On the chaotic character of the stochastic heat equation, II*

Daniel Conus
Lehigh University

Mathew Joseph
University of Utah

Davar Khoshnevisan
University of Utah

Shang-Yuan Shiu
Academica Sinica

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Abstract

Consider the stochastic heat equation $\partial_t u = (\varkappa/2)\Delta u + \sigma(u)\dot{F}$, where the solution $u := u_t(x)$ is indexed by $(t\,, x) \in (0\,,\infty) \times \mathbf{R}^d$, and $\dot{F}$ is a centered Gaussian noise that is white in time and has spatially-correlated coordinates. We analyze the large-$\|x\|$ fixed-t behavior of the solution u in different regimes, thereby study the effect of noise on the solution in various cases. Among other things, we show that if the spatial correlation function f of the noise is of Riesz type, that is $f(x) \propto \|x\|^{-\alpha}$, then the "fluctuation exponents" of the solution are ψ for the spatial variable and $2\psi - 1$ for the time variable, where $\psi := 2/(4-\alpha)$. Moreover, these exponent relations hold as long as $\alpha \in (0\,, d \wedge 2)$; that is precisely when Dalang's theory [12] implies the existence of a solution to our stochastic PDE. These findings bolster earlier physical predictions [22, 23].

Keywords: The stochastic heat equation, chaos, intermittency, the parabolic Anderson model, the KPZ equation, critical exponents.

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1 Introduction

Consider the nonlinear stochastic heat equation,

$$\frac{\partial}{\partial t} u_t(x) = \frac{\varkappa}{2}(\Delta u_t)(x) + \sigma(u_t(x))\dot{F}_t(x), \tag{SHE}$$

where $\varkappa > 0$ is a viscosity constant, $\sigma : \mathbf{R} \to \mathbf{R}$ is globally Lipschitz continuous, and $\{\dot{F}_t(x)\}_{t>0,x\in\mathbf{R}^d}$ is a centered generalized Gaussian random field [20,

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Chapter 2, §2.7] with covariance measure

$$\mathrm{Cov}\left(\dot{F}_t(x)\,,\dot{F}_s(y)\right) = \delta_0(t-s)f(x-y) \tag{1.1}$$

of the convolution type. We also assume, mostly for the sake of technical simplicity, that the initial function $u_0 : \mathbf{R}^d \to \mathbf{R}$ is nonrandom, essentially bounded, and measurable. In particular, we assume the following once and for all:

$$\text{Throughout this paper, we assume that } \|u_0\|_{L^\infty(\mathbf{R}^d)} < \infty, \tag{1.2}$$

and that the correlation function f is sufficiently nice that there exists a unique strong solution to (SHE); see the next section for the technical details.

Our first result (Theorem 2.1) tells us that if the initial function u_0 decays at infinity faster than exponentially, then the solution $x \mapsto u_t(x)$ is typically globally bounded at all nonrandom times $t > 0$. The remainder of this paper is concerned with showing that if by contrast u_0 remains uniformly away from zero, then the typical structure of the random function $x \mapsto u_t(x)$ is quite different from the behavior outlined in Theorem 2.1. In particular, our results show that the solution to (SHE) depends in a very sensitive way on the structure of the initial function u_0. [This property explains the appearance of "chaos" in the title of the paper.]

Hereforth, we assume tacitly that u_0 is bounded uniformly away from zero and infinity. We now describe the remaining contributions of this paper [valid for such choices of u_0].

Loosely speaking, $\dot{F}_t(x)$ is nothing but white noise in the time variable t, and has a homogenous spatial correlation function f for its space variable x. In a companion paper [10] we study (SHE) in the case that $\dot{F}$ is replaced with space-time white noise; that is the case where we replace the covariance measure with $\delta_0(t-s)\delta_0(x-y)$. In that case, the solution exists only when $d = 1$ [12, 26, 28]. Before we describe the results of [10], let us introduce some notation.

Let $h, g : \mathbf{R}^d \to \mathbf{R}_+$ be two functions. We write: (a) "$h(x) \succ g(x)$" when $\limsup_{\|x\|\to\infty}[h(x)/g(x)]$ is bounded below by a constant; (b) "$h(x) \asymp g(x)$" when $h(x) \succ g(x)$ and $g(x) \succ h(x)$ both hold; and finally (c) "$h(x) \overset{(\log)}{\approx} g(x)$" means that $\log h(x) \asymp \log g(x)$.

Armed with this notation, we can describe some of the findings of [10] as follows:

1. If σ is bounded uniformly away from zero, then $u_t(x) \succ \varkappa^{-1/12}(\log\|x\|)^{1/6}$ a.s. for all times $t > 0$, where the constant in "$\succ$" does not depend on $\varkappa$;

2. If σ is bounded uniformly away from zero and infinity, then $u_t(x) \asymp \varkappa^{-1/4}(\log\|x\|)^{1/2}$ a.s. for all $t > 0$, where the constant in "$\asymp$" holds uniformly for all $\varkappa \geqslant \varkappa_0$ for every fixed $\varkappa_0 > 0$; and

3. If $\sigma(z) = cz$ for some $c > 0$—and (SHE) is in that case called the "parabolic Anderson model" [7]—then

$$u_t(x) \overset{(\log)}{\approx} \exp\left(\frac{(\log\|x\|)^\psi}{\varkappa^{2\psi-1}}\right), \tag{1.3}$$

for $\psi = 2/3$ and $2\psi - 1 = 1/3$, valid a.s. for all $t > 0$.[1]

Coupled with the results of [18], the preceding facts show that the solution to the stochastic heat equation (SHE), driven by space-time white noise, depends sensitively on the choice of the initial data.

Let us emphasize that these findings [and the subsequent ones of the present paper] are remarks about the effect of the noise on the solution to the PDE (SHE). Indeed, it is easy to see that if $u_0(x)$ is identically equal to one—this is permissible in the present setup—then the distribution of $u_t(x)$ is independent of x. Therefore, the limiting behaviors described above cannot be detected by looking at the distribution of $u_t(x)$ alone for a fixed x. Rather it is the correlation between $u_t(x)$ and $u_t(y)$ that plays an important role.

The goal of the present paper is to study the effect of disorder on the "intermittent" behavior of the solution to (SHE); specifically, we consider spatially-homogeneous correlation functions of the form $f(x-y)$ that are fairly nice, and think of the viscosity coefficient $\varkappa$ as small, but positive. Dalang's theory [12] can be used to show that the stochastic PDE (SHE) has a solution in all dimensions if $f(0) < \infty$; and it turns out that typically the following are valid, as $\|x\| \to \infty$:

1′. If σ is bounded uniformly away from zero, then $u_t(x) \succ (\log \|x\|)^{1/4}$ for all times $t > 0$, uniformly for all $\varkappa > 0$ small;

2′. If σ is bounded uniformly away from zero and infinity, then $u_t(x) \asymp (\log \|x\|)^{1/2}$ for all $t > 0$, uniformly for all $\varkappa > 0$ small; and

3′. If $\sigma(z) = cz$ for some $c > 0$ [the parabolic Anderson model] then (1.3) holds with $\psi = 1/2$ and $2\psi - 1 = 0$, for all $t > 0$ and uniformly for all $\varkappa > 0$ small.

Thus, we find that for nice bounded correlation functions, the level of disorder [as measured by $1/\varkappa$] does not play a role in determining the asymptotic large-$\|x\|$ behavior of the solution, whereas it does for $f(x-y) = \delta_0(x-y)$. In other words, 1′, 2′, and 3′ are in sharp contrast to 1, 2, and 3 respectively. This contrast can be explained loosely as saying that when f is nice, the model is "mean field"; see in particular the application of the typically-crude inequality (4.29), which is shown to be sharp in this context.

One can think of the viscosity coefficient $\varkappa$ as "inverse time" by making analogies with finite-dimensional diffusions. As such, (1.3) suggests a kind of space-time scaling that is valid universally for many choices of initial data u_0; interestingly enough this very scaling law [ψ versus $2\psi - 1$] has been predicted in the physics literature [23, 22], and several parts of it have been proved rigorously in recent works by Balázs, Quastel, and Seppäläinen [2] and Amir, Corwin, and Quastel [1] in a large-t fixed-x regime.

[1] Even though the variable x is one-dimensional here, we write "$\|x\|$" in place of "$|x|$" because we revisit (1.3) in the next few paragraphs and consider the case that $x \in \mathbf{R}^d$ for $d \geqslant 1$.

We mentioned that (1.3) holds for $\psi = 2/3$ [space-time white noise] and $\psi = 1/2$ [f nice and bounded]. In the last portion of this paper we prove that there are models—for the correlation function f of the noise $\dot{F}$—that satisfy (1.3) for every $\psi \in (1/2\,, 2/3)$ in dimension $d = 1$ and for every $\psi \in (1/2\,, 1)$ in dimension $d \geqslant 2$. It is possible that these results reinforce the "superuniversality" predictions of Kardar and Zhang [23].

We conclude the introduction by setting forth some notation that will be used throughout, and consistently.

Let $p_t(z)$ denote the heat kernel for $(\varkappa/2)\Delta$ on $\mathbf{R}^d$; that is,

$$p_t(z) := \frac{1}{(2\pi\varkappa t)^{d/2}} \exp\left(-\frac{\|z\|^2}{2\varkappa t}\right) \qquad (t > 0,\ z \in \mathbf{R}^d). \tag{1.4}$$

We will use the Banach norms on random fields as defined in [19]. Specifically, we define, for all $k \geqslant 1$, $\delta > 0$, and random fields Z,

$$\mathcal{M}_\delta^{(k)}(Z) := \sup_{\substack{t\geqslant 0\\ x\in\mathbf{R}^d}} \left[\mathrm{e}^{-\delta t}\|Z_t(x)\|_k\right], \tag{1.5}$$

where we write

$$\|Z\|_k := \left(\mathrm{E}\left(|Z|^k\right)\right)^{1/k} \text{ whenever } Z \in L^k(\mathrm{P}) \text{ for some } k \in [1\,,\infty). \tag{1.6}$$

Throughout, $\mathcal{S}$ denotes the collection of all rapidly-decreasing Schwarz test functions from $\mathbf{R}^d$ to $\mathbf{R}$, and our Fourier transform is normalized so that

$$\hat{g}(\xi) = \int_{\mathbf{R}^d} \mathrm{e}^{ix\cdot\xi} g(x)\,\mathrm{d}x \qquad \text{for all } g \in L^1(\mathbf{R}^d). \tag{1.7}$$

On several occasions, we apply the Burkholder–Davis–Gundy inequality [4, 5, 6] for continuous $L^2(\mathrm{P})$ martingales: If $\{X_t\}_{t\geqslant 0}$ is a continuous $L^2(\mathrm{P})$ martingale with running maximum $X_t^* := \sup_{s\in[0,t]}|X_s|$ and quadratic variation process $\langle X\rangle$, then for all real numbers $k \geqslant 2$ and $t > 0$,

$$\|X_t^*\|_k \leqslant \|4k\langle X\rangle_t\|_{k/2}^{1/2}. \tag{BDG}$$

The factor $4k$ is the asymptotically-optimal bound of Carlen and Kree [8] for the sharp constant in the Burkholder–Davis–Gundy inequality that is due to Davis [14]. We will also sometimes use the notation

$$\underline{u}_0 := \inf_{x\in\mathbf{R}^d} u_0(x), \qquad \overline{u}_0 := \sup_{x\in\mathbf{R}^d} u_0(x). \tag{1.8}$$

2 Main results

Throughout, we assume tacitly that $\hat{f}$ is a measurable function [which then is necessarily nonnegative] and

$$\int_{\mathbf{R}^d} \frac{\hat{f}(\xi)}{1 + \|\xi\|^2}\,\mathrm{d}\xi < \infty. \tag{2.1}$$

Condition (2.1) ensures the existence of an a.s.-unique predictable random field $u = \{u_t(x)\}_{t>0, x\in\mathbf{R}^d}$ that solves (SHE) in the mild form [12].[2] That is, u solves the following random integral equation for all $t > 0$ and $x \in \mathbf{R}^d$:

$$u_t(x) = (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x)\sigma(u_s(y))\, F(\mathrm{d}s\,\mathrm{d}y) \qquad \text{a.s.} \tag{2.2}$$

We note that, because f is positive definite, Condition (2.1) is verified automatically [for all $d \geqslant 1$] when f is a bounded function. In fact, it has been shown in Foondun and Khoshnevisan [17] that Dalang's condition (2.1) is equivalent to the condition that the correlation function f has a *bounded potential* in the sense of classical potential theory. Let us recall what this means next: Define R_β to be the β-potential corresponding to the convolution semigroup defined by $\{p_t\}_{t>0}$; that is, R_β is the linear operator that is defined via setting

$$(R_\beta\phi)(x) := \int_0^\infty \mathrm{e}^{-\beta t}(p_t * \phi)(x)\,\mathrm{d}t \qquad (t > 0,\ x \in \mathbf{R}^d), \tag{2.3}$$

for all measurable $\phi : \mathbf{R}^d \to \mathbf{R}_+$. Then, Dalang's condition (2.1) is equivalent to the condition that $R_\beta f$ is a bounded function for one, hence all, $\beta > 0$; and another equivalent statement [the maximum principle] is that

$$(R_\beta f)(0) < \infty \quad \text{for one, hence all, } \beta > 0. \tag{2.4}$$

See [17, Theorem 1.2] for details.

Our first main result states that if u_0 decays at infinity faster than exponentially, then a mild condition on f ensures that the solution to (SHE) is bounded at all times.

Theorem 2.1. *Suppose* $\limsup_{\|x\|\to\infty} \|x\|^{-1}\log|u_0(x)| = -\infty$ *and* $\int_0^1 s^{-a}(p_s * f)(0)\,\mathrm{d}s < \infty$ *for some* $a \in (0, 1/2)$. *Also assume* $\sigma(0) = 0$. *Then* $\sup_{x\in\mathbf{R}^d} |u_t(x)| < \infty$ *a.s. for all* $t > 0$. *In fact,* $\sup_{x\in\mathbf{R}^d} |u_t(x)| \in L^k(\mathrm{P})$ *for all* $t > 0$ *and* $k \in [2, \infty)$.

Our condition on f is indeed mild, as the following remark shows.

Remark 2.2. Suppose that there exist constants $A \in (0, \infty)$ and $\alpha \in (0, d\wedge 2)$ such that $\sup_{\|x\|>z} f(x) \leqslant Az^{-\alpha}$ for all $z > 0$. [Just about every correlation function that one would like to consider has this property.] Then we can deduce from the form of the heat kernel that for all $r, s > 0$,

$$\begin{aligned}
(p_s * f)(0) &\leqslant (2\pi\varkappa s)^{-d/2}\cdot\int_{\|x\|\leqslant r} f(x)\,\mathrm{d}x + \sup_{\|x\|>r} f(x) \\
&\leqslant (2\pi\varkappa s)^{-d/2}\cdot\sum_{k=0}^\infty \int_{2^{-k-1}r<\|x\|\leqslant 2^{-k}r} f(x)\,\mathrm{d}x + \frac{A}{r^\alpha} \\
&\leqslant \frac{\text{const}}{s^{d/2}}\cdot\sum_{k=0}^\infty \left(2^{-k-1}r\right)^{d-\alpha} + \frac{A}{r^\alpha} \leqslant \text{const}\cdot\left[\frac{r^{d-\alpha}}{s^{d/2}} + r^{-\alpha}\right].
\end{aligned} \tag{2.5}$$

[2] Dalang's theory assumes that f is continuous away from the origin; this continuity condition can be removed [17, 26].

We optimize over $r > 0$ to find that $(p_s * f)(0) \leqslant \text{const} \cdot s^{-\alpha/2}$. In particular, $(R_\beta f)(0) < \infty$ for all $\beta > 0$, and $\int_0^1 s^{-a}(p_s * f)(0)\,\mathrm{d}s < \infty$ for some $a \in (0, 1/2)$. $\square$

Recall that the initial function u_0 is assumed to be bounded throughout. For the remainder of our analysis we study only bounded initial functions that also satisfy $\inf_{x\in\mathbf{R}^d} u_0(x) > 0$. And we study only correlation functions f that have the form $f = h * \tilde{h}$ for some nonnegative function $h \in W^{1,2}_{loc}(\mathbf{R}^d)$, where $\tilde{h}(x) := h(-x)$ denotes the reflection of h, and $W^{1,2}_{loc}(\mathbf{R}^d)$ denotes the vector space of all locally integrable functions $g : \mathbf{R}^d \to \mathbf{R}$ whose Fourier transform is a function that satisfies

$$\int_{\|x\|<r} \|x\|^2\, |\hat{g}(x)|^2\,\mathrm{d}x < \infty \qquad \text{for all } r > 0. \tag{2.6}$$

Because $L^2(\mathbf{R}^d) \subset W^{1,2}_{loc}(\mathbf{R}^d)$, Young's inequality tells us that $f := h * \tilde{h}$ is positive definite and continuous, provided that $h \in L^2(\mathbf{R}^d)$; in that case, we have also that $\sup_{x\in\mathbf{R}^d} |f(x)| = f(0) < \infty$. And the condition that $h \in L^2(\mathbf{R}^d)$ cannot be relaxed, as there exist many choices of nonnegative $h \in W^{1,2}_{loc}(\mathbf{R}^d) \setminus L^2(\mathbf{R}^d)$ for which $f(0) = \infty$; see Example 3.2 below. We remark also that (2.1) holds automatically when $h \in L^2(\mathbf{R}^d)$.

First, let us consider the case that $h \in L^2(\mathbf{R}^d)$ is nonnegative [so that f is nonnegative, bounded and continuous, and (2.1) is valid automatically]. According to the theory of Walsh [28], (SHE) has a mild solution $u = \{u_t(x)\}_{t>0,x\in\mathbf{R}^d}$—for all $d \geqslant 1$—that has continuous trajectories and is unique up to evanescence among all predictable random fields that satisfy $\sup_{t\in(0,T)} \sup_{x\in\mathbf{R}^d} \mathrm{E}(|u_t(x)|^2) < \infty$ for all $T > 0$. In particular, u solves (2.2) almost surely for all $t > 0$ and $x \in \mathbf{R}^d$, where the stochastic integral is the one defined by Walsh [28] and Dalang [12].

Our next result describes the behavior of that solution, for nice choices of $h \in L^2(\mathbf{R}^d)$, when viewed very far away from the origin.

Theorem 2.3. *Consider* (SHE) *where* $\inf_{x\in\mathbf{R}^d} u_0(x) > 0$*, and suppose* $f = h*\tilde{h}$ *for a nonnegative* $h \in L^2(\mathbf{R}^d)$ *that satisfies the following for some* $a > 0$*:* $\int_{\|z\|>n} [h(z)]^2\,\mathrm{d}z = O(n^{-a})$ *as* $n \to \infty$*. If* σ *is bounded uniformly away from zero, then*

$$\limsup_{\|x\|\to\infty} \frac{|u_t(x)|}{(\log\|x\|)^{1/4}} > 0 \qquad \textit{a.s.} \quad \textit{for all } t > 0. \tag{2.7}$$

If σ *is bounded uniformly away from zero and infinity, then*

$$0 < \limsup_{\|x\|\to\infty} \frac{|u_t(x)|}{(\log\|x\|)^{1/2}} < \infty \qquad \textit{a.s.} \quad \textit{for all } t > 0. \tag{2.8}$$

Remark 2.4. Our derivation of Theorem 2.3 will in fact yield a little more information. Namely, that the limsups in (2.7) and (2.8) are both bounded below by a constant $c(\varkappa) := c(t\,, \varkappa\,, f\,, d)$ which satisfies $\inf_{\varkappa\in(0,\varkappa_0)} c(\varkappa) > 0$ for all $\varkappa_0 > 0$; and the limsup in (2.8) is bounded above by a constant that does not depend on the viscosity coefficient $\varkappa$. $\square$

If $g_1, g_2, \ldots$ is a sequence of independent standard normal random variables, then it is well known that $\limsup_{n\to\infty}(2\log n)^{-1/2} g_n = 1$ a.s. Now choose and fix some $t > 0$. Because $\{u_t(x)\}_{x\in\mathbf{R}^d}$ is a centered Gaussian process when σ is a constant, the preceding theorem suggests that the asymptotic behavior of $x \mapsto u_t(x)$ is the same as in the case that σ is a constant; and that behavior is "Gaussian." This "Gaussian" property continues to hold if we replace $\dot{F}$ by space-time white noise—that is formally when $f = \delta_0$; see [10]. Next we exhibit "non Gaussian" behavior by considering the following special case of (SHE):

$$\frac{\partial}{\partial t} u_t(x) = \frac{\varkappa}{2}(\Delta u_t)(x) + u_t(x)\dot{F}_t(x). \tag{PAM}$$

This is the socalled "parabolic Anderson model," and arises in many different contexts in mathematics and theoretical physics [7, Introduction].

Theorem 2.5. *Consider* (PAM) *when* $\inf_{x\in\mathbf{R}^d} u_0(x) > 0$ *and* $f = h * \tilde{h}$ *for some nonnegative function* $h \in L^2(\mathbf{R}^d)$ *that satisfies the following for some* $a > 0$*:* $\int_{\|z\|>n}[h(z)]^2\,\mathrm{d}z = O(n^{-a})$ *as* $n \to \infty$*, Then for every* $t > 0$ *there exist positive and finite constants* $\underline{A}_t(\varkappa) := \underline{A}(t\,, \varkappa\,, d\,, f\,, a)$ *and* $\overline{A}_t = \overline{A}(t\,, d\,, f(0)\,, a)$ *such that with probability one*

$$\underline{A}_t(\varkappa) \leqslant \limsup_{\|x\|\to\infty} \frac{\log u_t(x)}{(\log\|x\|)^{1/2}} \leqslant \overline{A}_t. \tag{2.9}$$

Moreover: (i) There exists $\varkappa_0 := \varkappa_0(f\,, d) \in (0\,,\infty)$ *such that* $\inf_{\varkappa\in(0,\varkappa_0)} \underline{A}_t(\varkappa) > 0$ *for all* $t > 0$*; and (ii) If* $f(x) > 0$ *for all* $x \in \mathbf{R}^d$*, then* $\inf_{\varkappa\in(0,\varkappa_1)} \underline{A}_t(\varkappa) > 0$ *for all* $\varkappa_1 > 0$*.*

The conclusion of Theorem 2.5 is that, under the condition of that theorem, and if the viscosity coefficient $\varkappa$ is sufficiently small, then for all $t > 0$,

$$\frac{\underline{B}}{\varkappa^{2\psi-1}} \leqslant \limsup_{\|x\|\to\infty} \frac{\log u_t(x)}{(\log\|x\|)^{\psi}} \leqslant \frac{\overline{B}}{\varkappa^{2\psi-1}} \qquad \text{a.s.}, \tag{2.10}$$

with nontrivial constants $\underline{B}$ and $\overline{B}$ that depend on $(t\,, d\,, f)$—but *not* on $\varkappa$—and $\psi = 1/2$. Loosely speaking, the preceding and its proof together imply that

$$\sup_{\|x\|<R} u_t(x) \overset{(\log)}{\approx} \mathrm{e}^{\mathrm{const}\cdot(\log R)^{1/2}}, \tag{2.11}$$

for all $\varkappa$ small and R large. This informal assertion was mentioned earlier in Introduction.

In [10] we have proved that if $\dot{F}$ is replaced with space-time white noise—that is, loosely speaking, when $f = \delta_0$—then (2.10) holds with $\psi = 2/3$. That is,

$$\sup_{\|x\|<R} u_t(x) \overset{(\log)}{\approx} \mathrm{e}^{\mathrm{const}\cdot(\log R)^{2/3}/\varkappa^{1/3}}, \tag{2.12}$$

for all $\varkappa > 0$ and R large.

In some sense these two examples signify the extremes among all choices of possible correlations. One might wonder if there are other correlation models that interpolate between the mentioned cases of $\psi = 1/2$ and $\psi = 2/3$. Our next theorem shows that the answer is "yes for every $\psi \in (1/2\,, 2/3)$ when $d = 1$ and every $\psi \in (1/2\,, 1)$ when $d \geqslant 2$." However, our construction requires us to consider certain correlation functions f that have the form $h * \tilde{h}$ for some $h \in W^{1,2}_{loc}(\mathbf{R}^d) \setminus L^2(\mathbf{R}^d)$.

In fact, we choose and fix some number $\alpha \in (0\,, d)$, and consider correlation functions of the Riesz type; namely,

$$f(x) := \text{const} \cdot \|x\|^{-\alpha} \qquad \text{for all } x \in \mathbf{R}^d. \tag{2.13}$$

It is not hard to check that f is a correlation function that has the form $h * \tilde{h}$ for some $h \in W^{1,2}_{loc}(\mathbf{R}^d)$, and $h \notin L^2(\mathbf{R}^d)$; see also Example 3.2 below. Because the Fourier transform of f is proportional to $\|\xi\|^{-(d-\alpha)}$, (2.1) is equivalent to the condition that $0 < \alpha < \min(d\,, 2)$, and Dalang's theory [12] tells us that if $u_0 : \mathbf{R}^d \to \mathbf{R}$ is bounded and measurable, then (SHE) has a solution [that is also unique up to evanescence], provided that $0 < \alpha < \min(d\,, 2)$. Moreover, when σ is a constant, (SHE) has a solution if and only if $0 < \alpha < \min(d\,, 2)$.

Our next result describes the "non Gaussian" asymptotic behavior of the solution to the parabolic Anderson model (PAM) under these conditions.

Theorem 2.6. *Consider* (PAM) *when* $\inf_{x\in\mathbf{R}^d} u_0(x) > 0$*. If* $f(x) = \text{const} \cdot \|x\|^{-\alpha}$ *for some* $\alpha \in (0\,, d \wedge 2)$*, then for every* $t > 0$ *there exist positive and finite constants* $\underline{B}$ *and* $\overline{B}$*—both depending only on* $(t\,, d\,, \alpha)$*—such that* (2.10) *holds with* $\psi := 2/(4-\alpha)$*; that is, for all* $t > 0$*,*

$$\frac{\underline{B}}{\varkappa^{\alpha/(4-\alpha)}} \leqslant \limsup_{\|x\|\to\infty} \frac{\log u_t(x)}{(\log\|x\|)^{2/(4-\alpha)}} \leqslant \frac{\overline{B}}{\varkappa^{\alpha/(4-\alpha)}} \qquad a.s. \tag{2.14}$$

Remark 2.7. We mention here that the constants in the above theorems might depend on u_0 but *only* through $\inf_{x\in\mathbf{R}^d} u_0(x)$ and $\sup_{x\in\mathbf{R}^d} u_0(x)$. We will not keep track of this dependence. Our primary interest is the dependence on $\varkappa$. $\square$

An important step in our arguments is to show that if $x_1, \ldots, x_N$ are sufficiently spread out then typically $u_t(x_1), \ldots, u_t(x_N)$ are sufficiently close to being independent. This amounts to a sharp estimate for the socalled "correlation length." We estimate that, roughly using the arguments of [10], devised for the space-time white noise. Those arguments are in turn using several couplings [16, 24], which might be of some interest. We add that the presence of spatial correlations adds a number of subtle [but quite serious] technical problems to this program.

3 A coupling of the noise

3.1 A construction of the noise

Let $W := \{W_t(x)\}_{t\geqslant 0, x\in\mathbf{R}^d}$ denote $(d+1)$-parameter Brownian sheet. That is, W is a centered Gaussian random field with the following covariance structure: For all $s,t\geqslant 0$ and $x,y\in\mathbf{R}^d$,

$$\mathrm{Cov}\left(W_t(x)\,,W_s(y)\right) = (s\wedge t)\cdot\prod_{j=1}^d (|x_j|\wedge|y_j|)\mathbf{1}_{(0,\infty)}(x_jy_j). \tag{3.1}$$

Define $\mathcal{F}_t$ to be the sigma-algebra generated by all random variables of the form $W_s(x)$, as s ranges over $[0\,,t]$ and x over $\mathbf{R}^d$. As is standard in stochastic analysis, we may assume without loss of generality that $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfy the "usual conditions" of the general theory of stochastic processes [15, Chapter 4].

If $h\in L^2(\mathbf{R}^d)$, then we may consider the mean-zero Gaussian random field $\{(h*W_t)(x)\}_{t\geqslant 0, x\in\mathbf{R}^d}$ that is defined as the following Wiener integral:

$$(h*W_t)(x) := \int_{\mathbf{R}^d} h(x-z)\,W_t(\mathrm{d}z). \tag{3.2}$$

It is easy to see that the covariance function of this process is given by

$$\mathrm{Cov}\left((h*W_t)(x)\,,(h*W_s)(y)\right) = (s\wedge t)f(x-y), \tag{3.3}$$

where we recall, from the introduction, that $f := h*\tilde{h}$. In this way we can define an isonormal noise $F^{(h)}$ via the following: For every $\phi\in\mathcal{S}$ [the usual space of all test functions of rapid decrease],

$$F_t^{(h)}(\phi) := \int_{(0,t)\times\mathbf{R}^d} \phi(x)(h*\mathrm{d}W_s)(x)\,\mathrm{d}x \qquad (t>0). \tag{3.4}$$

It is easy to see that the following form of the stochastic Fubini theorem holds:

$$F_t^{(h)}(\phi) = \int_{(0,t)\times\mathbf{R}^d} (\phi*\tilde{h})(x)\,W(\mathrm{d}s\,\mathrm{d}x). \tag{3.5}$$

[Compute the $L^2(\mathrm{P})$-norm of the difference.] In particular, $\{F_t^{(h)}(\phi)\}_{t\geqslant 0}$ is a Brownian motion [for each fixed $\phi\in\mathcal{S}$], normalized so that

$$\mathrm{Var}\left(F_1^{(h)}(\phi)\right) = \int_{\mathbf{R}^d}\left|(\phi*\tilde{h})(x)\right|^2\mathrm{d}x = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^d}|\hat{\phi}(\xi)|^2\hat{f}(\xi)\,\mathrm{d}\xi. \tag{3.6}$$

[The second identity is a consequence of Plancherel's theorem, together with the fact that $|\hat{h}(\xi)|^2 = \hat{f}(\xi)$.]

3.2 An extension

Suppose $h \in L^2(\mathbf{R}^d)$, and that the underlying correlation function is described by $f := h * \tilde{h}$. Consider the following probability density function on $\mathbf{R}^d$:

$$\varrho(x) := \prod_{j=1}^{d} \left(\frac{1 - \cos x_j}{\pi x_j^2} \right) \qquad \text{for } x \in \mathbf{R}^d. \tag{3.7}$$

We may build an approximation $\{\varrho_n\}_{n \geqslant 1}$ to the identity as follows: For all real numbers $n \geqslant 1$ and for every $x \in \mathbf{R}^d$,

$$\varrho_n(x) := n^d \varrho(nx), \quad \text{so that} \quad \hat{\varrho}_n(\xi) = \prod_{j=1}^{d} \left(1 - \frac{|\xi_j|}{n} \right)^+, \tag{3.8}$$

for all $\xi \in \mathbf{R}^d$.

Lemma 3.1. *If $h \in L^2(\mathbf{R}^d)$, then for all $\phi \in \mathcal{S}$ and integers $n, m \geqslant 1$,*

$$\begin{aligned} &\mathrm{E}\left(\sup_{t \in (0,T)} \left| F_t^{(h*\varrho_{n+m})}(\phi) - F_t^{(h*\varrho_n)}(\phi) \right|^2 \right) \\ &\qquad\qquad \leqslant \frac{16 d^2 T}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\phi}(\xi)|^2 \left(1 \wedge \frac{\|\xi\|^2}{n^2} \right) \hat{f}(\xi)\, \mathrm{d}\xi. \end{aligned} \tag{3.9}$$

Proof. By the Wiener isometry and Doob's maximal inequality, the left-hand side of the preceding display is bounded above by $4TQ$, where

$$\begin{aligned} Q &:= \int_{\mathbf{R}^d} \left| \left(\phi * \widetilde{h * \varrho_{n+m}} \right)(x) - \left(\phi * \widetilde{h * \varrho_n} \right)(x) \right|^2 \mathrm{d}x \\ &= \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{\phi}(\xi)|^2 \left| \hat{\varrho}_{n+m}(\xi) - \hat{\varrho}_n(\xi) \right|^2 \hat{f}(\xi)\, \mathrm{d}\xi; \end{aligned} \tag{3.10}$$

we have appealed to the Plancherel's theorem, together with the fact that $\hat{f}(\xi) = |\hat{h}(\xi)|^2$. Because

$$0 \leqslant 1 - \hat{\varrho}_n(\xi) \leqslant 1 - \left(\left(1 - \frac{1}{n} \max_{1 \leqslant j \leqslant d} |\xi_j| \right)^+ \right)^d \leqslant \frac{d\|\xi\|}{n}, \tag{3.11}$$

it follows from the triangle inequality that $|\hat{\varrho}_{n+m}(\xi) - \hat{\varrho}_n(\xi)| \leqslant 2d\|\xi\|/n$. This implies the lemma, because we also have $|\hat{\varrho}_{n+m}(\xi) - \hat{\varrho}_n(\xi)| \leqslant \|\varrho_{n+m}\|_{L^1(\mathbf{R}^d)} + \|\varrho_n\|_{L^1(\mathbf{R}^d)} = 2 \leqslant 2d$. $\square$

Lemma 3.1 has the following consequence: Suppose $h \in W^{1,2}_{loc}(\mathbf{R}^d)$, and $f := h * \tilde{h}$ in the sense of generalized functions. Because $h \in W^{1,2}_{loc}(\mathbf{R}^d)$, the dominated convergence theorem tells us that

$$\lim_{n \to \infty} \int_{\mathbf{R}^d} |\hat{\phi}(\xi)|^2 \left(1 \wedge \frac{\|\xi\|^2}{n^2} \right) \hat{f}(\xi)\, \mathrm{d}\xi = 0 \quad \text{for all } \phi \in \mathcal{S}. \tag{3.12}$$

Consequently, $F_t^{(h)}(\phi) := \lim_{n\to\infty} F_t^{(h*\varrho_n)}(\phi)$ exists in $L^2(\mathrm{P})$, locally uniformly in t. Because $L^2(\mathrm{P})$-limits of centered Gaussian random fields are themselves Gaussian, it follows that $F^{(h)} := \{F_t^{(h)}(\phi)\}_{t\geqslant 0,\phi\in\mathcal{S}}$ is a centered Gaussian random field, and $\{F_t^{(h)}\}_{t\geqslant 0}$ is a Brownian motion scaled in order to satisfy (3.6). We mention also that, for these very reasons, $F^{(h)}$ satisfies (3.5) a.s. for all $t \geqslant 0$ and $\phi \in \mathcal{S}$. The following example shows that one can construct the Gaussian random field $F^{(h)}$ even when $h \in W^{1,2}_{loc}(\mathbf{R}^d)$ is not in $L^2(\mathbf{R}^d)$.

Example 3.2 (Riesz kernels)**.** We are interested in correlation functions of the Riesz type: $f(x) = c_0 \cdot \|x\|^{-\alpha}$, where $x \in \mathbf{R}^d$ [and of course $\alpha \in (0\,, d)$ so that f is locally integrable]. If is well known that $\hat{f}(\xi) = c_1 \cdot \|\xi\|^{-(d-\alpha)}$ for a positive and finite constant c_1 that depends only on $(d\,, \alpha\,, c_0)$. We may define $h \in L^1_{loc}(\mathbf{R}^d)$ via $\hat{h}(\xi) := c_1^{1/2} \cdot \|\xi\|^{-(d-\alpha)/2}$. It then follows that $f = h * \tilde{h}$; and it is clear from the fact that $\hat{f} = |\hat{h}|^2$ that $h \in W^{1,2}_{loc}(\mathbf{R}^d)$ if and only if $\int_{\|\xi\|<1} \|\xi\|^2 \hat{f}(\xi)\,\mathrm{d}\xi < \infty$, which is satisfied automatically because $\alpha \in (0\,, d)$. □

Of course, even more general Gaussian random fields can be constructed using only general theory. What is important for the sequel is that here we have constructed a random-field-valued *stochastic process* $(t\,, h) \mapsto F_t^{(h)}$; i.e., the random fields $\{F_t^{(h)}(\phi)\}_{\phi\in\mathcal{S}}$ are all coupled together as $(t\,, h)$ ranges over the index set $(0\,, \infty) \times W^{1,2}_{loc}(\mathbf{R}^d)$.

3.3 A coupling of stochastic convolutions

Suppose $Z := \{Z_t(x)\}_{t\geqslant 0, x\in\mathbf{R}^d}$ is a random field that is predictable with respect to the filtration $\mathcal{F}$, and satisfies the following for all $t > 0$ and $x \in \mathbf{R}^d$:

$$\int_0^t \mathrm{d}s \iint_{\mathbf{R}^d\times\mathbf{R}^d} \mathrm{d}y\,\mathrm{d}z\; p_{t-s}(y-x)p_{t-s}(z-x)\,|\mathrm{E}\,(Z_s(y)Z_s(z))|\, f(y-z) < \infty. \tag{3.13}$$

Then we may apply the theories of Walsh [28, Chapter 2] and Dalang [12] to the martingale measure $(t\,, A) \mapsto F_t^{(h)}(\mathbf{1}_A)$, and construct the stochastic convolution $p * Z\dot{F}^{(h)}$ as the random field

$$\left(p * Z\dot{F}^{(h)}\right)_t(x) := \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x)Z_s(y)\, F^{(h)}(\mathrm{d}s\,\mathrm{d}y). \tag{3.14}$$

Also, we have the following Itô-type isometry:

$$\begin{aligned}
&\mathrm{E}\left(\left|\int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x)Z_s(y)\, F^{(h)}(\mathrm{d}s\,\mathrm{d}y)\right|^2\right) \qquad (3.15)\\
&= \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y \int_{\mathbf{R}^d} \mathrm{d}z\; p_{t-s}(y-x)p_{t-s}(z-x)\mathrm{E}\,[Z_s(y)Z_s(z)]\, f(y-z).
\end{aligned}$$

If $h : \mathbf{R}^d \to \mathbf{R}_+$ is nonnegative and measurable, then we define, for all real numbers $n \geqslant 1$,

$$h_n(x) := h(x)\hat{\varrho}_n(x) \qquad \text{for every } x \in \mathbf{R}^d. \tag{3.16}$$

Some important features of this construction are that: (a) $0 \leqslant h_n \leqslant h$ pointwise; (b) $h_n \to h$ as $n \to \infty$, pointwise; (c) every h_n has compact support; and (d) if $h \in W^{1,2}_{loc}(\mathbf{R}^d)$, then $h_n \in W^{1,2}_{loc}(\mathbf{R}^d)$ for all $n \geqslant 1$.

For the final results of this section we consider only nonnegative functions $h \in L^2(\mathbf{R}^d)$ that satisfy the following [relatively mild] condition:

$$\sup_{r>0}\left[r^a \cdot \int_{\|x\|>r} [h(x)]^2 \, \mathrm{d}x\right] < \infty \qquad \text{for some } a > 0. \tag{3.17}$$

Lemma 3.3. *If $h \in L^2(\mathbf{R}^d)$ satisfies* (3.17)*, then there exists $b \in (0\,,2)$ such that*

$$\sup_{n\geqslant 1}\left[n^b \cdot \int_{\mathbf{R}^d}\left(1 \wedge \frac{\|x\|^2}{n^2}\right)[h(x)]^2 \, \mathrm{d}x\right] < \infty. \tag{3.18}$$

Proof. We may—and will—assume, without loss of generality, that (3.17) holds for some $a \in (0\,,2)$. Then, thanks to (3.17),

$$\begin{aligned}\int_{\|x\|\leqslant n} \frac{\|x\|^2}{n^2}[h(x)]^2 \, \mathrm{d}x &\leqslant \sum_{k=0}^{\infty} 4^{-k} \int_{2^{-k-1}n<\|x\|\leqslant 2^{-k}n} [h(x)]^2 \, \mathrm{d}x \\ &\leqslant \text{const} \cdot \sum_{k=0}^{\infty} 4^{-k}\left(2^{-k-1}n\right)^{-a},\end{aligned} \tag{3.19}$$

and this is $O(n^{-a})$ since $a \in (0\,,2)$. The lemma follows readily from this. □

Proposition 3.4. *If $h \in L^2(\mathbf{R}^d)$ is nonnegative and satisfies* (3.17)*, then for all predictable random fields that satisfy* (3.13)*, and for all $\delta > 1$, $x \in \mathbf{R}^d$, $n \geqslant 1$, and $k \geqslant 2$,*

$$\mathcal{M}_\delta^{(k)}\left(p * Z\dot{F}^{(h)} - p * Z\dot{F}^{(h_n)}\right) \leqslant C\sqrt{\frac{k}{n^b}}\ \mathcal{M}_\delta^{(k)}(Z) \tag{3.20}$$

for some positive constant C which does not depend on $\varkappa$, where b is the constant introduced in Lemma 3.3 and $\mathcal{M}_\delta^{(k)}$ is defined in (1.5).

Remark 3.5. This proposition has a similar appearance as Lemma 3.1. However, note that here we are concerned with correlations functions of the form $q * \tilde{q}$ where $q := h\hat{\varrho}_n$, whereas in Lemma 3.1 we were interested in $q = h * \varrho_n$. The methods of proof are quite different. □

Proof. The present proof follows closely renewal-theoretic ideas that were developed in [19]. Because we wish to appeal to the same method several more times in the sequel, we describe nearly all the details once, and then refer to the present discussion for details in later applications of this method.

Eq. (3.5) implies that $p * Z\dot{F}^{(h)} - p * Z\dot{F}^{(h_n)} = p * Z\dot{F}^{(D)}$ a.s., where $D := h - h_n = h(1 - \hat{\varrho}_n) \geqslant 0$. According to (BDG),

$$\begin{aligned} &\mathrm{E}\left(\left|\int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x) Z_s(y) F^{(D)}(\mathrm{d}s\,\mathrm{d}y)\right|^k\right) && (3.21)\\ &\leqslant \mathrm{E}\left(\left|4k\int_0^t \mathrm{d}s \iint_{\mathbf{R}^d\times\mathbf{R}^d} \mathrm{d}y\,\mathrm{d}z\ p_{t-s}(y-x)p_{t-s}(z-x)\mathcal{Z} f^{(D)}(y-z)\right|^{k/2}\right), \end{aligned}$$

where $\mathcal{Z} := |Z_s(y)Z_s(z)|$ and $f^{(D)} := D * \tilde{D}$; we observe that $f^{(D)} \geqslant 0$. The classical Minkowski inequality for integrals implies that $\| \int_{(0,t)\times\mathbf{R}^d\times\mathbf{R}^d}(\cdots)\|_{k/2} \leqslant \int_{(0,t)\times\mathbf{R}^d\times\mathbf{R}^d} \|\cdots\|_{k/2}$. Therefore, it follows that

$$\begin{aligned} &\mathrm{E}\left(\left|\int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x) Z_s(y) F^{(D)}(\mathrm{d}s\,\mathrm{d}y)\right|^k\right) && (3.22)\\ &\leqslant \left|4k\int_0^t \mathrm{d}s \iint_{\mathbf{R}^d\times\mathbf{R}^d} \mathrm{d}y\,\mathrm{d}z\ p_{t-s}(y-x)p_{t-s}(z-x) f^{(D)}(z-y)\|Z_s(y)Z_s(z)\|_{k/2}\right|^{k/2}. \end{aligned}$$

Young's inequality shows that the function $f^{(D)} = D * \tilde{D}$ is bounded uniformly from above by

$$\begin{aligned} \|D\|^2_{L^2(\mathbf{R}^d)} &= \|h(1-\hat{\varrho}_n)\|^2_{L^2(\mathbf{R}^d)} && (3.23)\\ &\leqslant \left(\frac{d}{n}\right)^2 \int_{|z|_\infty \leqslant n} [\|z\| h(z)]^2 dz + \int_{|z|_\infty > n} [h(z)]^2 dz = O(n^{-b}), \end{aligned}$$

where $|z|_\infty := \max_{1\leqslant j\leqslant n} |z_j|$; see also Lemma 3.3. Therefore

$$\begin{aligned} &\mathrm{E}\left(\left|\int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x) Z_s(y) F^{(D)}(\mathrm{d}s\,\mathrm{d}y)\right|^k\right) && (3.24)\\ &= O(n^{-bk/2})\left|k\int_0^t \mathrm{d}s \iint_{\mathbf{R}^d\times\mathbf{R}^d} \mathrm{d}y\,\mathrm{d}z\ p_{t-s}(y-x)p_{t-s}(z-x)\|Z_s(y)Z_s(z)\|_{k/2}\right|^{k/2}. \end{aligned}$$

According to the Cauchy–Schwarz inequality, $\|Z_s(y)Z_s(z)\|^{1/2}_{k/2}$ is bounded above by $\sup_{w\in\mathbf{R}^d} \|Z_s(w)\|_k \leqslant \mathrm{e}^{\delta s}\mathcal{M}_\delta^{(k)}(Z)$, and the proposition follows. □

4 Moment and tail estimates

In this section we state and prove a number of inequalities that will be needed subsequently. Our estimates are developed in different subsections for the different cases of interest [e.g., σ bounded, $\sigma(u) \propto u$, $f = h * \tilde{h}$ for $h \in L^2(\mathbf{R}^d)$, $f(x) \propto \|x\|^{-\alpha}$, etc.]. Although the techniques vary from one subsection to the next, the common theme of this section is that all bounds are ultimately derived by establishing moment inequalities of one sort or another.

4.1 An upper bound in the general $h \in L^2(\mathbf{R}^d)$ case

Proposition 4.1. *Let u denote the solution to* (SHE), *where $f := h * \tilde{h}$ for some nonnegative $h \in L^2(\mathbf{R}^d)$. Then, for all $t > 0$ there exists a positive and finite constant $\gamma = \gamma(d\,, f(0)\,, t)$—independent of $\varkappa$—such that for all $\lambda > \mathrm{e}$,*

$$\sup_{x \in \mathbf{R}^d} \mathrm{P}\left\{u_t(x) > \lambda\right\} \leqslant \gamma^{-1} \mathrm{e}^{-\gamma(\log \lambda)^2}. \tag{4.1}$$

Proof. Because $|(p_t * u_0)(x)| \leqslant \|u_0\|_{L^\infty(\mathbf{R}^d)}$ uniformly in $x \in \mathbf{R}^d$, we can appeal to (BDG) and (2.2) in order to obtain

$$\begin{aligned} \|u_t(x)\|_k &\leqslant \|u_0\|_{L^\infty(\mathbf{R}^d)} + \left\| \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-s)\sigma(u_s(y)) F^{(h)}(\mathrm{d}s\,\mathrm{d}y) \right\|_k \\ &\leqslant \|u_0\|_{L^\infty(\mathbf{R}^d)} + 2\sqrt{k} \left(\mathrm{E}\left[\left(\int_0^t \mathrm{d}s \iint\limits_{\mathbf{R}^d\times\mathbf{R}^d} \mathrm{d}y\,\mathrm{d}z\ \mathcal{Q} \right)^{k/2} \right] \right)^{1/k}, \end{aligned} \tag{4.2}$$

where $\mathcal{Q} := f(y-z)p_{t-s}(y-x)p_{t-s}(z-x)\sigma(u_s(y))\sigma(u_s(z))$; see the proof of Proposition 3.4 for more details on this method. Since $|\mathcal{Q}|$ is bounded above by $\mathcal{W} := f(0)p_{t-s}(y-x)p_{t-s}(z-x)|\sigma(u_s(y)) \cdot \sigma(u_s(z))|$ we find that

$$\|u_t(x)\|_k \leqslant \|u_0\|_{L^\infty(\mathbf{R}^d)} + \left(4k \int_0^t \mathrm{d}s \iint\limits_{\mathbf{R}^d\times\mathbf{R}^d} \mathrm{d}y\,\mathrm{d}z\ \|\mathcal{W}\|_{k/2} \right)^{1/2}, \tag{4.3}$$

Because $|\sigma(z)| \leqslant |\sigma(0)| + \mathrm{Lip}_\sigma|z|$ for all $z \in \mathbf{R}$, we may apply the Cauchy–Schwarz inequality to find that $\|u_t(x)\|_k$ is bounded above by

$$\begin{aligned} &\|u_0\|_{L^\infty(\mathbf{R}^d)} + \left(4k \cdot f(0) \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y\ p_{t-s}(y-x) \|\sigma(u_s(y))\|_k^2 \right)^{1/2} \\ &\leqslant \|u_0\|_{L^\infty(\mathbf{R}^d)} + \left(4k \cdot f(0) \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y\ p_{t-s}(y-x) \left[|\sigma(0)| + \mathrm{Lip}_\sigma \|u_s(y)\|_k\right]^2 \right)^{1/2}. \end{aligned} \tag{4.4}$$

We introduce a parameter $\delta > 0$ whose value will be chosen later on. It follows from the preceding and some algebra that

$$\|u_t(x)\|_k^2 \leqslant 2\|u_0\|_{L^\infty(\mathbf{R}^d)}^2 + 16kf(0)\left(|\sigma(0)|^2 t + \mathrm{Lip}_\sigma^2 \mathrm{e}^{2\delta t} \mathcal{A}\right), \tag{4.5}$$

where $\mathcal{A} := \int_0^t \mathrm{d}s\, \mathrm{e}^{-2\delta(t-s)} \int_{\mathbf{R}^d} \mathrm{d}y\ p_{t-s}(y-x)\mathrm{e}^{-2\delta s}\|u_s(y)\|_k^2$. Note that

$$\mathcal{A} \leqslant \int_0^t \mathrm{d}s\, \mathrm{e}^{-2\delta(t-s)} \int_{\mathbf{R}^d} \mathrm{d}y\ p_{t-s}(y-x) \left[\mathcal{M}_\delta^{(k)}(u)\right]^2 \leqslant \frac{1}{2\delta}\left[\mathcal{M}_\delta^{(k)}(u)\right]^2. \tag{4.6}$$

Therefore, for all $\delta > 0$ and $k \geqslant 2$, $[\mathcal{M}_\delta^{(k)}(u)]^2$ is bounded above by

$$2\|u_0\|_{L^\infty(\mathbf{R}^d)}^2 + 16kf(0)\left(|\sigma(0)|^2 \sup_{t\geqslant 0}\left[t\mathrm{e}^{-2\delta t}\right] + \frac{\mathrm{Lip}_\sigma^2}{2\delta}\left[\mathcal{M}_\delta^{(k)}(u)\right]^2\right). \tag{4.7}$$

Let us choose $\delta := \left(1 \vee 16f(0)\mathrm{Lip}_\sigma^2\right)k$ to find that $\mathcal{M}_\delta^{(k)}(u)^2 \leqslant (4\sup_{x\in\mathbf{R}^d} u_0(x)^2 + Ck)$ for some constant $C > 0$ that does not depend on k, and hence,

$$\sup_{x\in\mathbf{R}^d} \|u_t(x)\|_k \leqslant \mathrm{const}\cdot\sqrt{k}\,\mathrm{e}^{\left(1\vee 16f(0)\mathrm{Lip}_\sigma^2\right)kt}. \tag{4.8}$$

Lemma 3.4 of [10] then tells us that there exists $\gamma := \gamma(t) > 0$ sufficiently small [how small depends on t but not on $(\varkappa, x)$] such that $\mathrm{E}[\exp(\gamma(\log_+ u_t(x))^2)] < \infty$. Therefore, the proposition follows from Chebyshev's inequality. $\square$

4.2 Lower bounds for $h \in L^2(\mathbf{R}^d)$ when σ is bounded

Lemma 4.2. *Let u denote the solution to* (SHE)*, where σ is assumed to be bounded uniformly away from zero and infinity and $\inf_{x\in\mathbf{R}^d} u_0(x) > 0$. If $f = h * \tilde{h}$ for some nonnegative $h \in L^2(\mathbf{R}^d)$, then for all $t > 0$ there exist positive and finite constants $c_1 = c_1(\varkappa, t, d, f)$ and $c_2 = c_2(t, d, f)$—independent of $\varkappa$—such that uniformly for all $\lambda > \mathrm{e}$,*

$$c_1^{-1}\mathrm{e}^{-c_1\lambda^2} \leqslant \inf_{x\in\mathbf{R}^d} \mathrm{P}\left\{|u_t(x)| > \lambda\right\} \leqslant \sup_{x\in\mathbf{R}^d} \mathrm{P}\left\{|u_t(x)| > \lambda\right\} \leqslant c_2^{-1}\mathrm{e}^{-c_2\lambda^2}. \tag{4.9}$$

Furthermore, $\sup_{\varkappa\in(0,\varkappa_0)} c_1(\varkappa) < \infty$ for all $\varkappa_0 < \infty$.

Proof. Choose and fix an arbitrary $\tau > 0$, and consider the continuous $L^2(\mathrm{P})$ martingale $\{M_t\}_{t\in[0,\tau]}$ defined by

$$M_t := (p_\tau * u_0)(x) + \int_{(0,t)\times\mathbf{R}^d} p_{\tau-s}(y-x)\sigma(u_s(y))\, F^{(h)}(\mathrm{d}s\,\mathrm{d}y), \tag{4.10}$$

as t ranges within $(0, \tau)$. By Itô's formula, for all even integers $k \geqslant 2$,

$$M_t^k = (p_\tau * u_0)(x)^k + k\int_0^t M_s^{k-1}\,\mathrm{d}M_s + \binom{k}{2}\int_0^t M_s^{k-2}\,\mathrm{d}\langle M\rangle_s. \tag{4.11}$$

The final integral that involves quadratic variation can be written as

$$\int_0^t M_s^{k-2}\left[\int_{\mathbf{R}^d}\mathrm{d}y\int_{\mathbf{R}^d}\mathrm{d}z\ p_{\tau-s}(y-x)p_{\tau-s}(z-x)f(z-y)\mathcal{Z}\right]\mathrm{d}s, \tag{4.12}$$

where $\mathcal{Z} := \sigma(u_s(y))\sigma(u_s(z)) \geqslant \epsilon_0^2$ for some $\epsilon_0 > 0$. This is because σ is uniformly bounded away from 0. Thus, the last integral in (4.11) is bounded below by

$$\begin{aligned}\epsilon_0^2 \int_0^t M_s^{k-2} \left[\int_{\mathbf{R}^d} \mathrm{d}y \int_{\mathbf{R}^d} \mathrm{d}z \; p_{\tau-s}(y-x) p_{\tau-s}(z-x) f(z-y)\right] \mathrm{d}s \\ = \epsilon_0^2 \int_0^t M_s^{k-2} \left(p_{\tau-s}\,, p_{\tau-s} * f\right)_{L^2(\mathbf{R}^d)} \mathrm{d}s,\end{aligned} \tag{4.13}$$

where $\langle a\,, b\rangle_{L^2(\mathbf{R}^d)} := \int_{\mathbf{R}^d} a(x)b(x)\,\mathrm{d}x$ denotes the usual inner product on $L^2(\mathbf{R}^d)$. This leads us to the recursive inequality,

$$\mathrm{E}(M_t^k) \geqslant \left(\inf_{x\in\mathbf{R}^d} u_0(x)\right)^k + \binom{k}{2}\epsilon_0^2 \cdot \int_0^t \mathrm{E}(M_s^{k-2}) \langle p_{\tau-s}\,, p_{\tau-s} * f\rangle_{L^2(\mathbf{R}^d)}\, \mathrm{d}s. \tag{4.14}$$

Next, consider the Gaussian process $\{\zeta_t\}_{t\geqslant 0}$ defined by

$$\zeta_t := \epsilon_0 \int_{(0,t)\times\mathbf{R}^d} p_{\tau-s}(y-x)\, F^{(h)}(\mathrm{d}s\,\mathrm{d}y) \quad (0 < t < \tau). \tag{4.15}$$

We may iterate, as was done in [10, proof of Proposition 3.6], in order to find that

$$\mathrm{E}(M_t^k) \geqslant \mathrm{E}\left(\left[\inf_{x\in\mathbf{R}^d} u_0(x) + \zeta_t\right]^k\right) \geqslant \mathrm{E}\left(\zeta_t^k\right) \geqslant \left(\mathrm{const} \cdot k\, \mathrm{E}\left[\zeta_t^2\right]\right)^{k/2}. \tag{4.16}$$

Now $\mathrm{E}(\zeta_t^2) = \epsilon_0^2 \int_0^t \langle p_{\tau-s}\,, p_{\tau-s} * f\rangle_{L^2(\mathbf{R}^d)} \mathrm{d}s$. Since $p_{\tau-s} \in \mathcal{S}$ for all $s \in (0\,,\tau)$, Parseval's identity applies, and it follows that

$$\langle p_{\tau-s}\,, p_{\tau-s} * f\rangle_{L^2(\mathbf{R}^d)} = \frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} \hat{f}(\xi) \mathrm{e}^{-\varkappa(\tau-s)\|\xi\|^2}\, \mathrm{d}\xi. \tag{4.17}$$

Therefore,

$$\begin{aligned}\mathrm{E}(\zeta_\tau^2) &= \frac{\epsilon_0^2}{(2\pi)^d} \int_{\mathbf{R}^d} \hat{f}(\xi) \left[\frac{1-\mathrm{e}^{-\varkappa\tau\|\xi\|^2}}{\varkappa\|\xi\|^2}\right] \mathrm{d}\xi \\ &\geqslant \frac{\epsilon_0^2}{2(2\pi)^d} \int_{\mathbf{R}^d} \frac{\hat{f}(\xi)}{\tau^{-1} + \varkappa\|\xi\|^2}\, \mathrm{d}\xi.\end{aligned} \tag{4.18}$$

This requires only the elementary bound $(1-\mathrm{e}^{-z})/z \geqslant (2(1+z))^{-1}$, valid for all $z > 0$. Since $M_t = u_t(x)$ when $t = \tau$, it follows that

$$c(\varkappa)\sqrt{k} \leqslant \inf_{x\in\mathbf{R}^d} \|u_t(x)\|_k, \tag{4.19}$$

for all $k \geqslant 2$, where $c(\varkappa) = c(t\,,\varkappa\,,f\,,d)$ is positive and finite, and has the additional property that

$$\inf_{\varkappa\in(0,\varkappa_0)} c(\varkappa) > 0 \qquad \text{for all } \varkappa_0 > 0. \tag{4.20}$$

Similar arguments reveal that

$$\sup_{x\in\mathbf{R}^d} \|u_t(x)\|_k \leqslant c'\sqrt{k}, \tag{4.21}$$

for all $k \geqslant 2$, where c' is a positive and finite constant that depends only on $(t\,, f\,, d)$. The result follows from the preceding two moment estimates (see [10] for details). $\square$

Lemma 4.3. *Let u denote the solution to* (SHE)*, where σ is assumed to be bounded uniformly away from zero and* $\inf_{x\in\mathbf{R}^d} u_0(x) > 0$. *If $f = h * \tilde{h}$ for some nonnegative $h \in L^2(\mathbf{R}^d)$, then for all $t > 0$ there exists a positive and finite constant $a(\varkappa) := a(\varkappa\,, t\,, d\,, f)$ such that uniformly for every $\lambda > \mathrm{e}$,*

$$\mathrm{P}\{|u_t(x)| \geqslant \lambda\} \geqslant \frac{\exp\left(-a(\varkappa)\lambda^4\right)}{\sqrt{a(\varkappa)}}. \tag{4.22}$$

Furthermore, $\sup_{\varkappa\in(0,\varkappa_0)} a(\varkappa) < \infty$ *for all $\varkappa_0 > 0$.*

Proof. The proof of this proposition is similar to the proof of Proposition 3.7 in the companion paper [10], and uses the following elementary fact [called the "Paley–Zygmund inequality"]: If $Z \in L^2(\mathrm{P})$ is nonnegative and $\epsilon \in (0\,,1)$, then

$$\mathrm{P}\left\{Z > (1-\epsilon)\mathrm{E}Z\right\} \geqslant \frac{(\epsilon \mathrm{E}Z)^2}{\mathrm{E}(Z^2)}. \tag{4.23}$$

This is a ready consequence of the Cauchy–Schwarz inequality.

Note, first, that the moment bound (4.19) continues to hold for a constant $c(\varkappa) = c(t\,,\varkappa\,,f\,,d)$ that satisfies (4.20). We can no longer apply (4.21), however, since that inequality used the condition that σ is bounded above; a property that need not hold in the present setting. Fortunately, the general estimate (4.8) is valid with "const" not depending on $\varkappa$. Therefore, we appeal to the Paley–Zygmund inequality (4.23) to see that

$$\mathrm{P}\left\{|u_t(x)| \geqslant \frac{1}{2}\|u_t(x)\|_{2k}\right\} \geqslant \frac{\left[\mathrm{E}\left(|u_t(x)|^{2k}\right)\right]^2}{4\mathrm{E}\left(|u_t(x)|^{4k}\right)} \geqslant \mathrm{const}\cdot[c(\varkappa)]^2\mathrm{e}^{-Ck^2}, \tag{4.24}$$

as $k \to \infty$, where $C \in (0\,,\infty)$ does not depend on $(k\,,\varkappa)$. Since $\|u_t(x)\|_{2k} \geqslant c(\varkappa)\cdot\sqrt{2k}$, it follows that $\mathrm{P}\{|u_t(x)| \geqslant c(\varkappa)\cdot\sqrt{k/2}\} \geqslant \exp(-C'k^2)$ as $k \to \infty$ for some C' which depends only on t. We obtain the proposition by considering λ between $c(\varkappa)\cdot\sqrt{k/2}$ and $c(\varkappa)\cdot\sqrt{(k+1)/2}$. $\square$

4.3 A lower bound for the parabolic Anderson model for $h \in L^2(\mathbf{R}^d)$

Throughout this subsection we consider u to be the solution to the parabolic Anderson model (PAM) in the case that $\inf_{x\in\mathbf{R}^d} u_0(x) > 0$.

Proposition 4.4. *There exists a constant $\Lambda_d \in (0\,,\infty)$—depending only on d—such that for all $t, \varkappa > 0$ and $k \geqslant 2$,*

$$\left[\inf_{x\in\mathbf{R}^d} u_0(x)\right]^k \mathrm{e}^{\Lambda_d a_t k^2} \leqslant \mathrm{E}\left(|u_t(x)|^k\right) \leqslant \left[\sup_{x\in\mathbf{R}^d} u_0(x)\right]^k \mathrm{e}^{tf(0)k^2}, \tag{4.25}$$

where $a_t = a_t(f\,,\varkappa) > 0$ for all $t, \varkappa > 0$, and is defined by

$$a_t := \sup_{\delta>0}\left[\frac{\delta^2}{4\varkappa}\left(1 \wedge \frac{4\varkappa t}{\delta^2}\right)\inf_{x\in B(0,\delta)} f(x)\right]. \tag{4.26}$$

This proves, in particular, that the exponent estimate $\left(1 \vee 16f(0)\mathrm{Lip}_\sigma^2\right) k^2 t$, derived more generally in (4.8), is sharp—up to a constant—as a function of k.

The proof of Proposition 4.4 hinges on the following, which by itself is a ready consequence of a moment formula of Conus [9]; see also [3, 21] for related results and special cases.

Lemma 4.5 ([9])**.** *For all $t > 0$, and $x \in \mathbf{R}^d$, we have the following inequalities*

$$\begin{aligned}
\mathrm{E}\left(|u_t(x)|^k\right) &\geqslant \left[\inf_{x\in\mathbf{R}^d} u_0(x)\right]^k \cdot \mathrm{E}\exp\left(\sum\sum_{1\leqslant i\neq j\leqslant k}\int_0^t f\left(\sqrt{\varkappa}\left[b_r^{(i)} - b_r^{(j)}\right]\right)\mathrm{d}r\right),\\
\mathrm{E}\left(|u_t(x)|^k\right) &\leqslant \left[\sup_{x\in\mathbf{R}^d} u_0(x)\right]^k \cdot \mathrm{E}\exp\left(\sum\sum_{1\leqslant i\neq j\leqslant k}\int_0^t f\left(\sqrt{\varkappa}\left[b_r^{(i)} - b_r^{(j)}\right]\right)\mathrm{d}r\right),
\end{aligned} \tag{4.27}$$

where $b^{(1)}, b^{(2)}, \ldots$ denote independent standard Brownian motions in $\mathbf{R}^d$.

Proof of Proposition 4.4. The upper bound for $\mathrm{E}(|u_t(x)|^k)$ follows readily from Lemma 4.5 and the basic fact that f is maximized at the origin.

In order to establish the lower bound recall that f is continuous and $f(0) > 0$. Because $f(x) \geqslant q\mathbf{1}_{B(0,\delta)}(x)$ for all $\delta > 0$, with $q = q(\delta) := \inf_{x\in B(0,\delta)} f(x)$, it follows that if $b^{(1)}, \ldots, b^{(k)}$ are independent d-dimensional Brownian motions, then

$$\begin{aligned}
&\sum\sum_{1\leqslant i\neq j\leqslant k}\int_0^t f\left(\sqrt{\varkappa}\left[b_r^{(i)} - b_r^{(j)}\right]\right)\mathrm{d}r\\
&\qquad\geqslant q\sum\sum_{1\leqslant i\neq j\leqslant k}\int_0^t \mathbf{1}_{B(0,\delta/\sqrt{\varkappa})}\left(b_r^{(i)} - b_r^{(j)}\right)\mathrm{d}r\\
&\qquad\geqslant q\sum\sum_{1\leqslant\neq j\leqslant k}\int_0^t \mathbf{1}_{B(0,\delta/(2\sqrt{\varkappa}))}(b_r^{(i)})\mathbf{1}_{B(0,\delta/2\sqrt{\varkappa})}(b_r^{(j)})\,\mathrm{d}r.
\end{aligned} \tag{4.28}$$

Recall Jensen's inequality,

$$\mathrm{E}(\mathrm{e}^Z) \geqslant \mathrm{e}^{\mathrm{E}Z}, \tag{4.29}$$

valid for all nonnegative random variables Z. Because of (4.29), Lemma 4.5 and the preceding, we can conclude that

$$\begin{aligned}\mathrm{E}\left(|u_t(x)|^k\right) &\geqslant I^k \cdot \mathrm{E}\exp\left(q\sum\sum_{1\leqslant i\neq j\leqslant k}\int_0^t \mathbf{1}_{B(0,\delta/(2\sqrt{\varkappa}))}(b_r^{(i)})\mathbf{1}_{B(0,\delta/2\sqrt{\varkappa})}(b_r^{(j)})\,\mathrm{d}r\right)\\ &= I^k\cdot\exp\left(qk(k-1)\cdot\int_0^t\left[G\left(\frac{\delta}{2\sqrt{\varkappa}\sqrt{r}}\right)\right]^2\mathrm{d}r\right),\end{aligned} \tag{4.30}$$

where $I := \inf u_0$ and $G(z) := (2\pi)^{-d/2}\int_{\|x\|\leqslant z}\mathrm{e}^{-\|x\|^2/2}\,\mathrm{d}x$ for all $z > 0$. Because $k(k-1) \geqslant k^2/4$ for all $k \geqslant 2$, and we find that $\mathrm{E}(|u_t(x)|^k) \geqslant I^k\cdot\exp(A_\delta k^2)$, where A_δ is defined as

$$\frac{q}{4}\int_0^t\left[G\left(\frac{\delta}{2\sqrt{\varkappa}\sqrt{r}}\right)\right]^2\mathrm{d}r = \inf_{x\in B(0,\delta)} f(x)\cdot\int_0^t\left[\frac{1}{2}G\left(\frac{\delta}{2\sqrt{\varkappa}\sqrt{r}}\right)\right]^2\mathrm{d}r. \tag{4.31}$$

Finally, we observe that

$$0 < \tilde{\Lambda}_d := \inf_{z>0}\left[\frac{\frac{1}{2}G(z)}{1\wedge z^d}\right]^{1/2} < \infty. \tag{4.32}$$

A few lines of computation yield the bound, $\sup_{\delta>0} A_\delta \geqslant \tilde{\Lambda}_d a_t$. The lemma follows from this by readjusting and relabeling the constants. □

5 Localization when $h \in L^2(\mathbf{R}^d)$ satisfies (3.17)

Throughout this section we assume that $h \in L^2(\mathbf{R}^d)$ is nonnegative and satisfies condition (3.17). Moreover, we let u denote the solution to (SHE).

In order to simplify the notation we define, for every $x := (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$ and $a \in \mathbf{R}_+$,

$$[x-a\,, x+a] := [x_1-a\,, x_1+a]\times\cdots\times[x_d-a\,, x_d+a]. \tag{5.1}$$

That is, $[x-a\,, x+a]$ denotes the ℓ^∞ ball of radius a around x.

Given an arbitrary $\beta > 0$, define $U^{(\beta)}$ to be the solution to the random integral equation

$$\begin{aligned}&U_t^{(\beta)}(x)\\ &= (p_t * u_0)(x) + \int_{(0,t)\times[x-\beta\sqrt{t},x+\beta\sqrt{t}]} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right)F^{(h_\beta)}(\mathrm{d}s\,\mathrm{d}y),\end{aligned} \tag{5.2}$$

where h_β is defined in (3.16). A comparison with the mild form (2.2) of the solution to (SHE) shows that $U^{(\beta)}$ is a kind of "localized" version of u. Our goal is to prove that if β is sufficiently large, then $U_t^{(\beta)}(x) \approx u_t(x)$.

The method of Dalang [12] can be used to prove that the predictable random field $U^{(\beta)}$ exists, is unique up to a modification, and satisfies the estimate $\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}^d}\mathrm{E}(|U_t^{(\beta)}(x)|^k) < \infty$ for every $T > 0$ and $k \geqslant 2$. Furthermore, the method of Foondun and Khoshnevisan [19] shows that, in fact $U^{(\beta)}$ satisfies a similar bound as does u in (4.8). Namely, there exists a constant $D_1 \in (0\,,\infty)$—depending on σ and t—such that for all $t > 0$ and $k \geqslant 2$,

$$\sup_{\beta>0}\sup_{x\in\mathbf{R}^d}\mathrm{E}\left(|U_t^{(\beta)}(x)|^k\right) \leqslant D_1\mathrm{e}^{D_1k^2t}. \tag{5.3}$$

We skip the details of the proofs of these facts, as they require only simple modifications to the methods of [12, 19].

Remark 5.1. We emphasize that D_1 depends only on $(t\,, f(0)\,, d, \sigma)$. In particular, it can be chosen to be independent of $\varkappa$. In fact, D_1 has exactly the same parameter dependencies as the upper bound for the moment estimate in (4.8); and the two assertions holds for very much the same reasons. □

Lemma 5.2. *For every $T > 0$ there exists finite and positive constants G_* and F_*—depending only on $(T\,, f(0)\,, d\,, \varkappa\,, b\,, \sigma)$—such that for sufficiently large $\beta > 0$ and $k \geqslant 1$,*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}^d}\mathrm{E}\left(\left|u_t(x) - U_t^{(\beta)}(x)\right|^k\right) \leqslant \frac{G_*^k k^{k/2}\exp(F_*k^2)}{\beta^{kb/2}}, \tag{5.4}$$

where $b \in (0\,,2)$ was introduced in Lemma 3.3.

Proof. By the triangle inequality,

$$\begin{aligned}
&\left\|u_t(x) - U_t^{(\beta)}(x)\right\|_k \qquad (5.5)\\
&\leqslant \left\|u_t(x) - V_t^{(\beta)}(x)\right\|_k + \left\|V_t^{(\beta)}(x) - Y_t^{(\beta)}(x)\right\|_k + \left\|Y_t^{(\beta)}(x) - U_t^{(\beta)}(x)\right\|_k,
\end{aligned}$$

where

$$V_t^{(\beta)}(x) := (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right)F^{(h)}(\mathrm{d}s\,\mathrm{d}y), \tag{5.6}$$

and

$$Y_t^{(\beta)}(x) := (p_t * u_0)(x) + \int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right)F^{(h_\beta)}(\mathrm{d}s\,\mathrm{d}y). \tag{5.7}$$

In accord with (3.24) and (5.3),

$$\|V^{(\beta)} - Y^{(\beta)}\|_k \leqslant \text{const}\cdot\sqrt{\frac{kt}{\beta^b}}\mathrm{e}^{D_1tk} \tag{5.8}$$

where we remind that D_1 is a constant that does not depend on $\varkappa$. Next we bound the quantity $\|Y^{(\beta)} - U^{(\beta)}\|_k$, using the Burkholder–Davis–Gundy inequality, (BDG) and obtain the following:

$$\begin{aligned}
&\left\|Y_t^{(\beta)}(x) - U_t^{(\beta)}(x)\right\|_k \\
&= \left\|\int_{(0,t)\times[x-\beta\sqrt{t},x+\beta\sqrt{t}]^c} p_{t-s}(y-x)\sigma\left(U_s^{(\beta)}(y)\right) F^{(h_\beta)}(\mathrm{d}s\,\mathrm{d}y)\right\|_k \\
&\leqslant \text{const}\cdot\sqrt{kf(0)}\left(\int_0^t \mathrm{d}s\int_{[x-\beta\sqrt{t},x+\beta\sqrt{t}]^c}\mathrm{d}y\int_{[x-\beta\sqrt{t},x+\beta\sqrt{t}]^c}\mathrm{d}z\ \mathcal{W}\right)^{1/2},
\end{aligned}\tag{5.9}$$

where

$$\mathcal{W} := p_{t-s}(y-x)p_{t-s}(z-x)\left(1+\left\|U_s^{(\beta)}(y)\right\|_k\right)\left(1+\left\|U_s^{(\beta)}(z)\right\|_k\right).\tag{5.10}$$

Therefore, (5.3) implies that

$$\left\|Y_t^{(\beta)}(x) - U_t^{(\beta)}(x)\right\|_k \leqslant D_2\mathrm{e}^{D_2tk}\sqrt{kf(0)\cdot\tilde{\mathcal{W}}},\tag{5.11}$$

where $D_2\in(0\,,\infty)$ depends only on d, $f(0)$, and t, and

$$\tilde{\mathcal{W}} := \int_0^t \mathrm{d}s\left(\int_{[x-\beta\sqrt{t},\,x+\beta\sqrt{t}]^c}\mathrm{d}y\ p_{t-s}(y-x)\right)^2.\tag{5.12}$$

Before we proceed further, let us note that

$$\int_{\substack{z\in\mathbf{R}:\\|z|>\beta\sqrt{t}}}\frac{\mathrm{e}^{-z^2/(2\varkappa(t-s))}}{\sqrt{2\pi\varkappa(t-s)}}\,\mathrm{d}z \leqslant 2\cdot\exp\left(-\frac{\beta^2t}{4\varkappa(t-s)}\right).\tag{5.13}$$

Using the above in (5.11), we obtain

$$\left\|Y_t^{(\beta)}(x) - U_t^{(\beta)}(x)\right\|_k \leqslant 2D_2\mathrm{e}^{D_2tk}\sqrt{ktf(0)}\exp\left(-\frac{d\beta^2}{4\varkappa}\right).\tag{5.14}$$

Next we estimate $\|u_t(x) - V_t^{(\beta)}(x)\|_k$. An application of (BDG) yields

$$\begin{aligned}
&\left\|u_t(x) - V_t^{(\beta)}(x)\right\|_k \\
&\leqslant \left\|\int_{(0,t)\times\mathbf{R}^d} p_{t-s}(y-x)\left\{\sigma(u_s(y)) - \sigma(U_s^{(\beta)}(y))\right\}F^{(h)}(\mathrm{d}s\,\mathrm{d}y)\right\|_k \\
&\leqslant 2\sqrt{k}\left\|\int_0^t\mathrm{d}s\int_{\mathbf{R}^d}\mathrm{d}y\int_{\mathbf{R}^d}\mathrm{d}z\ f(y-z)p_{t-s}(y-x)p_{t-s}(z-x)\mathcal{Q}\right\|_{k/2},
\end{aligned}\tag{5.15}$$

where $\mathcal{Q} := |\sigma(u_s(y)) - \sigma(U_s^{(\beta)}(y))| \cdot |\sigma(u_s(z)) - \sigma(U_s^{(\beta)}(z))|$. Since σ is Lipschitz continuous, it follows from Minkowski's inequality that

$$\left\| u_t(x) - V_t^{(\beta)}(x) \right\|_k^2 \leqslant 4\mathrm{Lip}_\sigma^2 k f(0) \int_0^t \mathcal{Q}_s^* \, \mathrm{d}s, \tag{5.16}$$

where $\mathcal{Q}_s^* := \sup_{y\in\mathbf{R}^d} \|u_s(y) - U_s^{(\beta)}(y)\|_k^2$. Equations (5.5), (5.8) and (5.14) together imply that $\mathcal{Q}_t^* \leqslant \mathrm{const}\cdot kt\beta^{-b}\mathrm{e}^{\mathrm{const}\cdot kt} + \mathrm{const}\cdot kf(0)\cdot\int_0^t \mathcal{Q}_s^* \, \mathrm{d}s$. Therefore,

$$\mathcal{Q}_t^* \leqslant \mathrm{const} \cdot \left(\frac{tk\mathrm{e}^{\mathrm{const}\cdot kt}}{\beta^b} \right) \qquad \text{for all } t > 0, \tag{5.17}$$

owing to Gronwall's inequality. Because "const" does not depend on $(k\,,t)$, we take both sides to the power $k/2$ in order to finish the proof. $\square$

Now, let us define $U_t^{(\beta,n)}$ to be the nth Picard-iteration approximation of $U_t^{(\beta)}(x)$. That is , $U_t^{(\beta,0)}(x) := u_0(x)$, and for all $l \geqslant 0$,

$$\begin{aligned} &U_t^{(\beta,l+1)}(x) \\ &:= (p_t * u_0)(x) + \int_{(0,t)\times[x-\beta\sqrt{t},x+\beta\sqrt{t}]} p_{t-s}(y-x)\sigma\left(U_s^{(\beta,l)}(y)\right) F^{(h_\beta)}(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{5.18}$$

Lemma 5.3. *For every $T > 0$ there exists finite and positive constants G and F—depending only on $(T\,,f(0)\,,d\,,\varkappa\,,b\,,\sigma)$—such that for sufficiently large $\beta > 0$ and $k \geqslant 1$,*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbf{R}^d} \mathrm{E}\left(\left| u_t(x) - U_t^{(\beta,[\log\beta]+1)}(x) \right|^k \right) \leqslant \frac{G^k k^{k/2} \exp(Fk^2)}{\beta^{kb/2}}, \tag{5.19}$$

where $b \in (0\,,2)$ was introduced in Lemma 3.3.

Proof. The method of Foondun and Khoshnevisan [17] can be used to show that if $\delta := D'k$ for a sufficiently-large positive and finite constant D', then

$$\mathcal{M}_\delta^{(k)}\left(U^{(\beta)} - U^{(\beta,n)} \right) \leqslant \mathrm{const}\cdot\mathrm{e}^{-n} \qquad \text{for all } n \geqslant 0 \text{ and } k \in [2\,,\infty). \tag{5.20}$$

To elaborate, we replace the u^n of Ref. [17, (5.36)] by our $U^{(\beta,n)}$ and obtain

$$\|U^{(\beta,n+1)} - U^{(\beta,n)}\|_{k,\theta} \leqslant \|U^{(\beta,n)} - U^{(\beta,n-1)}\|_{k,\theta} \cdot Q(k\,,\theta), \tag{5.21}$$

where $\|X\|_{k,\theta} := \{\sup_{t\geqslant 0}\sup_{x\in\mathbf{R}} \mathrm{e}^{-\theta t}\mathrm{E}(|X_t(x)|^k)\}^{1/k} = \mathcal{M}_{\theta/k}^{(k)}(X)$, for all random fields $\{X_t(x)\}_{t>0,x\in\mathbf{R}^d}$, and $Q(k\,,\theta)$ is defined in Theorem 1.3 of [17]. We recall from [17] that $Q(k\,,\theta)$ satisfies the following bounds:

$$Q(k\,,\theta) \leqslant \sqrt{4k\mathrm{Lip}_\sigma^2 \cdot \Upsilon\left(\frac{2\theta}{k}\right)} \leqslant \mathrm{const}\cdot\frac{k\|h\|_{L^2(\mathbf{R}^d)}}{\theta^{1/2}}. \tag{5.22}$$

[The function Υ is defined in [17, (1.8)].] Therefore, it follows readily from these bounds that if $\theta := D''k^2$ for a large enough $D''>0$, then

$$\|U^{(\beta,n+1)} - U^{(\beta,n)}\|_{k,\theta} \leqslant \mathrm{e}^{-1}\|U^{(\beta,n)} - U^{(\beta,n-1)}\|_{k,\theta}. \tag{5.23}$$

We obtain (5.20) from this inequality.

Finally we set $n := [\log\beta]+1$ and apply the preceding together with Lemma 5.2 to finish the proof. □

For every $x,\, y \in \mathbf{R}^d$, let us define

$$D(x\,,y) := \min_{1\leqslant l\leqslant d} |x_l - y_l|. \tag{5.24}$$

Lemma 5.4. *Choose and fix $\beta \geqslant 1$, $t>0$ and let $n := [\log\beta]+1$. Also fix $x^{(1)}, x^{(2)}, \dots \in \mathbf{R}^d$ such that $D(x^{(i)}\,,x^{(j)}) \geqslant 2n\beta(1+\sqrt{t})$. Then $\{U_t^{(\beta,n)}(x^{(j)})\}_{j\in\mathbf{Z}}$ are independent random variables.*

Proof. The lemma follows from the recursive definition of the $U^{(\beta,n)}$'s. Indeed, $U_t^{(\beta,n)}(x)$ depends on $U_s^{(\beta,n-1)}(y)$, $y \in [x-\beta\sqrt{t}\,, x+\beta\sqrt{t}]$, $s\in[0,t]$. An induction argument shows that $U_t^{(\beta,n)}(x)$ depends only on the values of $U_s^{(\beta,1)}(y)$, as y varies in $[x-(n-1)\beta\sqrt{t}\,, x+(n-1)\beta\sqrt{t}]$ and s in $[0\,,t]$.

Finally, we observe that $\{U_s^{(\beta,1)}(x)\}_{s\in[0,t],\, x\in\mathbf{R}^d}$ is a Gaussian random field that has the property that $U_s^{(\beta,1)}(x)$ and $U_s^{(\beta,1)}(x')$ are independent whenever $D(x\,,x') \geqslant 2\beta(1+\sqrt{t})$. [This assertion follows from a direct covariance calculation in conjunction with the fact that $(h_\beta * \tilde{h}_\beta)(z) = 0$ when $D(0,z)\geqslant 2\beta$]. □

6 Proof of Theorem 2.1

In this section we prove our first main theorem (Theorem 2.1). It is our first proof primarily because the following derivation is the least technical and requires that we keep track of very few parameter dependencies in our inequalities.

Define for all $k\in[2\,,\infty)$, $\beta>0$, and predictable random fields Z,

$$\mathcal{Y}_\beta^{(k)}(Z) := \sup_{\substack{t>0\\ x\in\mathbf{R}^d}} \left[\exp\left(-\beta t + \sqrt{\frac{\beta}{8\varkappa}}\;\|x\|\right)\cdot \|Z_t(x)\|_k\right]. \tag{6.1}$$

Let us begin by developing a weighted Young's inequality for stochastic convolutions. This is similar in spirit to the results of Conus and Khoshnevisan [11], extended to the present setting of correlated noise. However, entirely new ideas are needed in order to develop this result; therefore, we include a complete proof.

Proposition 6.1 (A weighted stochastic Young inequality)**.** *Let $Z := \{Z_t(x)\}_{t>0,x\in\mathbf{R}^d}$ be a predictable random field. Then for all real numbers $k\in[2\,,\infty)$ and $\beta>0$,*

$$\mathcal{Y}_\beta^{(k)}\left(p * Z\dot{F}\right) \leqslant \mathcal{Y}_\beta^{(k)}(Z)\cdot\sqrt{2^d k (R_{\beta/4}f)(0)}, \tag{6.2}$$

where R_β is the resolvent operator defined in (2.3)*.*

Proof. For the sake of typographical ease we write $c = c(\beta) := \sqrt{\beta/(8\varkappa)}$ throughout the proof.

Our derivation of (3.22) yields the following estimate:

$$\begin{aligned}
&\left\| \left(p * Z\dot{F} \right)_t (x) \right\|_k^2 \\
&\qquad \leqslant 4k \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y \int_{\mathbf{R}^d} \mathrm{d}z \; f(y-z) p_{t-s}(y-x) p_{t-s}(z-x) \cdot \mathcal{Z},
\end{aligned} \tag{6.3}$$

where $\mathcal{Z} := \|Z_s(y) \cdot Z_s(z)\|_{k/2} \leqslant \|Z_s(y)\|_k \cdot \|Z_s(z)\|_k$. Consequently, for all $\beta > 0$,

$$\begin{aligned}
&\left\| \left(p * Z\dot{F} \right)_t (x) \right\|_k^2 \\
&\leqslant 4k \left[\mathcal{Y}_\beta^{(k)}(Z) \right]^2 \cdot \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y \int_{\mathbf{R}^d} \mathrm{d}z \; f(y-z) P_s(y\,, y-x) P_s(z\,, z-x),
\end{aligned} \tag{6.4}$$

where $P_s(a\,, b) := \mathrm{e}^{\beta s - c\|a\|} p_{t-s}(b)$ for all $s > 0$ and $a \in \mathbf{R}^d$. Since $\|y\| \geqslant \|x\| - \|x-y\|$ and $\|z\| \geqslant \|x\| - \|x-z\|$, it follows that

$$\begin{aligned}
&\left\| \left(p * Z\dot{F} \right)_t (x) \right\|_k^2 \\
&\qquad \leqslant 4k \mathrm{e}^{2\beta t - 2c\|x\|} \left[\mathcal{Y}_\beta^{(k)}(Z) \right]^2 \cdot \int_0^\infty \mathrm{e}^{-\beta s} \left(Q_s * Q_s * f \right)(0) \, \mathrm{d}s,
\end{aligned} \tag{6.5}$$

where

$$Q_s(a) := \mathrm{e}^{-(\beta s/2) + c\|a\|} p_s(a) \qquad \text{for all } s > 0 \text{ and } a \in \mathbf{R}^d. \tag{6.6}$$

Clearly,

$$\text{if } \frac{\beta s}{2} \geqslant c\|a\|, \text{ then } \quad Q_s(a) \leqslant p_s(a). \tag{6.7}$$

Now consider the case that $(\beta s/2) < c\|a\|$. Then,

$$c\|a\| - \frac{\|a\|^2}{2s\varkappa} = -\frac{\|a\|^2}{2s\varkappa}\left(1 - \frac{2s\varkappa c}{\|a\|} \right) < -\frac{\|a\|^2}{2s\varkappa}\left(1 - \frac{4\varkappa c^2}{\beta} \right) = -\frac{\|a\|^2}{4s\varkappa}. \tag{6.8}$$

We can exponentiate the preceding to see that, in the case that $(\beta s/2) < c\|a\|$,

$$Q_s(a) \leqslant \frac{\mathrm{e}^{-(\beta s/2) - \|a\|^2/(4s\varkappa)}}{(2\pi\varkappa s)^{d/2}} \leqslant 2^{d/2} p_{2s}(a). \tag{6.9}$$

Since $p_s(a) \leqslant 2^{d/2} p_{2s}(a)$ for all $s > 0$ and $a \in \mathbf{R}^d$, we deduce from (6.7) and (6.9) that (6.9) holds for all $s > 0$ and $a \in \mathbf{R}^d$. Therefore, the Chapman–Kolmogorov equation implies that $Q_s * Q_s \leqslant 2^d p_{4s}$, and hence

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-\beta s} \left(Q_s * Q_s * f \right)(0) \, \mathrm{d}s &\leqslant 2^d \int_0^\infty \mathrm{e}^{-\beta s} \left(p_{4s} * f \right)(0) \, \mathrm{d}s \\
&= 2^{d-2} (R_{\beta/4} f)(0).
\end{aligned} \tag{6.10}$$

The proposition now follows from (6.5). $\square$

Next we state and prove an elementary estimate for the heat semigroup.

Lemma 6.2. *Suppose* $\phi : \mathbf{R}^d \to \mathbf{R}$ *is a measurable function and* $L(c) := \sup_{x\in\mathbf{R}^d}(\mathrm{e}^{c\|x\|}|\phi(x)|)$ *is finite for some* $c > 0$. *Then* $\mathcal{Y}^{(k)}_{8c^2\varkappa}(p * \phi) \leqslant 2^{d/2}L(c)$ *for all* $k \in [2\,,\infty)$.

Proof. Let us define $\beta := 8c^2\varkappa$, so that $c = \sqrt{\beta/(8\varkappa)}$. Then,

$$\begin{aligned}
\mathrm{e}^{-\beta t+c\|x\|}\,|(p_t * \phi)(x)| &= \int_{\mathbf{R}^d} \mathrm{e}^{-\beta t+c\|x\|}p_t(x-y)|\phi(y)|\,\mathrm{d}y\\
&\leqslant \int_{\mathbf{R}^d} \mathrm{e}^{-\beta t+c\|x-y\|}p_t(x-y)\cdot \mathrm{e}^{c\|y\|}|\phi(y)|\,\mathrm{d}y \qquad (6.11)\\
&\leqslant L(c)\int_{\mathbf{R}^d} \mathrm{e}^{-\beta t+c\|z\|}p_t(z)\,\mathrm{d}z \leqslant L(c)\int_{\mathbf{R}^d} Q_t(z)\,\mathrm{d}z,
\end{aligned}$$

where the function $Q_t(z)$ is defined in (6.6). We apply (6.9) to deduce from this that $\mathrm{e}^{-\beta t+c\|x\|}|(p_t * \phi)(x)| \leqslant 2^{d/2}L(c)\int_{\mathbf{R}^d} p_{2t}(z)\,\mathrm{d}z = 2^{d/2}L(c)$. Optimize over t and x to finish. □

We will next see how to combine the preceding results in order to establish the rapid decay of the moments of the solution to (SHE) as $\|x\| \to \infty$.

Proposition 6.3. *Recall that* $u_0 : \mathbf{R}^d \to \mathbf{R}$ *is a bounded and measurable function and* $\sigma(0) = 0$. *If, in addition,* $\limsup_{\|x\|\to\infty} \|x\|^{-1}\log|u_0(x)| = -\infty$, *then*

$$\limsup_{\|x\|\to\infty} \frac{\log \mathrm{E}(|u_t(x)|^k)}{\|x\|} < 0 \quad \textit{for all } t > 0 \textit{ and } k \in [2\,,\infty). \qquad (6.12)$$

Proof. For all $t > 0$ and $x \in \mathbf{R}^d$, define $u^{(0)}_t(x) := u_0(x)$, and

$$u^{(l+1)}_t(x) := (p_t * u_0)(x) + \left(p * \left(\sigma \circ u^{(l)}\right)\dot{F}\right)_t(x) \quad \text{for all } l \geqslant 0. \qquad (6.13)$$

That is, $u^{(l)}$ is the l^{th} level in the Picard iteration approximation to the solution u. By the triangle inequality,

$$\begin{aligned}
\mathcal{Y}^{(k)}_\beta\left(u^{(l+1)}\right) &\leqslant \mathcal{Y}^{(k)}_\beta(p * u_0) + \mathcal{Y}^{(k)}_\beta\left(\left(p * \left(\sigma\circ u^{(l)}\right)\dot{F}\right)\right)\\
&\leqslant \mathcal{Y}^{(k)}_\beta(p * u_0) + \mathcal{Y}^{(k)}_\beta\left(\sigma \circ u^{(l)}\right)\cdot\sqrt{2^dk(R_{\beta/4}f)(0)};
\end{aligned} \qquad (6.14)$$

see Proposition 6.1. Because $|\sigma(z)| \leqslant \mathrm{Lip}_\sigma|z|$ for all $z \in \mathbf{R}^d$, it follows from the triangle inequality that

$$\mathcal{Y}^{(k)}_\beta\left(u^{(l+1)}\right) \leqslant \mathcal{Y}^{(k)}_\beta(p * u_0) + \mathcal{Y}^{(k)}_\beta\left(u^{(l)}\right)\cdot\sqrt{2^d\mathrm{Lip}^2_\sigma k(R_{\beta/4}f)(0)}. \qquad (6.15)$$

By the dominated convergence theorem, $\lim_{q\to\infty}(R_qf)(0) = 0$. Therefore, we may choose β large enough to ensure that the coefficient of $\mathcal{Y}^{(k)}_\beta(u^{(l)})$ in the preceding is at most $1/2$. The following holds for this choice of β:

$$\sup_{l\geqslant 0}\mathcal{Y}_\beta^{(k)}\left(u^{(l+1)}\right)\leqslant 2\mathcal{Y}_\beta^{(k)}\left(p*u_0\right)\leqslant 2^{(d+2)/2}\sup_{x\in\mathbf{R}^d}\left(\mathrm{e}^{\|x\|\sqrt{\beta/(8\varkappa)}}|u_0(x)|\right); \tag{6.16}$$

we have applied Lemma 6.2 in order to deduce the final inequality. According to the theory of Dalang [12], $u_t^{(l)}(x)\to u_t(x)$ in probability as $l\to\infty$, for all $t>0$ and $x\in\mathbf{R}^d$. Therefore, Fatou's lemma implies that

$$\mathcal{Y}_\beta^{(k)}(u)\leqslant 2^{(d+2)/2}\sup_{x\in\mathbf{R}^d}\left(\mathrm{e}^{\|x\|\sqrt{\beta/(8\varkappa)}}|u_0(x)|\right); \tag{6.17}$$

whence follows the result [after some arithmetic]. $\square$

Next we introduce a fairly crude estimate for the spatial oscillations of the solution to (SHE), in the sense of $L^k(\mathrm{P})$. We begin with an estimate of $L^1(\mathbf{R}^d)$-derivatives of the heat kernel. This is without doubt a well-known result, though we could not find an explicit reference. In any event, the proof is both elementary and short; therefore we include it for the sake of completeness.

Lemma 6.4. *For all $s>0$ and $x\in\mathbf{R}^d$,*

$$\int_{\mathbf{R}^d}|p_s(y-x)-p_s(y)|\,\mathrm{d}y\leqslant\text{const}\cdot\left(\frac{\|x\|}{\sqrt{\varkappa s}}\wedge 1\right), \tag{6.18}$$

where the implied constant does not depend on $(s\,,x)$.

Proof. For s fixed, let us define

$$\mu_d(r)=\mu_d(r\,;s):=\sup_{\substack{z\in\mathbf{R}^d\\ \|z\|\leqslant r}}\int_{\mathbf{R}^d}|p_s(y-z)-p_s(y)|\,\mathrm{d}y\qquad\text{for all } r>0. \tag{6.19}$$

First consider the case that $d=1$. In that case, we may use the differential equation $p_s'(w)=-(w/\varkappa s)p_s(w)$ in order to see that

$$\begin{aligned}\mu_1(|x|)&=\sup_{z\in(0,|x|)}\int_{-\infty}^{\infty}\left|\int_{y-z}^{y}p_s'(w)\,\mathrm{d}w\right|\,\mathrm{d}y\\ &\leqslant\frac{1}{\varkappa s}\sup_{z\in(0,|x|)}\int_{-\infty}^{\infty}\mathrm{d}y\int_{y-z}^{y}\mathrm{d}w\ |w|p_s(w)=\frac{|x|}{\varkappa s}\int_{-\infty}^{\infty}|w|p_s(w)\,\mathrm{d}w\\ &=\sqrt{\frac{2}{\pi\varkappa s}}\,|x|\qquad\qquad\text{for all } x\in\mathbf{R}.\end{aligned} \tag{6.20}$$

For general d, we can integrate one coordinate at a time and then apply the triangle inequality to see that for all $x:=(x_1\,,\ldots,x_d)\in\mathbf{R}^d$, $\mu_d(\|x\|)\leqslant\sum_{j=1}^d\mu_1(\|x\|)\leqslant\sqrt{2/(\pi\varkappa s)}\,d\|x\|$. Because $|p_s(y-x)-p_s(y)|\leqslant p_s(y-x)+p_s(y)$, we also have $\mu_d(\|x\|)\leqslant 2$. $\square$

Proposition 6.5. *Let us assume that: (i)* $\limsup_{\|x\|\to\infty} \|x\|^{-1}\log|u_0(x)| = -\infty$, *(ii)* $\sigma(0)=0$, *and (iii)* $\int_0^1 s^{-a}(p_s * f)(0)\,\mathrm{d}s < \infty$ *for some* $a\in(0\,,1/2)$. *Then for all* $t>0$ *and* $k\in[2\,,\infty)$ *there exists a constant* $C\in(1\,,\infty)$ *such that uniformly for all* $x,x'\in\mathbf{R}^d$ *that satisfy* $\|x-x'\|\leqslant 1$,

$$\mathrm{E}\left(|u_t(x)-u_t(x')|^k\right) \leqslant C\exp\left(-\frac{\|x\|\wedge\|x'\|}{C}\right)\cdot\|x-x'\|^{ak/4}. \tag{6.21}$$

Proof. First of all, we note that

$$\begin{aligned}|(p_t*u_0)(x)-(p_t*u_0)(x')| &\leqslant \|u_0\|_{L^\infty(\mathbf{R}^d)}\cdot\int_{\mathbf{R}^d}|p_t(y-x)-p_t(y-x')|\,\mathrm{d}y\\ &\leqslant \mathrm{const}\cdot\|x-x'\|;\end{aligned} \tag{6.22}$$

see Lemma 6.4. Now we may use this estimate and the same argument that led us to (3.22) in order to deduce that for all $\ell\in[2\,,\infty)$,

$$\begin{aligned}\|u_t(x)-u_t(x')\|_\ell^2 &\leqslant \mathrm{const}\cdot\|x-x'\|^2\\ &\quad+\mathrm{const}\cdot\int_0^t\mathrm{d}s\int_{\mathbf{R}^d}\mathrm{d}y\int_{\mathbf{R}^d}\mathrm{d}z\ f(y-z)\,\mathcal{A}\,\mathcal{B}_s(y)\,\mathcal{B}_s(z),\end{aligned} \tag{6.23}$$

where

$$\begin{aligned}&\mathcal{A}:=\mathcal{A}_s(y\,,z):=\|u_s(y)\cdot u_s(z)\|_{\ell/2}^2, &&\text{and}\\ &\mathcal{B}_s(w):=|p_{t-s}(w-x)-p_{t-s}(w-x')| &&\text{for all } w\in\mathbf{R}^d.\end{aligned} \tag{6.24}$$

According to [12], $\sup_{s\in[0,T]}\sup_{y,z\in\mathbf{R}^d}\mathcal{A}<\infty$. On the other hand, $(\mathcal{B}_s*f)(z)\leqslant 2\sup_{w\in\mathbf{R}^d}(p_{t-s}*f)(w)$, and the latter quantity is equal to $2(p_{t-s}*f)(0)$ since p_r*f is positive definite and continuous for all $r>0$ [whence is maximized at the origin]. We can summarize our efforts as follows:

$$\begin{aligned}&\|u_t(x)-u_t(x')\|_\ell^2\\ &\leqslant \mathrm{const}\cdot\|x-x'\|^2+\mathrm{const}\cdot\int_0^t(p_s*f)(0)\,\mathrm{d}s\int_{\mathbf{R}^d}\mathrm{d}z\ \ |p_s(z-x)-p_s(z-x')|\\ &\leqslant \mathrm{const}\cdot\|x-x'\|^2+\mathrm{const}\cdot\int_0^t(p_s*f)(0)\left(\frac{\|x-x'\|}{\sqrt{s}}\wedge 1\right)\mathrm{d}s;\end{aligned} \tag{6.25}$$

see Lemma 6.4 below, for instance. We remark that the implied constants do not depend on $(x\,,x')$. Since $r\wedge 1\leqslant r^{2a}$ for all $r>0$, it follows that

$$\|u_t(x)-u_t(x')\|_\ell \leqslant \mathrm{const}\cdot\|x-x'\|^{a/2}, \tag{6.26}$$

where the implied constant does not depend on $(x\,,x')$ as long as $\|x-x'\|\leqslant 1$ [say]. Next we write

$$\begin{aligned}&\mathrm{E}\left(|u_t(x)-u_t(x')|^k\right)\\ &\qquad\leqslant \mathrm{E}\left(|u_t(x)-u_t(x')|^{k/2}\cdot\{|u_t(x)|+|u_t(x')|\}^{k/2}\right)\\ &\qquad\leqslant \mathrm{const}\cdot\|u_t(x)-u_t(x')\|_k^{k/2}\left(\|u_t(x)\|_k^{k/2}\vee\|u_t(x')\|_k^{k/2}\right),\end{aligned} \tag{6.27}$$

by Hölder's inequality. Proposition 6.3 and Eq. (6.26) together complete our proof. □

Proposition 6.5 and a quantitative form of Kolmogorov's continuity lemma [13, pp. 10–12] readily imply the following.

Corollary 6.6. *Let us assume that: (i)* $\limsup_{\|x\|\to\infty}\|x\|^{-1}\log|u_0(x)| = -\infty$*, (ii)* $\sigma(0)=0$*, and (iii)* $\int_0^1 s^{-a}(p_s * f)(0)\,\mathrm{d}s<\infty$ *for some* $a\in(0\,,1/2)$*. Then for all* $t>0$ *and* $k\in[2\,,\infty)$ *there exists a constant* $C\in(1\,,\infty)$ *such that uniformly for all hypercubes* $T\subset\mathbf{R}^d$ *of sidelength* $2/\sqrt{d}$*,*

$$\mathrm{E}\left(\sup_{x,x'\in T}|u_t(x)-u_t(x')|^k\right)\leqslant C\exp\left(-\frac{1}{C}\inf_{z\in T}\|z\|\right). \tag{6.28}$$

Finally, we are in position to establish Theorem 2.1.

Proof of Theorem 2.1. Define

$$T(x):=\left\{y\in\mathbf{R}^d:\ \max_{1\leqslant j\leqslant d}|x_j-y_j|\leqslant\frac{2}{\sqrt{d}}\right\}\quad\text{for every } x\in\mathbf{R}^d. \tag{6.29}$$

Then, for all $t>0$ and $k\in[2\,,\infty)$, there exists a constant $c\in(0\,,1)$ such that uniformly for every $x\in\mathbf{R}^d$,

$$\begin{aligned}\mathrm{E}\left(\sup_{y\in T(x)}|u_t(y)|^k\right)&\leqslant 2^k\left\{\mathrm{E}\left(|u_t(x)|^k\right)+\mathrm{E}\left(\sup_{y\in T(x)}|u_t(y)-u_t(x)|^k\right)\right\}\\&\leqslant\frac{1}{c}\cdot\left\{\mathrm{e}^{-c\|x\|}+\exp\left(-c\inf_{y\in T(x)}\|y\|\right)\right\};\end{aligned} \tag{6.30}$$

see Proposition 6.3 and Corollary 6.6. Because $\inf_{y\in T(x)}\|y\|\geqslant\|x\|-1$ for all $x\in\mathbf{Z}^d$, the preceding is bounded by const $\cdot\exp(-\text{const}\cdot\|x\|)$, whence $\mathrm{E}(\sup_{z\in\mathbf{R}^d}|u_t(z)|^k)\leqslant\sum_{x\in\mathbf{Z}^d}\mathrm{E}(\sup_{y\in T(x)}|u_t(y)|^k)$ is finite. □

7 Proof of Theorem 2.3

Throughout this section, we assume that $f=h*\tilde{h}$ for some nonnegative function $h\in L^2(\mathbf{R}^d)$ that satisfies (3.17). Moreover, we let u denote the solution to (SHE).

7.1 The first part

Here and throughout we define for all $R,t>0$

$$u_t^*(R):=\sup_{\|x\|\leqslant R}|u_t(x)|. \tag{7.1}$$

As it turns out, it is easier to prove slightly stronger statements than (2.7) and (2.8). The following is the stronger version of (2.7).

Proposition 7.1. *If σ is bounded uniformly away from zero, then*

$$\liminf_{R\to\infty} \frac{u_t^*(R)}{(\log R)^{1/4}} > 0 \qquad a.s. \tag{7.2}$$

Proof. Let us introduce a free parameter $N \geqslant 1$, which is an integer that we will select carefully later on in the proof.

As before, let us denote $n = [\log \beta]+1$. For all $\theta, R > 0$ and $x^{(1)}, x^{(2)}, \cdots, x^{(N)} \in \mathbf{R}^d$, we may write

$$\begin{aligned} \mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_t(x^{(j)})| < \theta(\log R)^{1/4}\right\} &\leqslant \mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |U_t^{(\beta,n)}(x^{(j)})| < 2\theta(\log R)^{1/4}\right\} \\ &+ \mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_t(x^{(j)}) - U_t^{(\beta,n)}(x^{(j)})| > \theta(\log R)^{1/4}\right\}. \end{aligned} \tag{7.3}$$

We bound these quantities in order.

Suppose in addition that $D(x^{(i)}, x^{(j)}) \geqslant 2n\beta(1+\sqrt{t})$ whenever $i \neq j$, where $D(x, y)$ was defined in (5.24). Because of Lemma 5.4, the collection $\{U_t^{(\beta,n)}(x_j)\}_{j=1}^N$ is comprised of independent random variables. Consequently,

$$\begin{aligned} &\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |U_t^{(\beta,n)}(x^{(j)})| < 2\theta(\log R)^{1/4}\right\} \\ &\qquad\leqslant \left(\mathrm{P}\left\{\left|U_t^{(\beta,n)}(x^{(1)})\right| < 2\theta(\log R)^{1/4}\right\}\right)^N \leqslant (\mathcal{T}_1 + \mathcal{T}_2)^N, \end{aligned} \tag{7.4}$$

where

$$\begin{aligned} \mathcal{T}_1 &:= \sup_{x\in\mathbf{R}^d} \mathrm{P}\left\{|u_t(x)| < 3\theta(\log R)^{1/4}\right\}, \\ \mathcal{T}_2 &:= \sup_{x\in\mathbf{R}^d} \mathrm{P}\left\{|u_t(x) - U_t^{(\beta,n)}(x)| > \theta(\log R)^{1/4}\right\}. \end{aligned} \tag{7.5}$$

According to Lemma 4.3, $\mathcal{T}_1 \leqslant 1 - a(\varkappa)^{-\frac12} R^{-2(3\theta)^4 a(\varkappa)}$ for all R sufficiently large; and Lemma 5.3 implies that there exists a finite constant $m \geqslant 1$ such that uniformly for all $k, \beta \geqslant m$, $\mathcal{T}_2 \leqslant G^k k^{k/2} \mathrm{e}^{Fk^2} / (\theta^k \beta^{kb/2} (\log R)^{k/4}) \leqslant c_1(k) \beta^{-kb/2} (\log R)^{-k/4}$ for a finite and positive constant $c_1(k) := c_1(k, G, F, \theta)$. We combine the preceding to find that

$$(\mathcal{T}_1 + \mathcal{T}_2)^N \leqslant \left(1 - \frac{a(\varkappa)^{-\frac12}}{R^{2(3\theta)^4 a(\varkappa)}} + \frac{c_1(k)}{\beta^{kb/2}}\right)^N, \tag{7.6}$$

uniformly for all $k, \beta \geqslant m$. Because the left-hand side of (7.3) is bounded above by $(\mathcal{T}_1 + \mathcal{T}_2)^N + N\mathcal{T}_2$, it follows that

$$\begin{aligned} &\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_t(x^{(j)})| < \theta(\log R)^{1/4}\right\} \\ &\qquad\leqslant \left(1 - \frac{a(\varkappa)^{-\frac12}}{R^{2(3\theta)^4 a(\varkappa)}} + \frac{c_1(k)}{\beta^{kb/2}}\right)^N + \frac{c_1(k)N}{\beta^{kb/2}}, \end{aligned} \tag{7.7}$$

Now we choose the various parameters as follows: We choose $N := \lceil R^q \rceil^d$ and $\beta := R^{1-q}/\log R$, where $q \in (0\,,1)$ is fixed, and let $k \geqslant 2$ be the smallest integer so that $qd - \frac{1}{2}kb(1-q) < -2$ so that $N\beta^{-kb/2} \leqslant R^{-2}$. In a cube of side length $2(1+\sqrt{t})R$, there are at least N points separated by "D-distance" $2n\beta(1+\sqrt{t})$ where $n := [\log \beta] + 1$. Also choose $\theta > 0$ small enough so that $(3\theta)^4 a(\varkappa) < q$. For these choices of parameters, an application of the Borel–Cantelli lemma [together with a monotonicity argument] implies that $\liminf_{R\to\infty} (\log R)^{-1/4}\, u_t^*(R) > 0$ a.s. See [10] for more details of this kind of argument in a similar setting. $\square$

7.2 The second part

Similarly as in the proof of Theorem 2.1, we will need a result on the modulus of continuity of u.

Lemma 7.2. *If $\sup_{x\in\mathbf{R}} |\sigma(x)| < \infty$, then there exists a constant $C = C(t) \in (0\,,\infty)$ such that*

$$\mathrm{E}\left(|u_t(x) - u_t(x')|^{2k}\right) \leqslant \left(\frac{Ck}{\sqrt{\varkappa}}\right)^k \cdot \|x - x'\|^k, \tag{7.8}$$

uniformly for all $x, x' \in \mathbf{R}^d$ that satisfy $\|x - x'\| \leqslant (t\varkappa)^{1/2}$.

Proof. Let $S_0 := \sup_{z\in\mathbf{R}} |\sigma(z)|$. Because $|f(z)| \leqslant f(0)$ for all $z \in \mathbf{R}^d$, the optimal form of the Burkholder–Davis–Gundy inequality (BDG) and (6.22) imply that

$$\|u_t(x) - u_t(x')\|_{2k} \leqslant \mathrm{const} \cdot \|x - x'\| + 2S_0\sqrt{2kf(0)\, Q_t(x - x')}, \tag{7.9}$$

where

$$Q_t(w) := \int_0^t \mathrm{d}s \left(\int_{\mathbf{R}^d} \mathrm{d}y\ \ |p_{t-s}(y - w) - p_{t-s}(y)|\right)^2 \quad \text{for } w \in \mathbf{R}^d. \tag{7.10}$$

Lemma 6.4 and a small computation implies readily that $Q_t(w) \leqslant \mathrm{const} \cdot \|w\|\sqrt{t/\varkappa}$ whenever $\|w\| \leqslant (t\varkappa)^{1/2}$; and the lemma follows from these observations. $\square$

Lemma 7.3. *Choose and fix $t > 0$, and suppose that σ is bounded. Then there exists a constant $C \in (0\,,\infty)$ such that*

$$\mathrm{E}\left[\sup_{\substack{x,x'\in T:\\ \|x-x'\|\leqslant\delta}} \exp\left(\frac{\sqrt{\varkappa}|u_t(x) - u_t(x')|^2}{C\delta}\right)\right] \leqslant \frac{2}{\delta}, \tag{7.11}$$

uniformly for every $\delta \in (0\,,(t\varkappa)^{1/2}]$ and every cube $T \subset \mathbf{R}^d$ of side length at most 1.

As the proof is quite similar to the proof of [10, Lemma 6.2], we leave the verification to the reader. Instead we prove the following result, which readily implies (2.8), and thereby completes our derivation of Theorem 2.3.

Proposition 7.4. *If σ is bounded uniformly away from zero and infinity, then $u_t^*(R) \asymp (\log R)^{1/2}$ a.s.*

Proof. We may follow the proof of Proposition 7.1, but use Lemma 4.2 instead of Lemma 4.3, in order to establish that $\liminf_{R\to\infty}(\log R)^{-1/2}u_t^*(R) > 0$ a.s. We skip the details, as they involve making only routine changes to the proof of Proposition 7.1.

It remains to prove that

$$u_t^*(R) = O\left((\log R)^{1/2}\right) \qquad (R \to \infty) \quad \text{a.s. for all } t > 0. \tag{7.12}$$

It suffices to consider the case that $R \gg t$. Let us divide the cube $[0\,,R]^d$ into subcubes $\Gamma_1, \Gamma_2, \ldots$ such that the Γ_j's have common side length $a := \text{const}\cdot(t\varkappa)^{1/2}$ and the distance between any two points in Γ_j is at most $(t\varkappa)^{1/2}$. The total number N of such subcubes is $O(R^d)$.

We now apply Lemmas 7.3 and 4.2 as follows:

$$\begin{aligned}
&\mathrm{P}\left\{\sup_{x\in[0,R]^d}|u_t(x)| > 2b(\ln R)^{1/2}\right\} \qquad (7.13)\\
&\leqslant \mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u_t(x_j)| > b(\ln R)^{1/2}\right\} + \mathrm{P}\left\{\max_{1\leqslant j\leqslant N}\sup_{x,y\in\Gamma_j}|u_t(x)-u_t(y)| > b(\ln R)^{1/2}\right\}\\
&\leqslant \text{const}\cdot R^d \mathrm{e}^{-c_2 b^2\ln R} + \frac{\text{const}\cdot R^d}{(t\varkappa)^{1/2}\exp\left(b^2\ln(R)/Ct^{1/2}\right)}.
\end{aligned}$$

Consequently,

$$\sum_{m=1}^{\infty}\mathrm{P}\left\{\sup_{x\in[0,m]^d}|u_t(x)| > 2b(\ln m)^{1/2}\right\} < \infty, \tag{7.14}$$

provided that we choose b sufficiently large. This, the Borel–Cantelli Lemma, and a monotonicity argument together complete the proof of (7.12). □

8 Proof of Theorem 2.5

Let us first establish some point estimates for the tail probability of the solution u to (PAM). Throughout this subsection the assumptions of Theorem 2.5 are in force.

Lemma 8.1. *For every $t > 0$,*

$$\limsup_{\lambda\to\infty}\sup_{x\in\mathbf{R}^d}\frac{\log\mathrm{P}\{|u_t(x)|\geqslant\lambda\}}{(\log\lambda)^2} \leqslant -\frac{1}{4tf(0)}. \tag{8.1}$$

Additionally, for every $t > 0$,

$$\liminf_{\lambda\to\infty}\inf_{x\in\mathbf{R}^d}\frac{\log \mathrm{P}\{|u_t(x)|\geqslant\lambda\}}{(\log\lambda)^2}\geqslant -\frac{4tf(0)}{(\Lambda_d a_t)^2}, \tag{8.2}$$

where Λ_d and $a_t = a_t(f\,,\varkappa)$ were defined in Proposition 4.4.

Proof. Let $\log_+(z) := \log(z\vee \mathrm{e})$ for all real numbers z. Proposition 4.4 and Lemma 3.4 of the companion paper [10] together imply that if $0 < \gamma < (4tf(0))^{-1}$, then $\mathrm{E}\exp(\gamma|\log_+(u_t(x))|^2)$ is bounded uniformly in $x \in \mathbf{R}^d$. The first estimate of the lemma follows from this by an application of Chebyshev's inequality.

As regards the second bound, we apply the Paley–Zygmund inequality (4.23) in conjunction with Proposition 4.4 as follows:

$$\mathrm{P}\left\{|u_t(x)|\geqslant\frac12\|u_t(x)\|_{2k}\right\}\geqslant\frac{\left(\mathrm{E}\left(|u_t(x)|^{2k}\right)\right)^2}{4\mathrm{E}\left(|u_t(x)|^{4k}\right)}\geqslant\frac14\mathrm{e}^{k^2[8\Lambda_d a_t-16tf(0)]}\cdot\left(\underline{u}_0/\overline{u}_0\right)^{4k}. \tag{8.3}$$

Let us denote $\gamma = \gamma(\varkappa\,,t) := 16tf(0) - 8\Lambda_d a_t > 0$. A second application of Proposition 4.4 then yields the following pointwise bound:

$$\mathrm{P}\left\{|u_t(x)|\geqslant\frac{\underline{u}_0}{2}\mathrm{e}^{2\Lambda_d a_t k}\right\}\geqslant\frac14\mathrm{e}^{-\gamma k^2}\left(\underline{u}_0/\overline{u}_0\right)^{4k}. \tag{8.4}$$

The second assertion of the lemma follows from this and the trivial estimate $\gamma \leqslant 16tf(0)$, because we can consider λ between $\frac{\underline{u}_0}{2}\exp(2\Lambda_d a_t k)$ and $\frac{\underline{u}_0}{2}\exp(2\Lambda_d a_t(k-1))$. □

Owing to the parameter dependencies pointed out in Proposition 4.4, Theorem 2.5 is a direct consequence of the following result.

Proposition 8.2. *For the parabolic Anderson model, the following holds: For all $t > 0$, there exists a constant $\theta_t \in (0\,,\infty)$—independent of $\varkappa$—such that*

$$\frac{\Lambda_d a_t}{(8tf(0))^{1/2}}\leqslant\liminf_{R\to\infty}\frac{\log u_t^*(R)}{(\log R)^{1/2}}\leqslant\limsup_{R\to\infty}\frac{\log u_t^*(R)}{(\log R)^{1/2}}\leqslant\theta_t, \tag{8.5}$$

where Λ_d and $a_t = a_t(f\,,\varkappa)$ were defined in Proposition 4.4.

Proof. Choose and fix two positive and finite numbers a and b that satisfy the following:

$$a<\frac{1}{4tf(0)},\quad b>\frac{4tf(0)}{(\Lambda_d a_t)^2}. \tag{8.6}$$

According to Lemma 8.1, the following holds for all $\lambda > 0$ sufficiently large:

$$\mathrm{e}^{-b(\log\lambda)^2}\leqslant\mathrm{P}\left\{|u_t(x)|\geqslant\lambda\right\}\leqslant\mathrm{e}^{-a(\log\lambda)^2}. \tag{8.7}$$

Our goal is twofold: First, we would like to prove that with probability one $\log|u_t^*(R)| \asymp (\log R)^{1/2}$ as $R\to\infty$; and next to estimate the constants in "$\asymp$."

We first derive an almost sure asymptotic lower bound for $\log|u_t^*(R)|$.

Let us proceed as we did in our estimate of (7.3). We introduce free parameters $\beta, k, N \geqslant 1$ [to be chosen later] together with N points $x^{(1)}, \ldots, x^{(N)}$. We will assume that $D(x^{(i)}, x^{(j)}) \geqslant 2n\beta(1+\sqrt{t})$ where $D(x,y)$ was defined in (5.24) and $n := [\log \beta] + 1$ as in Lemma 5.4. If $\xi > 0$ is an arbitrary parameter, then our localization estimate (Lemma 5.3) yields the following for all sufficiently-large values of R [independently of N and β]:

$$
\begin{aligned}
&\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_t(x^{(j)})| < \mathrm{e}^{\xi\sqrt{\log R}}\right\} \\
&\leqslant \mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |U_t^{(\beta,n)}(x^{(j)})| < 2\mathrm{e}^{\xi\sqrt{\log R}}\right\} + \mathrm{P}\left\{\max_{1\leqslant j\leqslant N} \left|u_t(x^{(j)}) - U_t^{(\beta,n)}(x^{(j)})\right| > \mathrm{e}^{\xi\sqrt{\log R}}\right\} \\
&\leqslant \left(1 - \mathrm{P}\left\{\left|U_t^{(\beta,n)}(x^{(1)})\right| \geqslant 2\mathrm{e}^{\xi\sqrt{\log R}}\right\}\right)^N + \frac{NG^k k^{k/2}\mathrm{e}^{Fk^2}}{\beta^{kb/2}\mathrm{e}^{k\xi\sqrt{\log R}}}.
\end{aligned} \tag{8.8}
$$

And we estimate the remaining probability by similar means, viz.,

$$
\begin{aligned}
&\mathrm{P}\left\{\left|U_t^{(\beta,n)}(x^{(1)})\right| \geqslant 2\mathrm{e}^{\xi\sqrt{\log R}}\right\} \\
&\geqslant \mathrm{P}\left\{\left|u_t(x^{(1)})\right| \geqslant 3\mathrm{e}^{\xi\sqrt{\log R}}\right\} - \mathrm{P}\left\{\left|u_t(x^{(1)}) - U_t^{(\beta,n)}(x^{(1)})\right| > \mathrm{e}^{\xi\sqrt{\log R}}\right\} \\
&\geqslant \exp\left(-b\left\{\log\left(3\mathrm{e}^{\xi\sqrt{\log R}}\right)\right\}^2\right) - \frac{NG^k k^{k/2}\mathrm{e}^{Fk^2}}{\beta^{kb/2}\mathrm{e}^{k\xi\sqrt{\log R}}}.
\end{aligned} \tag{8.9}
$$

We now fix our parameters N and β as follows: First we choose an arbitrary $\theta \in (0,1)$, and then select $N := \lceil R^\theta \rceil^d$ and $\beta := R^{1-\theta}/\log R$. For these choices, we can apply (8.9) in (8.8) and deduce the bound

$$
\begin{aligned}
&\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_t(x_j)| < \mathrm{e}^{\xi\sqrt{\log R}}\right\} \\
&\leqslant \left(1 - \frac{\mathrm{const}}{R^{b\xi^2}} + \frac{G^k k^{k/2}\mathrm{e}^{Fk^2}(\log R)^k}{R^{(kb(1-\theta)-2\theta d)/2}\,\mathrm{e}^{\xi k\sqrt{\log R}}}\right)^N + \frac{G^k k^{k/2}\mathrm{e}^{Fk^2}(\log R)^k}{R^{(kb(1-\theta)-2\theta d)/2}\,\mathrm{e}^{\xi k\sqrt{\log R}}}.
\end{aligned} \tag{8.10}
$$

Now we choose our remaining parameters k and ξ so that $\frac{1}{2}kb(1-\theta) - \theta d > 2$ and $b\xi^2 < \theta/2$. In this way we obtain

$$
\mathrm{P}\left\{\max_{1\leqslant j\leqslant N} |u_t(x_j)| < \mathrm{e}^{\xi\sqrt{\log R}}\right\} \leqslant \exp\left(-CR^{\theta/2}\right) + \frac{C}{R^2}. \tag{8.11}
$$

In a cube of side length $2(1+\sqrt{t})R$, there are at least N points separated by "D-distance" $2(1+\sqrt{t})\beta n$. Therefore, the Borel–Cantelli Lemma and a monotonicity argument together imply that $\liminf_{R\to\infty} \exp\{-\xi(\log R)^{1/2}\} u_t^*(R) > 1$ almost surely. We can first let $\theta \downarrow 1$, then $\xi \uparrow (2b)^{-1/2}$, and finally $b \uparrow 4tf(0)/(\Lambda_d a_t)^2$—in this order—in order to complete our derviation of the stated a.s. asymptotic lower bound for $u_t^*(R)$.

For the other direction, we begin by applying (6.22) and (BDG):

$$\begin{aligned}\|u_t(x)-u_t(y)\|_{2k} &\leqslant \text{const}\cdot\|x-x'\| \qquad (8.12)\\ &+2\left(4kf(0)\int_0^t \|u_s(0)\|_{2k}^2\,\mathrm{d}s\left[\int_{\mathbf{R}^d}\mathrm{d}w\,|p_{t-s}(w-x)-p_{t-s}(w-y)|\right]^2\right)^{1/2}.\end{aligned}$$

We apply Proposition 4.4 to estimate $\|u_s(0)\|_{2k}$, and Lemma 6.4 to estimate the integral that involves the heat kernel. By arguments similar as in Lemma 7.2, we find that there exists $C=C(t)\in(0\,,\infty)$—independently of $(x\,,y\,,k\,,\varkappa)$—such that uniformly for all $x,y\in\mathbf{R}^d$ with $\|x-y\|\leqslant(t\varkappa)^{1/2}$,

$$\mathrm{E}\left(|u_t(x)-u_t(y)|^{2k}\right)\leqslant(Ck)^k\mathrm{e}^{4tf(0)k^2}\frac{\|x-y\|^k}{\varkappa^{k/2}}. \tag{8.13}$$

By arguments similar to the ones that led to (7.12) in the companion paper [10] we can show that

$$\mathrm{E}\left(\sup_{\substack{x,y\in T:\\ \|x-y\|\leqslant\sqrt{t\varkappa}}}|u_t(x)-u_t(y)|^{2k}\right)\leqslant C_1^k\mathrm{e}^{C_2k^2} \tag{8.14}$$

(where C_1 and C_2 depend only on t), uniformly over cubes T with side lengths at most 1. [The preceding should be compared with the result of Lemma 7.2.] Now that we are armed with (8.14), we may proceed to complete the proof of the theorem as follows: We split $[0\,,R]^d$ into subcubes of side length a each of which is contained in a ball of radius $\frac12(t\varkappa)^{1/2}$ centered around its midpoint. Let $\mathcal{C}_R$ denotes the collection of all mentioned subcubes and $\mathcal{M}_R$ the set of their midpoints. For all $\zeta>0$, we have:

$$\begin{aligned}\mathrm{P}\left\{u_t^*(R)>2\mathrm{e}^{\zeta\sqrt{\log R}}\right\} &\leqslant \mathrm{P}\left\{\max_{x\in\mathcal{M}_R}|u_t(x)|>\mathrm{e}^{\zeta\sqrt{\log R}}\right\} \qquad (8.15)\\ &\qquad+\mathrm{P}\left\{\sup_{T\in\mathcal{C}_R}\sup_{x,y\in T}|u_t(x)-u_t(y)|>\mathrm{e}^{\zeta\sqrt{\log R}}\right\},\\ &\leqslant O(R^d)\cdot\mathrm{P}\left\{|u_t(0)|>\mathrm{e}^{\zeta\sqrt{\log R}}\right\}\\ &\qquad+\sum_{T\in\mathcal{C}_R}\mathrm{P}\left\{\sup_{x,y\in T}|u_t(x)-u_t(y)|>\mathrm{e}^{\zeta\sqrt{\log R}}\right\}.\end{aligned}$$

We use the notation set forth in (8.7), together with (8.14), and deduce the following estimate:

$$\mathrm{P}\left\{u_t^*(R)>2\mathrm{e}^{\zeta\sqrt{\log R}}\right\}=O(R^d)\cdot\left[\mathrm{e}^{-a\zeta^2\log R}+\frac{C_1^k\mathrm{e}^{C_2k^2}}{\mathrm{e}^{2k\zeta\sqrt{\log R}}}\right], \tag{8.16}$$

as $R\to\infty$. Now choose $k:=[(\log R)^{1/2}]$ and ζ large so that the above is summable in R, as the variable R ranges over all positive integers. The Borel-Cantelli Lemma and a standard monotonicity argument together imply that with probability one, $\limsup_{R\to\infty}(\log R)^{-1/2}\log u_t^*(R) \leqslant \zeta$. [Now R is allowed to roam over all positive reals.] From the way in which ζ is chosen, it is clear that ζ does not depend on $\varkappa$. □

9 Riesz kernels

Now we turn to the case where the correlation function is of the Riesz form; more precisely, we have $f(x) = \text{const}\cdot\|x\|^{-\alpha}$ for some $\alpha\in(0\,,d\wedge 2)$. We begin this discussion by establishing some moment estimates for the solution u to (PAM). Before we being our analysis, let us recall some well-known facts from harmonic analysis (see for example [25]).

For all $b\in(0\,,d)$ define $\mathcal{R}_b(x) := \|x\|^{-b}$ ($x\in\mathbf{R}^d$). This is a rescaled Riesz kernel with index $b\in(0\,,d)$; it is a locally integrable function whose Fourier transform is defined, for all $\xi\in\mathbf{R}^d$, as

$$\hat{\mathcal{R}}_b(\xi) = C_{d,d-b}\mathcal{R}_{d-b}(\xi), \quad \text{where} \quad C_{d,p} := \frac{\pi^{d/2}2^{d-p}\Gamma((d-p)/2)}{\Gamma(p/2)}. \tag{9.1}$$

We may note that the correlation function f considered in this section is proportional to $\mathcal{R}_\alpha$. We note also that the Fourier transform of (9.1) is understood in the sense of generalized functions. Suppose next that $a,b\in(0\,,d)$ satisfy $a+b<d$, and note that $\hat{\mathcal{R}}_{d-a}(\xi)\hat{\mathcal{R}}_{d-b}(\xi) = \{C_{d,a}C_{d,b}/C_{d,a+b}\}\hat{\mathcal{R}}_{d-(a+b)}(\xi)$. In other words, whenever $a,b,a+b\in(0\,,d)$,

$$\mathcal{R}_{d-a} * \mathcal{R}_{d-b} = \frac{C_{d,a}C_{d,b}}{C_{d,a+b}}\mathcal{R}_{d-(a+b)}, \tag{9.2}$$

where the convolution is understood in the sense of generalized functions.

9.1 Riesz-kernel estimates

We now begin to develop several inequalities for the solution u to (PAM) in the case that $f(x) = \text{const}\cdot\|x\|^{-\alpha} = \text{const}\cdot\mathcal{R}_\alpha(x)$.

Proposition 9.1. *There exists positive and finite constants $\underline{c} = \underline{c}(\alpha\,,d)$ and $\bar{c} = \bar{c}(\alpha\,,d)$ such that*

$$\underline{u}_0^k \exp\left(\underline{c}t\frac{k^{(4-\alpha)/(2-\alpha)}}{\varkappa^{\alpha/(2-\alpha)}}\right) \leqslant \mathrm{E}\left(|u_t(x)|^k\right) \leqslant \overline{u}_0^k \exp\left(\bar{c}t\frac{k^{(4-\alpha)/(2-\alpha)}}{\varkappa^{\alpha/(2-\alpha)}}\right), \tag{9.3}$$

uniformly for all $x\in\mathbf{R}^d$, $t,\varkappa>0$, and $k\geqslant 2$, where $\underline{u}_0$ and $\overline{u}_0$ are defined in (1.8).

Remark 9.2. We are interested in what Proposition 9.1 has to say in the regime in which t is fixed, $\varkappa\approx 0$, and $k\approx\infty$. However, let us spend a few extra lines and emphasize also the following somewhat different consequence of Proposition 9.1. Define for all $k \geqslant 2$,

$$
\begin{aligned}
\underline{\lambda}(k) &:= \liminf_{t\to\infty} \inf_{x\in\mathbf{R}^d} \frac{1}{t}\log \mathrm{E}\left(|u_t(x)|^k\right)\\
\overline{\lambda}(k) &:= \limsup_{t\to\infty} \sup_{x\in\mathbf{R}^d} \frac{1}{t}\log \mathrm{E}\left(|u_t(x)|^k\right).
\end{aligned} \tag{9.4}
$$

These are respectively the lower and upper uniform Lyapunov $L^k(\mathrm{P})$-exponents of the parabolic Anderson model driven by Riesz-type correlations. Convexity alone implies that if $\underline{\lambda}(k_0) > 0$ for some $k_0 > 0$, and if $\overline{\lambda}(k) < \infty$ for all $k \geqslant k_0$, then $\underline{\lambda}(k)/k$ and $\overline{\lambda}(k)/k$ are both strictly increasing for $k > k_0$. Proposition 9.1 implies readily that the common of these increasing sequences is ∞. In fact, we have the following sharp growth rates, which appear to have not been known previously:

$$
\frac{\underline{c}}{\varkappa^{\alpha/(2-\alpha)}} \leqslant \liminf_{k\to\infty} \frac{\underline{\lambda}(k)}{k^{2/(2-\alpha)}} \leqslant \limsup_{k\to\infty} \frac{\overline{\lambda}(k)}{k^{2/(2-\alpha)}} \leqslant \frac{\bar{c}}{\varkappa^{\alpha/(2-\alpha)}}. \tag{9.5}
$$

These bounds can be used to study further the large-time intermittent structure of the solution to the parabolic Anderson model driven by Riesz-type correlations. We will not delve into this matter here. □

Proof. Recall that $f(x) = A \cdot \|x\|^{-\alpha}$; we will, without incurring much loss of generality, that $A = 1$.

We first derive the lower bound on the moments of $u_t(x)$. Let $\{b^{(j)}\}_{j=1}^k$ denote k independent standard Brownian motions in $\mathbf{R}^d$. We may apply Lemma 4.5 to see that

$$
\mathrm{E}\left(|u_t(x)|^k\right) \geqslant \underline{u}_0^k \mathrm{E}\left[\exp\left(\sum\sum_{1\leqslant i\neq j\leqslant k} \int_0^t \frac{\varkappa^{-\alpha/2}\,\mathrm{d}s}{\left\|b_s^{(i)} - b_s^{(j)}\right\|^\alpha}\right)\right]. \tag{9.6}
$$

We can use the preceding to obtain a large-deviations lower bound for the kth moment of $u_t(x)$ as follows: Note that $\int_0^t \|b_s^{(i)} - b_s^{(j)}\|^{-\alpha}\,\mathrm{d}s \geqslant (2\epsilon)^{-\alpha} t \mathbf{1}_{\Omega_\epsilon}$ a.s., where Ω_ϵ is defined as the event $\{\max_{1\leqslant l\leqslant k}\sup_{s\in[0,t]}\|b_s^{(l)}\| \leqslant \epsilon\}$. Therefore,

$$
\mathrm{E}\left(|u_t(x)|^k\right) \geqslant \underline{u}_0^k \sup_{\epsilon>0}\left[\exp\left(\frac{k(k-1)t}{(2\epsilon\sqrt{\varkappa})^\alpha}\right)\cdot \mathrm{P}(\Omega_\epsilon)\right]. \tag{9.7}
$$

Because of an eigenfunction expansion [27, Theorem 7.2, p. 126] there exist constants $\lambda_1 = \lambda_1(d) \in (0\,,\infty)$, and $c = c(d) \in (0\,,\infty)$ such that

$$
\mathrm{P}(\Omega_\epsilon) = \left(\mathrm{P}\left\{\sup_{s\in[0,t/\epsilon^2]} \|b_s^{(1)}\| \leqslant 1\right\}\right)^k \geqslant c^k \mathrm{e}^{-kt\lambda_1/\epsilon^2}, \tag{9.8}
$$

uniformly for all $k \geqslant 2$ and $\epsilon \in (0\,,t^{1/2}]$. And, in fact, λ_1 is the smallest positive eigenvalue of the Dirichlet Laplacian on the unit ball of $\mathbf{R}^d$. Thus,

$$\mathrm{E}\left(|u_t(x)|^k\right) \geqslant (c\underline{u}_0)^k \sup_{\epsilon\in(0,t^{1/2}]} \left[\exp\left(\frac{k(k-1)t}{(2\epsilon\sqrt{\varkappa})^\alpha} - \frac{kt\lambda_1}{\epsilon^2}\right)\right]. \tag{9.9}$$

The supremum of the expression inside the exponential is at least const $\cdot\, tk \cdot k^{2/(2-\alpha)}/\varkappa^{\alpha/(2-\alpha)}$, where "const" depends only on $(\alpha\,,d)$. This proves the asserted lower bound on the $L^k(\mathrm{P})$-norm of $u_t(x)$.

We adopt a different route for the upper bound. Let $\{\bar R_\lambda\}_{\lambda>0}$ denote the resolvent corresponding to $\sqrt{2}$ times a Brownian motion in $\mathbf{R}^d$ with diffusion coefficient $\varkappa$. In other words, $\bar R_\lambda f := \int_0^\infty \exp(-\lambda s)(p_{2s}*f)\,\mathrm{d}s = (1/2)(R_{\lambda/2}f)$. Next define

$$Q(k\,,\beta) := z_k\sqrt{(\bar R_{2\beta/k}f)(0)} \qquad \text{for all } \beta>0 \text{ and } k\geqslant 2, \tag{9.10}$$

where z_k is the optimal constant, due to Davis [14], in the Burkholder–Davis–Gundy inequality for the $L^k(\mathrm{P})$ norm of continuous martingales [4, 5, 6]. We can combine [17, Theorem 1.2] and Dalang's theorem [12] to conclude that, because the solution to (PAM) exists, $(\bar R_\lambda f)(0) < \infty$ for all $\lambda > 0$. The proof of [17, Theorem 1.3] and Eq. (5.35) therein [*loc. cit.*] together imply that if $Q(k\,,\beta) < 1$ then $\mathrm{e}^{-\beta t/k}\|u_t(x)\|_k \leqslant \overline{u}_0/(1-Q(k\,,\beta))$ uniformly for all $t>0$ and $x\in\mathbf{R}^d$. In particular, if $Q(k\,,\beta)\leqslant\frac12$, then

$$\mathrm{E}\left(|u_t(x)|^k\right) \leqslant \mathrm{e}^{\beta t}2^k\overline{u}_0^k. \tag{9.11}$$

According to Carlen and Kree [8], $z_k \leqslant 2\sqrt{k}$; this is the inequality that led also to (BDG). Therefore, (9.11) holds as soon as $k(\bar R_{2\beta/k}f)(0) < 1/16$. Because both Brownian motion and f satisfy scaling relations, a simple change of variables shows that $(\bar R_\lambda f)(0) = c_2\lambda^{-(2-\alpha)/2}\varkappa^{-\alpha/2}$, where c_2 is also a nontrivial constant that depends only on $(d\,,\alpha)$. Therefore, the condition $k(\bar R_{2\beta/k}f)(0) < 1/16$—shown earlier to be sufficient for (9.11)—is equivalent to the assertion that $\beta > k\cdot c_3k^{2/(2-\alpha)}/\varkappa^{\alpha/(2-\alpha)}$ for a nontrival constant c_3 that depends only on $(d\,,\alpha)$. Now we choose $\beta := 2k\cdot c_3k^{2/(2-\alpha)}/\varkappa^{\alpha/(2-\alpha)}$, plug this choice in (9.11), and deduce the upper bound. □

Before we proceed further, let us observe that, in accord with (9.2),

$$f(x) = \frac{\text{const}}{\|x\|^\alpha} = (h*h)(x) = (h*\tilde h)(x) \quad \text{with} \quad h(x) := \frac{\text{const}}{\|x\|^{(d+\alpha)/2}}, \tag{9.12}$$

where the convolution is understood in the sense of generalized functions.

As in (3.16), we can define $h_n(x) := h(x)\hat\varrho_n(x)$ and $f_n = (h-h_n)*(\tilde h-\tilde h_n)$.

Lemma 9.3. *For all $\eta\in(0\,,1\wedge\alpha)$ there exists a constant $A := A(d\,,\varkappa\,,\alpha\,,\eta)\in(0\,,\infty)$ such that $(p_s*f_n)(0)\leqslant An^{-\eta}\cdot s^{-(\alpha-\eta)/2}$ for all $s>0$ and $n\geqslant 1$.*

Proof. Because $f_n \leqslant h * (h - h_n)$, it follows that

$$
\begin{aligned}
(p_s * f_n)(0) &\leqslant [(p_s * h * h)(0) - (p_s * h * h_n)(0)] \\
&= \int_{\mathbf{R}^d} \mathrm{d}y \int_{\mathbf{R}^d} \mathrm{d}z\ p_s(z) h(y) h(y-z) \left(1 - \hat{\varrho}_n(y-z)\right) \qquad (9.13) \\
&\leqslant \text{const} \cdot \int_{\mathbf{R}^d} \frac{\mathrm{d}y}{\|y\|^{(d+\alpha)/2}} \int_{\mathbf{R}^d} \frac{p_s(z)\,\mathrm{d}z}{\|y-z\|^{(d+\alpha)/2}} \left(1 \wedge \frac{\|y-z\|}{n}\right);
\end{aligned}
$$

see (3.11).

Choose and fix some $\eta \in (0\,, 1 \wedge \alpha)$. Since $1 \wedge r \leqslant r^\eta$ for all $r > 0$,

$$
\begin{aligned}
(p_s * f_n)(0) &\leqslant \frac{\text{const}}{n^\eta} \cdot \int_{\mathbf{R}^d} \frac{\mathrm{d}y}{\|y\|^{(d+\alpha)/2}} \int_{\mathbf{R}^d} \frac{p_s(z)\,\mathrm{d}z}{\|y-z\|^{(d+\alpha-2\eta)/2}} \\
&= \frac{\text{const}}{n^\eta} \cdot \int_{\mathbf{R}^d} \mathrm{d}y \int_{\mathbf{R}^d} \mathrm{d}z\ \mathcal{R}_{(d+\alpha)/2}(y) p_s(z) \mathcal{R}_{(d+\alpha-2\eta)/2}(z-y) \\
&= \frac{\text{const}}{n^\eta} \cdot \int_{\mathbf{R}^d} \|z\|^{-\alpha+\eta} p_s(z)\,\mathrm{d}z, \qquad (9.14)
\end{aligned}
$$

by (9.2), because p_s is a rapidly-decreasing test function for all $s > 0$. A change of variable in the integral above proves the result. $\square$

Proposition 9.4. *For every $\eta \in (0\,, 1 \wedge \alpha)$, the following holds uniformly for every $k \geqslant 2$, $\delta > 0$, and all predictable random fields Z:*

$$
\mathcal{M}_\delta^{(k)}\left(p * ZF^{(h)} - p * ZF^{(h_n)}\right) \leqslant \text{const} \cdot \sqrt{\frac{k}{n^\eta \cdot \delta^{(2-\alpha+\eta)/2}}}\ \mathcal{M}_\delta^{(k)}(Z), \qquad (9.15)
$$

where the implied constant depends only on $(d\,, \varkappa\,, \alpha\,, \eta)$.

Remark 9.5. Proposition 9.4 compares to Proposition 3.4. $\square$

Proof. For notational simplicity, let us write

$$
\Xi := \left\| \left(p * ZF^{(h)}\right)_t(x) - \left(p * ZF^{(h_n)}\right)_t(x) \right\|_k. \qquad (9.16)
$$

We apply first (BDG), and then Minkowski's inequality, to see that for all $\delta > 0$,

$$
\begin{aligned}
\Xi^2 &\leqslant 4 \left[\mathcal{M}_\delta^{(k)}(Z)\right]^2 k \int_0^t \mathrm{d}s \int_{\mathbf{R}^d} \mathrm{d}y \int_{\mathbf{R}^d} \mathrm{d}z\ p_{t-s}(y) f_n(y-z) p_{t-s}(z) \\
&\leqslant 4\mathrm{e}^{2\delta t} \left[\mathcal{M}_\delta^{(k)}(Z)\right]^2 k \int_0^\infty \mathrm{e}^{-2\delta r} \left(p_{2r} * f_n\right)(0)\,\mathrm{d}r \\
&= 2\mathrm{e}^{2\delta t} \left[\mathcal{M}_\delta^{(k)}(Z)\right]^2 k \int_0^\infty \mathrm{e}^{-\delta s} \left(p_s * f_n\right)(0)\,\mathrm{d}s. \qquad (9.17)
\end{aligned}
$$

The appeal to Fubini's theorem is justified since: (i) p_r is a rapidly decreasing test function for all $r > 0$; (ii) $p_r * p_r = p_{2r}$ by the Chapman–Kolmogorov equation; and (iii) $p_r, f_n \geqslant 0$ pointwise for every $r > 0$ and $n \geqslant 1$. Now we apply Lemma 9.3 in order to find that for all $\eta \in (0\,,1\wedge\alpha)$,

$$\begin{aligned}\Xi^2 &\leqslant \text{const}\cdot\frac{\mathrm{e}^{2\delta t}k}{n^\eta}\left[\mathcal{M}_\delta^{(k)}(Z)\right]^2\int_0^\infty \mathrm{e}^{-\delta s}s^{-(\alpha-\eta)/2}\,\mathrm{d}s\\ &= \text{const}\cdot\frac{\mathrm{e}^{2\delta t}k}{n^\eta}\left[\mathcal{M}_\delta^{(k)}(Z)\right]^2\delta^{-(2-\alpha+\eta)/2}.\end{aligned}\tag{9.18}$$

Since the right-most term is independent of x, we can divide both sides by $\exp(2\delta t)$, optimize over t, and then take square root to complete the proof. $\square$

9.2 Localization for Riesz kernels

The next step in our analysis of Riesz-type correlations is to establish localization; namely results that are similar to those of Section 5 but which are applicable to the setting of Riesz kernels.

9.3 The general case

Recall the random fields $U^{(\beta)}$, $V^{(\beta)}$, and $Y^{(\beta)}$ respectively from (5.2), (5.6), and (5.7). We begin by studying the nonlinear problem (PAM) in the presence of noise whose spatial correlation is determined by $f(x) = \text{const}\cdot\|x\|^{-\alpha}$.

Proposition 9.6. *Let u denote the solution to* (PAM)*. For every $T > 0$ and $\eta \in (0\,,1\wedge\alpha)$ there exist finite and positive constants $\ell_i := \ell_i(d\,,\alpha\,,T,\varkappa\,,\eta)$ $[i = 1,2]$, such that uniformly for $\beta > 0$ and $k \geqslant 2$,*

$$\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}^d}\mathrm{E}\left(\left|u_t(x) - U_t^{(\beta)}(x)\right|^k\right) \leqslant \left(\frac{\ell_2 k}{\beta^\eta}\right)^{k/2}\mathrm{e}^{\ell_1 k^{(4-\alpha)/(2-\alpha)}}.\tag{9.19}$$

Proof. Notice that

$$\begin{aligned}V_t^{(\beta)}(x) &= (p_t * u_0)(x) + \left(p * U^{(\beta)}F^{(h)}\right)_t(x),\\ Y_t^{(\beta)}(x) &= (p_t * u_0)(x) + \left(p * U^{(\beta)}F^{(h_\beta)}\right)_t(x).\end{aligned}\tag{9.20}$$

Proposition 9.4 tells us that for all $\eta \in (0\,,1\wedge\alpha)$,

$$\mathcal{M}_\delta^{(k)}\left(V^{(\beta)} - Y^{(\beta)}\right) \leqslant C_1\cdot\sqrt{\frac{k}{\beta^\eta\cdot\delta^{(2-\alpha+\eta)/2}}}\,\mathcal{M}_\delta^{(k)}(U^{(\beta)}),\tag{9.21}$$

where C_1 is a positive and finite constant that depends only on $(d\,,\varkappa\,,\alpha\,,\eta)$. It follows from the definition (1.5) that

$$\mathcal{M}_\delta^{(k)}\left(V^{(\beta)} - Y^{(\beta)}\right) \leqslant \text{const}\cdot\sqrt{\frac{k\varkappa^{(2-\alpha)/2}}{\beta^\eta\cdot\delta^{(2-\alpha+\eta)/2}}}\mathcal{M}_\delta^{(k)}(U^{(\beta)}),\tag{9.22}$$

where "const" depends only on $(d\,,\varkappa\,,\alpha\,,\eta)$. In order to estimate the latter $\mathcal{M}_\delta^{(k)}$-norm we mimic the proof of the first inequality in Proposition 9.1 to see that, for the same constant $\bar{c}$ as in the latter proposition, $\log\|U_t^{(\beta)}(x)\|_k \leqslant \overline{u}_0 + \bar{c}tk^{2/(2-\alpha)}/\varkappa^{\alpha/(2-\alpha)}$ uniformly for all $x\in\mathbf{R}^d$, $t,\varkappa,\beta>0$, and $k\geqslant 2$. We omit the lengthy details because they involve making only small changes to the proof of the second inequality in Proposition 9.1. The end result is that

$$\mathcal{M}_\delta^{(k)}(U^{(\beta)}) \leqslant \sup_{t>0}\left[\overline{u}_0\exp\left\{-\delta t+\bar{c}t\frac{k^{2/(2-\alpha)}}{\varkappa^{\alpha/(2-\alpha)}}\right\}\right] = \overline{u}_0, \tag{9.23}$$

provided that

$$\delta > \bar{c}k^{2/(2-\alpha)}/\varkappa^{\alpha/(2-\alpha)}. \tag{9.24}$$

Therefore, the following is valid whenever δ satisfies (9.24):

$$\mathcal{M}_\delta^{(k)}\left(V^{(\beta)}-Y^{(\beta)}\right) \leqslant C_1\cdot\sqrt{\frac{k}{\beta^\eta\cdot\delta^{(2-\alpha+\eta)/2}}}, \tag{9.25}$$

where C_1 depends only on $(d\,,\varkappa\,,\alpha\,,\eta\,,\sigma(0)\,,\mathrm{Lip}_\sigma,\,\overline{u}_0)$.

In order to bound $\|Y_t^{(\beta)}(x)-U_t^{(\beta)}(x)\|_k$, we apply (BDG) and deduce that

$$\begin{aligned}
&\mathrm{E}\left(|Y_t^{(\beta)}(x)-U_t^{(\beta)}(x)|^k\right) &(9.26)\\
&\leqslant \mathrm{E}\left(\left|4k\int_0^t\mathrm{d}s\int_{[x-\beta\sqrt{t},x+\beta\sqrt{t}]^c}\mathrm{d}y\int_{[x-\beta\sqrt{t},x+\beta\sqrt{t}]^c}\mathrm{d}z\ h_\beta^{(*2)}(z-y)\mathcal{W}\right|^{k/2}\right)\\
&\leqslant\left(4k\int_0^t\mathrm{d}s\int_{[x-\beta\sqrt{t},\,x+\beta\sqrt{t}]^c}\mathrm{d}y\int_{[x-\beta\sqrt{t},\,x+\beta\sqrt{t}]^c}\mathrm{d}z\ f(z-y)\|\mathcal{W}\|_{k/2}\right)^{k/2},
\end{aligned}$$

where we have used Minkowski's inequality in the last bound. Here, $h_\beta^{(*2)} := h_\beta * \tilde{h}_\beta$, and $\mathcal{W} := p_{t-s}(y-x)p_{t-s}(z-x)|U_s^{(\beta)}(y)|\cdot|U_s^{(\beta)}(z)|$. In particular,

$$\|\mathcal{W}\|_{k/2} \leqslant p_{t-s}(y-x)p_{t-s}(z-x)\sup_{y\in\mathbf{R}^d}\left\|U_s^{(\beta)}(y)\right\|_k^2, \tag{9.27}$$

thanks to the Cauchy–Schwarz inequality. By the definition (1.5) of $\mathcal{M}_\delta^{(k)}$,

$$\sup_{w\in\mathbf{R}^d}\left\|U_s^{(\beta)}(w)\right\|_k \leqslant \mathrm{e}^{\delta s}\mathcal{M}_\delta^{(k)}\left(U^{(\beta)}\right)\qquad\text{for all } s>0. \tag{9.28}$$

Therefore,

$$\|\mathcal{W}\|_{k/2} \leqslant \text{const}\cdot\mathrm{e}^{2\delta s}p_{t-s}(y-x)p_{t-s}(z-x)\mathcal{M}_\delta^{(k)}(U^{(\beta)})^2. \tag{9.29}$$

Let us define

$$\Theta := \int_0^t\mathrm{d}s\iint_{\mathcal{A}\times\mathcal{A}}\mathrm{d}y\,\mathrm{d}z\quad f(z-y)\mathrm{e}^{2\delta s}p_{t-s}(y-x)p_{t-s}(z-x), \tag{9.30}$$

where we have written $\mathcal{A} := [x-\beta\sqrt{t}\,, x+\beta\sqrt{t}]^c$, for the sake of typographical ease. Our discussion so far implies that

$$\left\| Y_t^{(\beta)}(x) - U_t^{(\beta)}(x)\right\|_k \leqslant \text{const}\cdot\sqrt{k}\ \mathcal{M}_\delta^{(k)}(U^{(\beta)})\cdot\Theta^{1/2}. \tag{9.31}$$

We may estimate Θ as follows:

$$\begin{aligned}\Theta &\leqslant \int_0^t \sup_{w\in\mathbf{R}^d}(p_{t-s}*f)(w)\mathrm{e}^{2\delta s}\,\mathrm{d}s\int_{[x-\beta\sqrt{t},x+\beta\sqrt{t}]^c} p_{t-s}(y-x)\,\mathrm{d}y\\ &\leqslant \text{const}\cdot\int_0^t \sup_{w\in\mathbf{R}^d}(p_{t-s}*f)(w)\exp\left(-\frac{d\beta^2 t}{4\varkappa(t-s)}+2\delta s\right)\,\mathrm{d}s,\end{aligned} \tag{9.32}$$

where we used (5.13) and "const" depends only on $(d\,,\alpha)$. Because $p_{t-s}*f$ is a continuous positive-definite function, it is maximized at the origin. Thus, by scaling,

$$\sup_{w\in\mathbf{R}^d}(p_{t-s}*f)\,(w)\leqslant\frac{\text{const}}{(t-s)^{\alpha/2}\varkappa^{\alpha/2}}, \tag{9.33}$$

where "const" depends only on $(d\,,\alpha)$. Consequently,

$$\begin{aligned}\Theta &\leqslant \frac{\text{const}}{\varkappa^{\alpha/2}}\cdot\int_0^t\exp\left(-\frac{d\beta^2 t}{4\varkappa(t-s)}+2\delta s\right)\frac{\mathrm{d}s}{(t-s)^{\alpha/2}}\\ &\leqslant\frac{\text{const}}{\varkappa^{\alpha/2}}\cdot\mathrm{e}^{2\delta t}\,t^{(2-\alpha)/2}\int_0^1\mathrm{e}^{-d\beta^2/(4\varkappa s)}\,\frac{\mathrm{d}s}{s^{\alpha/2}}\leqslant\frac{\text{const}}{\varkappa^{\alpha/2}}t^{(2-\alpha)/2}\exp\left(2\delta t-\frac{d\beta^2}{4\varkappa}\right).\end{aligned} \tag{9.34}$$

It follows from the preceding discussion and (9.31) that

$$\mathcal{M}_\delta^{(k)}\left(Y^{(\beta)}-U^{(\beta)}\right)\leqslant\frac{\text{const}}{\varkappa^{\alpha/4}}\cdot\mathrm{e}^{-d\beta^2/(8\varkappa)}\sqrt{k}, \tag{9.35}$$

provided that δ satisfies (9.24).

Next we note that

$$\begin{aligned}&\|u_t(x)-V_t^{(\beta)}(x)\|_k\\ &\leqslant\left\|\int_{(0,t)\times\mathbf{R}^d}p_{t-s}(y-x)\left[u_s(y)-U_s^{(\beta)}(y)\right]F^{(h)}(\mathrm{d}s\,\mathrm{d}y)\right\|_k\\ &\leqslant\text{const}\cdot\sqrt{k\int_0^t\mathrm{d}s\int_{\mathbf{R}^d}\mathrm{d}y\int_{\mathbf{R}^d}\mathrm{d}z\ f(y-z)\widetilde{\mathcal{T}}},\end{aligned} \tag{9.36}$$

where

$$\begin{aligned}\widetilde{\mathcal{T}} &:= p_{t-s}(y-x)p_{t-s}(z-x)\left\|u_s(z)-U_s^{(\beta)}(z)\right\|_k\cdot\left\|u_s(y)-U_s^{(\beta)}(y)\right\|_k\\ &\leqslant p_{t-s}(y-x)p_{t-s}(z-x)\sup_{y\in\mathbf{R}^d}\left\|u_s(y)-U_s^{(\beta)}(y)\right\|_k^2.\end{aligned} \tag{9.37}$$

We then obtain

$$\|u_t(x) - V_t^{(\beta)}(x)\|_k \leqslant \text{const} \cdot \left(k \int_0^t \frac{\sup_y \|u_s(y) - U_s^{(\beta)}(y)\|_k^2}{((t-s)\varkappa)^{\alpha/2}}\, \mathrm{d}s \right)^{1/2}, \tag{9.38}$$

from similar calculations as before; see the derivation of (9.33). Consequently,

$$\begin{aligned}\mathcal{M}_\delta^{(k)}\left(u - V^{(\beta)}\right) &\leqslant \text{const} \cdot k^{1/2} \mathcal{M}_\delta^{(k)}\left(u - U^{(\beta)}\right) \left(\int_0^\infty \frac{\mathrm{e}^{-2\delta r}}{(\varkappa r)^{\alpha/2}}\, \mathrm{d}r \right)^{1/2} \\ &= \frac{\text{const} \cdot k^{1/2}}{\varkappa^{\alpha/4} \delta^{(2-\alpha)/4}} \mathcal{M}_\delta^{(k)}\left(u - U^{(\beta)}\right).\end{aligned} \tag{9.39}$$

Next we apply the decomposition (5.5) and the bounds in (9.39), (9.35), and (9.25) to see that

$$\begin{aligned}&\mathcal{M}_\delta^{(k)}\left(u - U^{(\beta)}\right) \\ &\leqslant \mathcal{M}_\delta^{(k)}\left(u - V^{(\beta)}\right) + \mathcal{M}_\delta^{(k)}\left(V^{(\beta)} - Y^{(\beta)}\right) + \mathcal{M}_\delta^{(k)}\left(Y^{(\beta)} - U^{(\beta)}\right) \\ &\leqslant \frac{\text{const} \cdot k^{1/2}}{\varkappa^{\alpha/4} \delta^{(2-\alpha)/4}} \mathcal{M}_\delta^{(k)}\left(u - U^{(\beta)}\right) + C_1 \cdot \sqrt{\frac{k}{\beta^\eta \cdot \delta^{(2-\alpha+\eta)/2}}} + \frac{\text{const} \cdot k^{1/2}}{\varkappa^{\alpha/4}} \mathrm{e}^{-d\beta^2/(8\varkappa)}.\end{aligned} \tag{9.40}$$

We now choose $\delta := Ck^{2/(2-\alpha)}/\varkappa^{\alpha/(2-\alpha)}$ with $C > \bar{c}$ so large that the coefficient of $\mathcal{M}_\delta^{(k)}(u - U^{(\beta)})$ in the preceding bound, is smaller that $1/2$. Because δ has a lower bound that holds uniformly for all $k \geqslant 1$, the preceding implies that

$$\mathcal{M}_\delta^{(k)}\left(u - U^{(\beta)}\right) \leqslant \text{const} \cdot \sqrt{k} \left[\beta^{-\eta/2} + \mathrm{e}^{-d\beta^2/(8\varkappa)} \right] \leqslant \text{const} \cdot \sqrt{k}\, \beta^{-\eta/2}, \tag{9.41}$$

which has the desired result. □

Recall the nth level Picard-iteration approximation $U_t^{(\beta,n)}(x)$ of $U_t^{(\beta)}(x)$ defined in (5.18). The next two lemmas are the Picard-iteration analogues of Lemmas 5.3 and 5.4.

Lemma 9.7. *For every $T > 0$ and $\eta \in (0, 1 \wedge \alpha)$ there exist finite and positive constants $\ell_i := \ell_i(d, \alpha, T, \varkappa, \eta, \sigma)$ $[i = 1, 2]$, such that uniformly for $\beta > 0$ and $k \geqslant 2$*

$$\sup_{t \in [0,T]} \sup_{x \in \mathbf{R}^d} \mathrm{E}\left(\left| u_t(x) - U_t^{(\beta,\, [\log \beta]+1)}(x) \right|^k \right) \leqslant \left(\frac{\ell_2 k}{\beta^\eta} \right)^{k/2} \mathrm{e}^{\ell_1 k^{(4-\alpha)/(2-\alpha)}}. \tag{9.42}$$

Lemma 9.8. *Choose and fix $\beta \geqslant 1$, $t > 0$ and $n \geqslant 1$. Also fix $x^{(1)}, x^{(2)}, \cdots \in \mathbf{R}^d$ such that $D(x^{(i)}, x^{(j)}) \geqslant 2n\beta(1 + \sqrt{t})$. Then $\{U_t^{(\beta,n)}(x^{(j)})\}_{j \in \mathbf{Z}}$ are independent random variables.*

We will skip the proofs, as they are entirely similar to the respective proofs of Lemmas 5.3 and 5.4, but apply the method of proof of Proposition 9.6 in place of Lemma 5.2.

9.4 Proof of Theorem 2.6

The proof of this theorem is similar to that of Theorem 2.5. Thanks to Proposition 9.1 and [10, Lemma 3.4], we have the following: There exist positive and finite constants $a < b$, independently of $\varkappa > 0$, such that for all $x \in \mathbf{R}^d$ and $\lambda > \mathrm{e}$,

$$a\mathrm{e}^{-b(\log\lambda)^{(4-\alpha)/2}\varkappa^{\alpha/2}} \leqslant \mathrm{P}\left\{|u_t(x)| \geqslant \lambda\right\} \leqslant b\mathrm{e}^{-a(\log\lambda)^{(4-\alpha)/2}\varkappa^{\alpha/2}}. \tag{9.43}$$

Define, for the sake of typographical ease,

$$\mathcal{E}_M := \mathcal{E}_{M,\varkappa}(R) := \exp\left(\frac{M\cdot(\log R)^{2/(4-\alpha)}}{\varkappa^{\alpha/(4-\alpha)}}\right) \qquad \text{for all } M > 0. \tag{9.44}$$

For the lower bound, we again choose N points $x^{(1)}, \ldots, x^{(N)}$ such that $D(x^{(i)}\,, x^{(j)}) \geqslant 2n\beta(1+\sqrt{t})$ whenever $i \neq j$; see (5.24) for the definition of $D(x\,,y)$. Let $n := [\log\beta]+1$ and choose and fix $\eta \in (0\,,1\wedge\alpha)$. We apply Proposition 9.6 and the independence of the $U^{(\beta,n)}(x^{(j)})$'s (Lemma 9.8) to see that

$$\begin{aligned}
&\mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u_t(x^{(j)})| < \mathcal{E}_M\right\}\\
&\leqslant \left(1-\mathrm{P}\left\{|U_t^{(\beta,n)}(x^{(1)})| \geqslant 2\mathcal{E}_M\right\}\right)^N + \text{const}\cdot\frac{N}{\beta^{k\eta/2}\mathcal{E}_M^k} \qquad (9.45)\\
&\leqslant \left(1-\left[\mathrm{P}\left\{|u_t(x^{(1)})| \geqslant 3\mathcal{E}_M\right\} - \text{const}\cdot\frac{N}{\beta^{k\eta/2}}\right]\right)^N + \text{const}\cdot\frac{N}{\beta^{k\eta/2}},
\end{aligned}$$

since $\mathcal{E}_M$ is large for R sufficently large. Notice that the implied constants depend on $(\varkappa\,,t\,,k\,,d,\alpha\,,\eta\,,\sigma)$. Now we choose the various parameters involved [and in this order]: Choose and fix some $\nu \in (0\,,1)$, and then set $N := \lceil R^\nu\rceil^d$ and $\beta := R^{1-\nu}$. The following is valid for all $M > 0$ sufficiently small, every k sufficiently large, and for the mentioned choices of N and β:

$$\mathrm{P}\left\{\max_{1\leqslant j\leqslant N}|u_t(x^{(j)})| < \mathcal{E}_M\right\} \leqslant \text{const}\cdot R^{-2}. \tag{9.46}$$

Borel-Cantelli Lemma and a simple monotonicity argument together yield the bound,

$$\liminf_{R\to\infty}\frac{\log u_t^*(R)}{(\log R)^{2/(4-\alpha)}} > \frac{C}{\varkappa^{\alpha/(4-\alpha)}} \qquad \text{a.s.}, \tag{9.47}$$

where C does not depend on $\varkappa$. For the other bound, we start with a modulus of continuity estimate, viz.,

$$\|u_t(x)-u_t(y)\|_{2k} \leqslant \text{const}\cdot\|x-y\| + \left(8k\int_0^t \sup_{a\in\mathbf{R}^d}\|u_s(a)\|_{2k}^2\ \mathcal{I}_s\,\mathrm{d}s\right)^{1/2}, \tag{9.48}$$

where $\mathcal{I}_s := \iint_{\mathbf{R}^d\times\mathbf{R}^d}\mathrm{d}w\,\mathrm{d}z\ |\mathcal{H}(w)\mathcal{H}(z)|f(w-z)$, for $\mathcal{H}(\xi) := p_{t-s}(\xi-x) - p_{t-s}(\xi-y)$ for all $\xi \in \mathbf{R}^d$. Because of Proposition 9.1, we can simplify our estimate to the following:

$$\|u_t(x)-u_t(y)\|_{2k} \leqslant \text{const}\cdot\|x-y\| + \overline{u}_0 \mathrm{e}^{2\overline{c}t(2k)^{2/(2-\alpha)}\varkappa^{-\alpha/(2-\alpha)}}\left(8k\int_0^t \mathcal{I}_s\,\mathrm{d}s\right)^{1/2}. \tag{9.49}$$

The simple estimate $\int_{\mathbf{R}^d}|\mathcal{H}(z)|f(w-z)\,\mathrm{d}z \leqslant 2\sup_{z\in\mathbf{R}^d}(p_{t-s}*f)(z)$, together with (9.33) yields

$$\mathcal{I}_s \leqslant \frac{\text{const}}{(t-s)^{\alpha/2}\varkappa^{\alpha/2}}\cdot\int_{\mathbf{R}^d}|\mathcal{H}(w)|\,\mathrm{d}w \leqslant \frac{\text{const}}{(t-s)^{\alpha/2}}\cdot\left(\frac{\|x-y\|}{(t-s)^{1/2}}\wedge 1\right), \tag{9.50}$$

where "const" does not depend on $(x\,,y\,,s\,,t)$, but might depend on $\varkappa$; see Lemma 6.4 for the last inequality. These remarks, and some computations together show that, uniformly for all $x,y\in\mathbf{R}^d$ that satisfy $\|x-y\|\leqslant 1\wedge t^{1/2}$, $\mathrm{E}(|u_t(x)-u_t(y)|^{2k})\leqslant C\|x-y\|^{\varpi k}$, where $C:=C(k\,,\varkappa\,,t\,,d\,,\alpha)$ is positive and finite and $\varpi=\min(1,2-\alpha)$. Now a quantitative form of the Kolmogorov continuity theorem [13, (39), p. 11] tells us that uniformly for all hypercubes $T\subset\mathbf{R}^d$ of side length $\leqslant d^{-1/2}(1\wedge t^{1/2})$, and for all $\delta\in(0\,,1\wedge t^{1/2})$,

$$\mathrm{E}\left(\sup_{\substack{x,y\in T\\ \|x-y\|\leqslant\delta}}|u_t(x)-u_t(y)|^{2k}\right)\leqslant \text{const}\cdot(\delta^{\varpi}k)^k\exp\left(\frac{\overline{c}t(2k)^{(4-\alpha)/(2-\alpha)}}{\varkappa^{\alpha/(2-\alpha)}}\right), \tag{9.51}$$

where "const" depends only on $(\varkappa\,,t\,,d\,,\alpha)$. We now split $[0\,,R]^d$ into subcubes of sidelength $\text{const}\cdot(1\wedge t^{1/2})$, each of which is contained in a ball of radius $(1\wedge t^{1/2})/2$. Let $\mathcal{C}_R$ denote the collection of mentioned subcubes and $\mathcal{M}_R$, the set of midpoints of these subcubes. We can then observe the following:

$$\mathrm{P}\left\{u_t^*(R)>2\mathcal{E}_M\right\}\leqslant\sum_{x\in\mathcal{M}_R}\mathrm{P}\left\{|u_t(x)|>\mathcal{E}_M\right\}+\sum_{T\in\mathcal{C}_R}\mathrm{P}\left\{\operatorname*{Osc}_T(u_t)>\mathcal{E}_M\right\}, \tag{9.52}$$

where $\mathrm{Osc}_T(g):=\sup_{x,y\in T}|g(x)-g(y)|$, and c depends only on $(t\,,d)$. In this way we find that

$$\mathrm{P}\left\{u_t^*(R)>2\mathcal{E}_M\right\}\leqslant AR^d\times\left[\frac{\mathrm{e}^{Ak^{(4-\alpha)/(2-\alpha)}\varkappa^{-\alpha/(2-\alpha)}}}{\mathrm{e}^{Mk(\log R)^{2/(4-\alpha)}\varkappa^{-\alpha/(4-\alpha)}}}\right], \tag{9.53}$$

where $A\in(0\,,\infty)$ is a constant that depends only on $(t\,,\varkappa\,,\alpha\,,d)$. Finally, we choose $k:=\varkappa^{\alpha/(4-\alpha)}(\log R)^{(2-\alpha)/(4-\alpha)}$ and M large enough to ensure that $\mathrm{P}\{u_t^*(R)>2\mathcal{E}_M\}=O(R^{-2})$ as $R\to\infty$. An application of Borel-Cantelli lemma proves the result. □

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References

[1] Amir, Gideon, Ivan Corwin, and Jeremy Quastel, Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions, *Comm. Pure Appl. Math.* **64** (2011) 466–537.

[2] Balázs, Marton, Jeremy Quastel, and Timo Seppäläinen, Scaling exponent for the Hopf–Cole solution of KPZ/stochastic Burgers, *Preprint* (2011).

[3] Bertini, Lorenzo and Cancrini, Nicoletta, The stochastic heat equation: Feynman–Kac formula and intermittence, *J. Statist. Physics* **78***(5/6)* (1994) 1377–1402.

[4] Burkholder, D. L., Martingale transforms, *Ann. Math. Statist.* **37** (1966) 1494–1504.

[5] Burkholder, D. L., Davis, B. J., and Gundy, R. F., Integral inequalities for convex functions of operators on martingales, In: *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability* II, 223–240, University of California Press, Berkeley, California, 1972.

[6] Burkholder, D. L. and Gundy, R. F., Extrapolation and interpolation of quasi-linear operators on martingales, *Acta Math.* **124** (1970), 249–304.

[7] Carmona, René A. and Molchanov, S. A., Parabolic Anderson Problem and Intermittency, *Memoires of the Amer. Math. Soc.* **108**, American Mathematical Society, Rhode Island, 1994.

[8] Carlen, Eric and Kree, Paul, L^p estimates for multiple stochastic integrals, *Ann. Probab.* **19***(1)* (1991) 354–368.

[9] Conus, Daniel, Moments for the Parabolic Anderson Model, an alternate proof to a result by Hu and Nualart. (preprint, 2011).

[10] Conus, Daniel, Joseph, Mathew, and Khoshnevisan, Davar, On the chaotic character of the stochastic heat equation, before the onset of intermittency, *Ann. Probab.* (to appear). Available electronically at `http://arxiv.org/abs/1104.0189`).

[11] Conus, Daniel, and Khoshnevisan, Davar, On the existence and position of the farthest peaks of a family of stochastic heat and wave equations, *Probab. Th. Related Fields* (to appear).

[12] Dalang, Robert C., Extending the martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e.'s, *Electron. J. Probab.* **4**, Paper no. 6 (1999) 29 pp. (electronic). Available electronically at `http://www.math.washington.edu/~ejpecp`.

[13] Dalang, Robert, Khoshnevisan, Davar, Mueller, Carl, Nualart, David, and Xiao, Yimin, *A Minicourse in Stochastic Partial Differential Equations* (2006). In: Lecture Notes in Mathematics, vol. 1962 (D. Khoshnevisan and F. Rassoul–Agha, editors) Springer–Verlag, Berlin, 2009

[14] Davis, Burgess, On the L^p norms of stochastic integrals and other martingales, *Duke Math. J.* **43***(4)* (1976) 697–704.

[15] Dellacherie, Claude and Meyer, Paul-André, *Probabilities and Potential, B. Theory of Martingales*, (translated from the French by J. P. Wilson), North-Holland Publishing Co., Amsterdam, 1982.

[16] Durrett, Richard, *Lecture Notes on Particle Systems and Percolation*, Wadsworth & Brooks Cole, Pacific Grove, CA, 1998.

[17] Foondun, Mohammud and Khoshnevisan, Davar, On the stochastic heat equation with spatially-colored random forcing, *Transactions of the Amer. Math. Soc.* (to appear). Available electronically at `http://arxiv.org/abs/1003.0348`

[18] Foondun, Mohammud and Khoshnevisan, Davar, On the global maximum of the solution to a stochastic heat equation with compact-support initial data, *Ann. Instit. Henri Poinc.* (to appear). Available electronically at `http://arxiv.org/abs/0901.3814`.

[19] Foondun, Mohammud and Khoshnevisan, Davar, Intermittence and nonlinear parabolic stochastic partial differential equations, *Electr. J. Probab.* **14**, Paper no. 12 (2009) 548–568 (electronic). Available electronically at `http://www.math.washington.edu/~ejpecp`.

[20] Gel′fand, I. M. and Vilenkin, N. Ya., *Generalized Functions*, vol. 4, (Applications of harmonic analysis, Translated from the Russian by Amiel Feinstein), Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977].

[21] Hu, Yaozhong, and David Nualart, Stochastic heat equation driven by fractional noise and local time, *Probab. Theory Related Fields* **143** (2009) 285–328.

[22] Kardar, Mehran, Giorgio Parisi, and Yi-Cheng Zhang, Dynamic scaling of growing interfaces, *Phys. Rev. Lett.* **56***(9)* (1985) 889–892.

[23] Kardar, Mehran and Yi-Cheng Zhang, Scaling of directed polymers in random media, *Phys. Rev. Lett.* **58***(20)* (1987) 2087–2090.

[24] Liggett, Thomas M., *Interacting Particle Systems*, Springer–Verlag, New York, 1985.

[25] Mattila, Pertti, *Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability* , Cambridge Studies in Advanced Mathematics, 1999.

[26] Peszat, Szymon and Jerzy Zabczyk, Nonlinear stochastic wave and heat equations, *Probab. Th. Rel. Fields* **116***(3)* (2000) 421–443.

[27] Port, Sidney C. and Stone, Charles J., *Brownian Motion and Classical Potential Theory*, Academic Press, New York, 1978.

[28] Walsh, John B., *An Introduction to Stochastic Partial Differential Equations*, in: École d'été de probabilités de Saint-Flour, XIV—1984, 265–439, Lecture Notes in Math., vol. 1180, Springer, Berlin, 1986.

Daniel Conus, Department of Mathematics, Lehigh University, Bethlehem, PA 18015.

Mathew Joseph and Davar Khoshnevisan,
Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090

Shang-Yuan Shiu, Institute of Mathematics, Academia Sinica, Taipei 10617

Emails & URLs:

`daniel.conus@lehigh.edu`,	`http://www.lehigh.edu/~dac311/`
`joseph@math.utah.edu`,	`http://www.math.utah.edu/~joseph/`
`davar@math.utah.edu`	`http://www.math.utah.edu/~davar/`
`shiu@math.sinica.edu.tw`	