every $t>0$, there exists a constant $A\in(0\,,\infty)$ such that*

$$\sup_{-\infty<x\neq x'<\infty}\frac{\mathrm{E}\left(|u(t\,,x)-u(t\,,x')|^{2p}\right)}{|x-x'|^p}\leqslant(Ap)^p\qquad\textit{for all } p\in[2\,,\infty).\tag{7.7}$$

Proof. We follows closely the proof of Lemma 6.1 of [12]. Fix $x, x' \in \mathbf{R}$ and define

$$M_\tau := \int_{(0,t)\times\mathbf{R}} (\Gamma_{t-s}(y-x) - \Gamma_{t-s}(y-x'))\sigma(u(s\,,y))\, W(\mathrm{d}s\,\mathrm{d}y). \tag{7.8}$$

Then, $\{M_\tau\}_{0\leqslant\tau\leqslant t}$ is a mean-zero continuous $L^p(\Omega)$-martingale for every $p \in [2\,,\infty)$. Moreover, its quadratic variation is bounded as follows:

$$\langle M\rangle_\tau \leqslant S_0^2 \int_0^\tau \mathrm{d}s \int_{-\infty}^\infty \mathrm{d}y\ |\Gamma_{t-s}(y-x) - \Gamma_{t-s}(y-x')|^2 \leqslant 2\tau S_0^2|x-x'|;$$

by (4.12). Because $u(t\,,x) - u(t\,,x') = M_t$, the Burkholder–Davis–Gundy inequality (2.1) implies the result. ☐

Next we transform the previous lemma into an estimate of sub-Gaussian moment bounds.

Lemma 7.4. *If* $0 < \epsilon_0 := \inf_{z\in\mathbf{R}} \sigma(z) \leqslant \sup_{z\in\mathbf{R}} \sigma(z) := S_0 < \infty$*, then for every* $t > 0$*, there exists a constant* $C = C_t \in (0\,,\infty)$ *such that*

$$\mathrm{E}\left[\sup_{\substack{x,x'\in I:\\ |x-x'|\leqslant\delta}} \exp\left(\frac{|u(t\,,x) - u(t\,,x')|^2}{C\delta}\right)\right] \leqslant \frac{2}{\delta}, \tag{7.9}$$

uniformly for every $\delta \in (0\,,1]$ *and every interval* $I \subset \mathbf{R}$ *of length at most one.*

Lemma 7.4 follows from Lemma 7.3 and a suitable form of Kolomogorov's continuity theorem. This type of technical argument appears in several places in the literature. Hence, we merely refer to the proof of [12, Lemma 6.2], where this sort of argument appears already in a different setting. Instead, we proceed with the more interesting

Proof of Theorem 7.2. We obtain lower bound by adapting the method of proof of Theorem 7.1. The only major required change is that we need to use Proposition 5.8 in place of Proposition 5.5. We also need to improve Proposition 6.1 in order to consider a moment bound that applies Proposition 5.7 instead of 5.1. After all this, (7.3) will turn into the following estimate:

$$\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u(t\,,x_j)| < \lambda\right\} \leqslant \left(1 - c_1\mathrm{e}^{-c_2(2\lambda)^2}\right)^N + NC^p(\tilde\mu_t p)^p\mathrm{e}^{-np}\lambda^{-p}. \tag{7.10}$$

Next we select the parameters judiciously: We take $\lambda := p$, $N := p\mathrm{e}^{c_2p^2}$, and $n = \varrho p$ for a sufficiently-large constant $\varrho \gg c_2$. In this way, (7.3) will read as follows:

$$\begin{aligned}\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u(t\,,x_j)| < p\right\} &\leqslant \mathrm{e}^{-c_1p} + \exp\left(c_2(2p)^2 + \log(p) + p\log(\tilde\mu_t) - \varrho p^2\right)\\ &\leqslant 2\mathrm{e}^{-c_1p}.\end{aligned}$$

A Borel-Cantelli type argument leads to the lower bound.

In order to establish the upper bound, let $R > 0$ be an integer and $x_j := -R + j$ for $j = 1, \ldots, 2R$. Then, we can write

$$\begin{aligned}\mathrm{P}&\left\{\sup_{x\in[-R,R]}|u(t\,,x)| > 2\alpha(\log R)^{1/2}\right\}\\ &\qquad\leqslant \mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R}|u(t\,,x_j)| > \alpha(\log R)^{1/2}\right\}\\ &\qquad\qquad + \mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R}\sup_{x\in(x_j,x_{j+1})}|u(t\,,x)-u(t\,,x_j)| > \alpha(\log R)^{1/2}\right\}.\end{aligned} \tag{7.11}$$

On one hand, Proposition 5.8 can be used to show that

$$\begin{aligned}\mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R}|u(t\,,x_j)| > \alpha(\log R)^{1/2}\right\} &\leqslant 2R\sup_{x\in\mathbf{R}}\mathrm{P}\left\{|u(t\,,x)| > \alpha(\log R)^{1/2}\right\}\\ &\leqslant \mathrm{const}\cdot R^{1-c\alpha^2/\kappa}.\end{aligned}$$

On the other hand, Chebyshev's inequality and Lemma 7.4 [with $\delta = 1$] together imply that

$$\mathrm{P}\left\{\max_{1\leqslant j\leqslant 2R}\sup_{x\in(x_j,x_{j+1})}|u(t\,,x)-u(t\,,x_j)| > \alpha(\log R)^{1/2}\right\} \leqslant \mathrm{const}\cdot R^{1-\alpha^2/C}.$$

Therefore, (7.11) has the following consequence:

$$\sum_{R=1}^{\infty}\mathrm{P}\left\{\sup_{x\in[-R,R]}|u(t\,,x)| > 2\alpha(\log R)^{1/2}\right\} \leqslant \sum_{R=1}^{\infty}R^{1-q\alpha^2}; \tag{7.12}$$

where

$$q := \min\left(c/\kappa\ ,\ 1/C\right).$$

The infinite sum in (7.12) converges when $\alpha > (2/q)^{1/2}$. Therefore, by an application of the Borel-Cantelli Lemma,

$$\limsup_{\substack{R\to\infty:\\ R\in\mathbf{Z}}}\frac{\sup_{x\in[-R,R]}|u(t\,,x)|}{(\log R)^{1/2}} \leqslant (8/q)^{1/2} \qquad \text{a.s.} \tag{7.13}$$

Clearly, $(8/q)^{1/2} \leqslant \kappa^{1/2}/c$ for all $\kappa > \kappa_0 := 8c^2/q$. A standard montonicity argument can be used to replace "$\limsup_{R\to\infty:\ R\in\mathbf{Z}}$" by "$\limsup_{R\to\infty}$." This concludes the proof. □

Among other things, Theorem 7.2 implies that if σ is bounded uniformly away from 0 and infinity, then the extrema of the solution u behave as they would for the linear stochastic wave equation; i.e., they grow as $(\log R)^{1/2}$. We have shown in [12, Theorem 1.2] that the same general phenomenon holds when the stochastic wave equation is replaced by the stochastic heat equation. We may notice however that the behavior in κ is quite different in the hyperbolic setting than in the parabolic case: Here, the extrema diminish as $\kappa^{1/2}$ as $\kappa \downarrow 0$; whereas they grow as $\kappa^{-1/4}$ in the parabolic case.

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