THE ANNALS OF APPLIED PROBABILITY, **9**, 629–667 (1999)

# Stochastic Calculus for Brownian Motion on a Brownian Fracture

By Davar Khoshnevisan* & Thomas M. Lewis
University of Utah & Furman University

Abstract. The impetus behind this work is a pathwise development of stochastic integrals with respect to iterated Brownian motion. We also provide a detailed analysis of the variations of iterated Brownian motion. These variations are linked to Brownian motion in random scenery and iterated Brownian motion itself.

**Keywords and Phrases.** Iterated Brownian Motion, Brownian Motion in Random Scenery, Stochastic integration, Sample–path Variations, Excursion Theory.

**1991 AMS Subject Classification.** Primary. 60H05; Secondary. 60Kxx, 60Fxx

**Short Title.** Stochastic Calculus for IBM

* Research supported by grants from the National Science Foundation and the National Security Agency

THE ANNALS OF APPLIED PROBABILITY, **9**, 629–667 (1999)

# 1. Introduction and Preliminaries

Heat flow on a fractal $\mathfrak{F}$ has been the subject of recent and vigorous investigations. See, for example, the survey article [2]. As in the more classical studies of heat flow on smooth manifolds (cf. [16]), a probabilistic interpretation of such problems comes from the description and analysis of "the canonical stochastic process" on $\mathfrak{F}$, which is usually called Brownian motion on $\mathfrak{F}$. One of the many areas of applications is heat flow along fractures. In this vein, see [14,17,20,21,22,26,39] These articles start with an idealized fracture (usually a simple geometric construct such as a comb) and proceed to the construction and analysis of Brownian motion on this fracture. Let us begin by attacking the problem from a different point of view. Namely, rather than considering a fixed idealized fracture, we begin with the following random idealization of a fracture: we assume that $\mathfrak{R}$ is a vertically homogeneous, two–dimensional rectangular medium with sides parallel to the axes. Then the our left–to–right random fracture $\mathfrak{R}$ looks like the graph of a one–dimensional Brownian motion. (To make this more physically sound, one needs some mild conditions on the local growth of the fracture together with the invariance principle of Donsker; see [6].) Approximating the Brownian graph by random walks and once again applying Donsker's invariance principle ([6]), it is reasonable to suppose that Brownian motion on a Brownian fracture is described by $(Y_t, Z_t)$, where $Y$ is a one–dimensional Brownian motion and $Z$ is the iterated Brownian motion built from $Y$. To construct $Z$, let $X^{\pm}$ be two independent one–dimensional Brownian motions which are independent of $Y$, as well (throughout this paper, we assume that all Brownian motions start at the origin). Let $X$ be the two–sided Brownian motion given by

$$X_t = \begin{cases} X^+(t), & \text{if } t \geqslant 0 \\ X^-(-t), & \text{if } t < 0. \end{cases}$$

Iterated Brownian motion $Z$ can be defined as

$$Z_t = X(Y_t).$$

As is customary, given a function $f$ (random or otherwise), we freely interchange between $f(t)$ and $f_t$ for typographical ease or for reasons of aesthetics.

The above model for Brownian motion on a Brownian fracture appears earlier (in a slightly different form) in [13]. Our model is further supported by the results of [11]. There, it is shown that iterated Brownian motion arises naturally as the (weak) limit of reflected Brownian motion in an infinitesimal fattening of the graph of a Brownian motion.

Recently iterated Brownian motion and its variants have been the subject of various works; see [1,4,5,8,9,10,11,13,15,23,24,25,29,30,38,40]. In addition to its relation to heat flow on fractures, iterated Brownian motion has a loose connection with the parabolic operator $\frac{1}{8}\Delta^2 - \partial/\partial t$; see [19] for details.

In this paper, we are concerned with developing a stochastic calculus for $Z$. It is not surprising that the key step in our analysis is a construction of stochastic integral processes of form $\int_0^t f(Z_s)dZ_s$, where $f$ is in a "nice" family of functions. Since $Z$ is not a semi-martingale, such a construction is necessarily non–trivial. (A folk theorem of C. Dellacherie essentially states that for $\int HdM$ to exist as an "integral" for a large class of $H$'s, $M$ need necessarily be a semi–martingale.) Our construction of $\int_0^t f(Z_s)dZ_s$ is reminiscent of the integrals of Stratonovich and Lebesgue. More precisely, for each nonnegative integer $n$, we divide space into an equipartition of mesh size $2^{-n/2}$. According to the times at which the Brownian motion $Y$ is in this partition, one obtains an induced random partition $\{T_{k,n}; 1 \leqslant k \leqslant 2^n t\}$ of the time interval $[0, t]$. One of the useful features of this random partition is that it uniformly approximates the more commonly used dyadic partition $\{k2^{-n}; 1 \leqslant k \leqslant 2^n t\}$. Having developed the partition, we show that

$$\int_0^t f(Z_s)dZ_s = \lim_{n\to\infty} \sum_{1 \leqslant k \leqslant 2^n t} f\Big(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\Big) \cdot \big(Z(T_{k+1,n}) - Z(T_{k,n})\big)$$

exists with probability one and can be explicitly identified in terms of other (better understood) processes. This material is developed in §2. The use of the midpoint rule in defining the stochastic integral is significant. The midpoint rule is a symmetric rule, and symmetry will play an important role in our analysis. As we will show later in this section, the analogous partial sum process based on the right–hand rule does not converge.

Based on Donsker's invariance principle, we have already argued that iterated Brownian motion is a reasonable candidate for the canonical process on a Brownian fracture. This viewpoint is further strengthened by our results in the remainder of this paper which are concerned with the variations of iterated Brownian motion. To explain these results, define — for smooth functions $f$,

$$V_n^{(j)}(f,t) = \sum_{1\leqslant k\leqslant 2^n t} f\Big(\frac{Z(T_{k+1,n})+Z(T_{k,n})}{2}\Big)\cdot \big(Z(T_{k+1,n})-Z(T_{k,n})\big)^j, \qquad (j=1,2,3,4)$$

When $f\equiv 1$, we will write $V_n^{(j)}(t)$ for $V_n^{(j)}(1,t)$, which we call the $j$–th **variation** of $Z$. A more traditional definition of the variation of iterated Brownian motion has been studied in [9]. In §3 and §4 we extend the results of [9] along the random partitions $\{T_{k,n}\}$. In fact, we prove that with probability one, for a nice function $f$,

$$\lim_{n\to\infty} 2^{-n/2} V_n^{(2)}(f,t) = \int_0^t f(Z_s)ds,$$
$$\lim_{n\to\infty} V_n^{(3)}(f,t) = 0,$$

and

$$\lim_{n\to\infty} V_n^{(4)}(f,t) = 3\int_0^t f(Z_s)ds.$$

Further refinements appear in the second–order analysis of these strong limit theorems. In essence, we show that appropriately normalized versions of $V_n^{(2)}(t) - 2^{n/2}t$ and $V_n^{(4)}(t)-3t$ converge in distribution to Kesten and Spitzer's Brownian motion in random scenery (see [27]), while an appropriately normalized version of $V_n^{(3)}(t)$ converges in distribution to iterated Brownian motion itself. Indeed, it can be shown that — after suitable normalizations — all even variations converge weakly to Brownian motion in random scenery while the odd variations converge weakly to iterated Brownian motion.

Our analysis of the variation of iterated Brownian motion indicates the failure of the right–hand rule in defining the stochastic integral. If $f$ is sufficiently smooth and has enough bounded derivatives, then, by Taylor's theorem, we have

$$\sum_{1\leqslant k\leqslant 2^n t} f\big(Z(T_{k,n})\big)\cdot\big(Z(T_{k+1,n})-Z(T_{k,n})\big)$$
$$= V_n^{(1)}(f,t) + \frac{1}{2}V_n^{(2)}(f',t) + \frac{1}{4}V_n^{(3)}(f'',t) + \frac{1}{12}V_n^{(4)}(f''',t) + o(1),$$

where $o(1)\to 0$ almost surely and in $L^2(\mathbb{P})$ as $n\to\infty$. It follows that

$$\lim_{n\to\infty}\left(\sum_{1\leqslant k\leqslant 2^n t} f\big(Z(T_{k,n})\big)\cdot\big(Z(T_{k+1,n})-Z(T_{k,n})\big) - \frac{1}{2}V_n^{(2)}(f',t)\right)$$
$$= \int_0^t f(Z_s)dZ_s + \frac{1}{4}\int_0^t f'''(Z_s)ds.$$

Consequently, the right–hand rule process will converge if and only if the associated quadratic variation process converges. However the quadratic variation process diverges whenever $f'$ is a positive function, to name an obvious, but by no means singular, example.

Our construction of $\int_0^t f(Z_s)dZ_s$ is performed pathwise and relies heavily on excursion theory for Brownian motion. It is interesting that a simplified version of our methods yields an excursion–theoretic construction of ordinary Itô integral processes of the type $\int_0^t f(Y_s)dY_s$ for Brownian motion $Y$ (see §5 for these results). While stochastic calculus treatments of excursion theory have been carried out in the literature (cf. [37]), ours appears to be the first attempt in the reverse direction.

A general pathwise approach to integration is carried out in [35]. This is based on a construction of Lévy–type stochastic areas. It would be interesting to see the connection between our results and those of [35].

We conclude this section by defining some notation which will be used throughout the paper. For any array $\{a_{i,n},\ j \in \mathbb{Z},\ n \geqslant 0\}$, we define $\Delta a_{j,n} = a_{j+1,n} - a_{j,n}$. Whenever a process $U$ has local times, we denote them by $L_t^x(U)$. This means that for any Borel function $f$ and all $t \geqslant 0$

$$\int_0^t f(U_s)ds = \int_{-\infty}^{\infty} f(a)L_t^a(U)da,$$

almost surely. We write $\mathbb{I}\{A\}$ for the indicator of a Borel set $A$. In other words, viewed as a random variable,

$$\mathbb{I}\{A\}(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \notin A. \end{cases}$$

Let $C^2(\mathbb{R})$ be the collection of all twice continuously differentiable functions, $f : \mathbb{R} \to \mathbb{R}$. By $C_b^2(\mathbb{R})$ we mean the collection of all $f \in C^2(\mathbb{R})$ such that $\|f\|_{C_b^2(\mathbb{R})} < \infty$, where

$$\|f\|_{C_b^2(\mathbb{R})} = \sup_x \big(|f(x)| + |f'(x)| + |f''(x)|\big). \tag{1.1}$$

It is easy to see that endowed with the norm $\|\cdots\|_{C_b^2(\mathbb{R})}$, $C_b^2(\mathbb{R})$ is a separable Banach space. For each integer $j$ and each nonnegative integer $n$, let $r_{j,n} = j2^{-n/2}$. Recalling that $X$ is a two–sided Brownian motion, we let

$$\begin{aligned} X_{j,n} &= X(r_{j,n}) \\ M_{j,n} &= \frac{X(r_{j+1,n}) + X(r_{j,n})}{2}. \end{aligned} \tag{1.2}$$

Finally, for any $p > -1$, $\mu_p$ will denote the absolute $p$–th moment of a standard normal distribution, that is,

$$\mu_p = (2\pi)^{-1/2} \int_{-\infty}^{\infty} |x|^p e^{-x^2/2} dx = \pi^{-1/2} 2^{p/2} \Gamma\Big(\frac{p+1}{2}\Big). \tag{1.3}$$

Acknowledgments. We thank Chris Burdzy for several interesting discussions on iterated Brownian motion. The presentation of this paper has been improved, largely due to the remarks of two anonymous referees to whom we wish to extend our thanks.

## 2. The Stochastic Integral

In this section we will define a stochastic integral with respect to iterated Brownian motion. For each $t > 0$, we will construct a sequence of partitions $\{T_{k,n},\ 0 \leqslant k \leqslant [2^n t]\}$ of the interval $[0,t]$ along which the partial sum process,

$$V_n^{(1)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right) \big(Z(T_{k+1,n}) - Z(T_{k,n})\big),$$

converges almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$, provided only that $f$ is sufficiently smooth. The limiting random variable is properly called the stochastic integral of $f(Z_s)$ with respect to $Z_s$ over the interval $[0,t]$ and will be denoted by $\int_0^t f(Z_s)dZ_s$. Our point of departure from the classical development of the stochastic integral is that the partitioning members $T_{k,n}$ are random variables, which we will define presently. For each integer $n \geqslant 0$ and each integer $j$, recall that $r_{j,n} = j2^{-n/2}$ and let

$$\mathcal{D}_n = \{r_{j,n},\, j \in \mathbb{Z}\}.$$

To define the elements of the $n$th partition, let $T_{0,n} = 0$ and, for each integer $k \geqslant 1$, let

$$T_{k,n} = \inf\{s > T_{k-1,n} :\ Y_s \in \mathcal{D}_n \setminus \{Y(T_{k-1,n})\}\}.$$

For future reference we observe that the process $\{Y(T_{k,n}),\, k \geqslant 0\}$ is a simple symmetric random walk on $\mathcal{D}_n$.

Here is the main result of this section.

**Theorem 2.1.** *Let $t > 0$ and $f \in C_b^2(\mathbb{R})$. Then*

$$V_n^{(1)}(f,t) \to \int_0^{Y_t} f(X_s)dX_s + \frac{1}{2}\mathrm{sgn}(Y_t)\int_0^{Y_t} f'(X_s)ds,$$

*almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.*

We have used the following natural definition for two–sided stochastic integrals:

$$\int_0^t f(X_s)dX_s = \begin{cases} \int_0^t f(X_s^+)dX_s^+, & \text{if } t \geqslant 0 \\ \int_0^{-t} f(X_s^-)dX_s^-, & \text{if } t < 0, \end{cases}$$

whenever the Itô integrals on the right exist.

**Remark 2.1.1.** For any $f \in C_b^2(\mathbb{R})$, define

$$\langle f, X\rangle(t) = \int_0^t f(X_s)dX_s + \frac{1}{2}\mathrm{sgn}(t)\int_0^t f'(X_s)ds.$$

Then $\{\langle f, X\rangle(t), t \in \mathbb{R}\}$ is the correct two–sided Stratonovich integral process of the integrand $f \circ X$. In the notation of §1, Theorem 2.1 asserts that

$$\int_0^t f(Z_s)dZ_s = \langle f, X\rangle(Y_t).$$

In other words, stochastic integration with respect to $Z$ is invariant under the natural composition map: $(X,Y) \mapsto Z$.

Before proceeding to the proof of Theorem 2.1, a few preliminary remarks and observations are in order. First we will demonstrate that the random partition $\{T_{k,n},\, k \in \mathbb{Z}\}$ approximates the dyadic partition $\{k/2^n,\, k \in \mathbb{Z}\}$ as $n$ tends to infinity.

**Lemma 2.2.** *Let $t > 0$. Then*

$$\sup_{0 \leqslant s \leqslant t} \left|T_{[2^n s],n} - s\right| \to 0$$

*almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.*

**Proof.** By the strong Markov property, $\{\Delta T_{j,n},\, j \geqslant 0\}$ is an i.i.d. sequence of random variables. Moreover, by Brownian scaling, $\Delta T_{1,n}$ has the same distribution as $2^{-n} T_{1,0}$. By Itô's formula, $\exp(\lambda Y_t - \lambda^2 t/2)$ is a mean-one martingale. Thus, by Doob's optional sampling theorem,

$$\mathbb{E}\big(\exp(-\lambda^2 T_{1,0}/2)\big) = \big(\cosh(\lambda)\big)^{-1}.$$

It follows that $\mathbb{E}(T_{1,0}) = 1$, $\mathbb{E}(T_{1,0}^2) = 5/3$; consequently, $\mathrm{var}(T_{1,0}) = 2/3$. Thus, by Brownian scaling,

$$\mathbb{E}(\Delta T_{1,n}) = 2^{-n} \quad \text{and} \quad \mathrm{var}(\Delta T_{1,n}) = \frac{2}{3} 2^{-2n}. \tag{2.1}$$

Given $0 \leqslant s \leqslant t$, we have

$$\begin{aligned}\sup_{0\leqslant s\leqslant t} \big|T_{[2^n s],n} - s\big| &\leqslant \sup_{0\leqslant s\leqslant t} \big|T_{[2^n s],n} - [2^n s]2^{-n}\big| + \sup_{0\leqslant s\leqslant t} \big|[2^n s]2^{-n} - s\big| \\ &\leqslant \max_{1\leqslant k\leqslant [2^n t]} |T_{k,n} - \mathbb{E}(T_{k,n})| + 2^{-n}.\end{aligned}$$

Since

$$T_{k,n} - \mathbb{E}(T_{k,n}) = \sum_{j=0}^{k-1} \big(\Delta T_{j,n} - \mathbb{E}(\Delta T_{j,n})\big),$$

we have, by Doob's maximal inequality and (2.1),

$$\begin{aligned}\mathbb{E}\left[\max_{1\leqslant k\leqslant [2^n t]} |T_{k,n} - \mathbb{E}(T_{k,n})|^2\right] &\leqslant 4 \sum_{j=0}^{[2^n t]-1} \mathrm{var}(\Delta T_{j,n}) \\ &= O(2^{-n}).\end{aligned}$$

In summary

$$\left\| \sup_{0\leqslant s\leqslant t} \big|T_{[2^n s],n} - s\big| \right\|_2 = O(2^{-n/2}), \tag{2.2}$$

which demonstrates the $L^2(\mathbb{P})$ convergence in question. The almost sure convergence follows from applications of Markov's inequality and the Borel-Cantelli lemma. ◇

For each $n \geqslant 0$, let

$$\begin{aligned}\tau_n &= \tau(n,t) = T_{[2^n t],n} \\ j^* &= j^*(n,t) = 2^{n/2} Y(\tau_n).\end{aligned}$$

In keeping with the notation that we have already developed, we have $r_{j^*,n} = Y(\tau_n)$.

**Lemma 2.3.** *Let $t > 0$. Then, as $n \to \infty$,*
(a) $\|Y(\tau_n) - Y(t)\|_2 = O(2^{-n/8})$;
(b) $\big\|\big(\mathrm{sgn}(Y(t)) - \mathrm{sgn}(Y(\tau_n))\big)\big(|Y(t)| + |Y(\tau_n)|\big)\big\|_2 = O(2^{-n/64})$.

**Proof.** For each integer $n \geqslant 1$, let $\varepsilon_n = \|\tau_n - t\|_2^{1/2}$. From (2.2) we have

$$\varepsilon_n = O(2^{-n/4}). \tag{2.3}$$

Observe that

$$\begin{aligned}\mathbb{E}(|Y(\tau_n)-Y(t)|^2) &= \mathbb{E}\big(|Y(\tau_n)-Y(t)|^2\mathbb{I}(|\tau_n-t|\leqslant\varepsilon_n)\big)\\ &\quad+\mathbb{E}\big(|Y(\tau_n)-Y(t)|^2\mathbb{I}(|\tau_n-t|>\varepsilon_n)\big)\\ &= A_n+B_n,\end{aligned}$$

with obvious notation. By (2.3) and the elementary properties of Browian motion, we have

$$\begin{aligned}A_n &\leqslant 2\mathbb{E}\left[\sup_{t\leqslant s\leqslant t+\varepsilon_n}|Y(s)-Y(t)|^2\right]\\ &= 2\varepsilon_n\mathbb{E}\left[\sup_{0\leqslant s\leqslant 1}|Y(s)|^2\right]\\ &= O(2^{-n/4}).\end{aligned}$$

Concerning $B_n$, observe that $\{2^{n/2}Y(T_{k,n}), k\geqslant 0\}$ is a simple symmetric random walk on $\mathbb{Z}$. As such,

$$\mathbb{E}\left[\left(2^{n/2}Y(\tau_n)\right)^4\right]=3[2^nt]^2-2[2^nt].$$

It follows that $\{\|Y(\tau_n)\|_4,\, n\geqslant 0\}$ is a bounded sequence. By the Hölder, Minkowski and Markov inequalities,

$$\begin{aligned}B_n &\leqslant \|Y(\tau_n)-Y(t)\|_4^2\sqrt{\mathbb{P}(|\tau_n-t|>\varepsilon_n)}\\ &\leqslant \big(\|Y(\tau_n)\|_4+\|Y(t)\|_4\big)^2\frac{\|\tau_n-t\|_2}{\varepsilon_n}\\ &= O(\varepsilon_n)\\ &= O(2^{-n/4}),\end{aligned}$$

which proves (a).

For each integer $n\geqslant 1$, let $\delta_n=\|Y(t)-Y(\tau_n)\|_2^{1/2}$ and observe that $\delta_n=O(2^{-n/16})$. By elementary considerations we obtain

$$|\mathrm{sgn}(Y(t))-\mathrm{sgn}((Y(\tau_n))|\leqslant 2\mathbb{I}(|Y(t)|\leqslant\delta_n)+2\mathbb{I}(|Y(\tau_n)|\leqslant\delta_n)+2\mathbb{I}(|Y(t)-Y(\tau_n)|>2\delta_n).$$

Consequently

$$\begin{aligned}\|\mathrm{sgn}(Y(t))-\mathrm{sgn}((Y(\tau_n))\|_4 &\leqslant 2\|\mathbb{I}(|Y(t)|\leqslant\delta_n)\|_4+2\|\mathbb{I}(|Y(\tau_n)|\leqslant\delta_n)\|_4\\ &\quad+2\|\mathbb{I}(|Y(t)-Y(\tau_n)|>2\delta_n)\|_4.\end{aligned}$$

We will obtain bounds for each of the terms on the right.

Since $t>0$, $|Y(t)|$ has a bounded density function. In particular,

$$\mathbb{P}(|Y(t)|\leqslant\delta_n)\leqslant\sqrt{\frac{2}{\pi t}}\,\delta_n.$$

This shows that $\|\mathbb{I}(|Y(t)|\leqslant\delta_n)\|_4\leqslant\sqrt[4]{2\delta_n}=O(2^{-n/64})$. Once again, let us observe that $\{2^{n/2}Y(T_{k,n}),\, k\geqslant 0\}$ is a simple symmetric random walk on $\mathbb{Z}$. Consequently, $\mathbb{E}(Y(\tau_n))=0$ and $\mathrm{var}(Y(\tau_n))=[2^nt]2^{-n}$. From the Berry–Esseen theorem we obtain the estimate

$$\mathbb{P}(|Y(\tau_n)|\leqslant\delta_n)\leqslant\mathbb{P}(|Y(1)|\leqslant\delta_n/\sqrt{[2^nt]2^{-n}})+\frac{C}{2^{n/2}},$$

where $C$ depends only on $t$. Arguing as above, we have $\|\mathbb{I}(|Y(\tau_n)|\leqslant\delta_n)\|_4=O(2^{-n/64})$.

By Markov's inequality,

$$\mathbb{P}(|Y(t)-Y(\tau_n)|>2\delta_n)\leqslant \frac{\|Y(t)-Y(\tau_n)\|_2}{4\delta_n^2} = \frac{1}{4}\delta_n,$$

which shows that $\|\mathbb{I}(|Y(t)-Y(\tau_n)|>2\delta_n)\|_4 = O(2^{-n/64})$. In summary, we have

$$\|\mathrm{sgn}(Y(t))-\mathrm{sgn}(Y(\tau_n))\|_4 = O(2^{-n/64}). \tag{2.4}$$

Finally, by the Hölder and Minkowski inequalities, we have

$$\big\|\big(\mathrm{sgn}(Y(t))-\mathrm{sgn}(Y(\tau_n))\big)\big(|Y(t)|+|Y(\tau_n)|\big)\big\|_2 \leqslant \|\mathrm{sgn}(Y(t))-\mathrm{sgn}(Y(\tau_n))\|_4\big(\|Y(t)\|_4+\|Y(\tau_n)\|_4\big).$$

As we have already observed, $\{\|Y(\tau_n)\|_4,\ n\geqslant 0\}$ is a bounded sequence. Thus, item (b) of this lemma follows from (2.4). ◇

We will adopt the following notation and definitions. For each integer $n\geqslant 0$, $j\in\mathbb{Z}$ and real number $t\geqslant 0$, let

$$U_{j,n}(t) = \sum_{k=0}^{[2^n t]-1} \mathbb{I}\big\{Y(T_{k,n})=r_{j,n}, Y(T_{k+1,n})=r_{j+1,n}\big\} \tag{2.5}$$

$$D_{j,n}(t) = \sum_{k=0}^{[2^n t]-1} \mathbb{I}\big\{Y(T_{k,n})=r_{j+1,n}, Y(T_{k+1,n})=r_{j,n}\big\}. \tag{2.6}$$

Thus, $U_{j,n}(t)$ and $D_{j,n}(t)$ denote the number of upcrossings and downcrossings of the interval $[r_{j,n}, r_{j+1,n}]$ within the first $[2^n t]$ steps of the random walk $\{Y(T_{k,n}),\ k\geqslant 0\}$, respectively.

As is customary, we will say that $\varphi:\mathbb{R}^2\to\mathbb{R}$ is *symmetric* provided that

$$\varphi(x,y)=\varphi(y,x)$$

for all $x,y\in\mathbb{R}$. We will say that $\varphi$ is *skew symmetric* provided that

$$\varphi(x,y)=-\varphi(y,x)$$

for all $x,y\in\mathbb{R}$. Recalling (1.2), we state and prove a useful real–variable lemma.

**Lemma 2.4.** *If $\varphi$ is symmetric, then*

$$\sum_{k=0}^{[2^n t]-1}\varphi\big(Z(T_{k,n}),Z(T_{k+1,n})\big) = \sum_{j\in\mathbb{Z}}\varphi(X_{j,n},X_{j+1,n})(U_{j,n}(t)+D_{j,n}(t)).$$

*If $\varphi$ is skew–symmetric, then*

$$\sum_{k=0}^{[2^n t]-1}\varphi\big(Z(T_{k,n}),Z(T_{k+1,n})\big) = \sum_{j\in\mathbb{Z}}\varphi(X_{j,n},X_{j+1,n})(U_{j,n}(t)-D_{j,n}(t)).$$

**Proof.** Since each step of the random walk $\{Y(T_{k,n}),\ k \geqslant 0\}$ is either and upcrossing or a downcrossing of some interval $[r_{j,n}, r_{j+1,n}]$, $j \in \mathbb{Z}$, it follows that

$$1 = \sum_{j\in\mathbb{Z}} \Big( \mathbb{I}\{Y(T_{k,n}) = r_{j,n}, Y(T_{k+1,n}) = r_{j+1,n}\} + \mathbb{I}\{Y(T_{k,n}) = r_{j+1,n}, Y(T_{k+1,n}) = r_{j,n}\} \Big).$$

Consequently

$$\sum_{k=0}^{[2^n t]-1} \varphi\big(Z(T_{k,n}), Z(T_{k+1,n})\big) = \sum_{j\in\mathbb{Z}} \sum_{k=0}^{[2^n t]-1} \varphi\big(Z(T_{k,n}), Z(T_{k+1,n})\big) \times \Big( \mathbb{I}\{Y(T_{k,n}) = r_{j,n}, Y(T_{k+1,n}) = r_{j+1,n}\} + \mathbb{I}\{Y(T_{k,n}) = r_{j+1,n}, Y(T_{k+1,n}) = r_{j,n}\} \Big).$$

Observe that from (2.5) and (2.6) we have

$$\sum_{k=0}^{[2^n t]-1} \varphi\big(Z(T_{k,n}), Z(T_{k+1,n})\big)\mathbb{I}\{Y(T_{k,n}) = r_{j,n}, Y(T_{k+1,n}) = r_{j+1,n}\} = \varphi(X_{j,n}, X_{j+1,n})U_{j,n}(t)$$

$$\sum_{k=0}^{[2^n t]-1} \varphi\big(Z(T_{k,n}), Z(T_{k+1,n})\big)\mathbb{I}\{Y(T_{k,n}) = r_{j+1,n}, Y(T_{k+1,n}) = r_{j,n}\} = \varphi(X_{j+1,n}, X_{j,n})D_{j,n}(t)$$

The remainder of the argument follows from the definitions of symmetric and skew symmetric. $\diamond$

Our next result will be used in conjunction with the decomposition developed in Lemma 2.3; its proof is easily obtained by observing that the upcrossings and downcrossings of the interval $[r_{j,n}, r_{j+1,n}]$ alternate.

**Lemma 2.5.** *Let $t > 0$. For each $j \in \mathbb{Z}$,*

$$U_{j,n}(t) - D_{j,n}(t) = \begin{cases} \mathbb{I}(0 \leqslant j < j^*) & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ -\mathbb{I}(j^* \leqslant j < 0) & \text{if } j^* < 0. \end{cases}$$

We will need a set of auxiliary processes. For $s \geqslant 0$, let

$$\widetilde{X}^{\pm}_s = X^{\pm}(r_{j,n}) \quad \text{when } r_{j,n} \leqslant s < r_{j+1,n}.$$

For $s \in \mathbb{R}$, let

$$\widetilde{X}_s = \begin{cases} \widetilde{X}^{+}_s & \text{if } s \geqslant 0 \\ \widetilde{X}^{-}_{-s} & \text{if } s < 0. \end{cases}$$

We will adopt the following conventions: given $t \in \mathbb{R}$, let

$$\int_0^t f(\widetilde{X}_s)dX_s = \begin{cases} \int_0^t f(\widetilde{X}^{+}_s)dX^{+}_s & \text{if } t \geqslant 0 \\ \int_0^{-t} f(\widetilde{X}^{-}_s)dX^{-}_s & \text{if } t < 0, \end{cases}$$

whenever the integrals on the right are defined. Due to the definition of $\{\widetilde{X}_s,\ s \in \mathbb{R}\}$, we have

$$\int_0^{r_{k,n}} f(\widetilde{X}_s)dX_s = \begin{cases} \sum_{j=0}^{k-1} f(X_{j,n}^+)\Delta X_{j,n}^+ & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ \sum_{j=0}^{|k|-1} f(X_{j,n}^-)\Delta X_{j,n}^- & \text{if } k < 0. \end{cases} \tag{2.7}$$

Similarly, by consideration of the cases, we obtain

$$\operatorname{sgn}(r_{k,n}) \int_0^{r_{k,n}} f(\widetilde{X}_s)ds = \begin{cases} \sum_{j=0}^{k-1} f(X_{j,n}^+)\Delta r_{j,n} & \text{if } k > 0 \\ 0 & \text{if } k = 0 \\ \sum_{j=0}^{|k|-1} f(X_{j,n}^-)\Delta r_{j,n} & \text{if } k < 0. \end{cases} \tag{2.8}$$

It will be convenient to rewrite the results of (2.8) in a modified form. For $k > 0$, it will be preferable to write

$$\begin{aligned} \operatorname{sgn}(r_{k,n}) \int_0^{r_{k,n}} f(\widetilde{X}_s)ds = \sum_{j=0}^{k-1} f(X_{j,n}^+)(\Delta X_{j,n}^+)^2 \\ + \sum_{j=0}^{k-1} f(X_{j,n}^+)\big(\Delta r_{j,n} - (\Delta X_{j,n}^+)^2\big). \end{aligned} \tag{2.9}$$

The obvious modifications should be made for the case $k < 0$.

**Proof of Theorem 2.1.** Recall (1.1)–(1.3). For each integer $n \geqslant 0$, let

$$\begin{aligned} \widetilde{V}_n^{(1)}(f,t) &= \int_0^{r_{j^*,n}} f(\widetilde{X}_s)dX_s + \frac{1}{2}\operatorname{sgn}(r_{j^*,n}) \int_0^{r_{j^*,n}} f'(\widetilde{X}_s)ds \\ \widehat{V}_n^{(1)}(f,t) &= \int_0^{Y(\tau_n)} f(X_s)dX_s + \frac{1}{2}\operatorname{sgn}(Y(\tau_n)) \int_0^{Y(\tau_n)} f'(X_s)ds \\ V^{(1)}(f,t) &= \int_0^{Y_t} f(X_s)dX_s + \frac{1}{2}\operatorname{sgn}(Y_t) \int_0^{Y_t} f'(X_s)ds. \end{aligned}$$

In this notation, we need to show that $V_n^{(1)}(f,t) \to V^{(1)}(f,t)$ almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. To this end, we have

$$\begin{aligned} \|V_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2 \leqslant \|V_n^{(1)}(f,t) - \widetilde{V}_n^{(1)}(f,t)\|_2 + \|\widetilde{V}_n^{(1)}(f,t) - \widehat{V}_n^{(1)}(f,t)\|_2 \\ + \|\widehat{V}_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2. \end{aligned}$$

We will estimate each of the terms on the right in order. We will begin by expressing $V_n^{(1)}(f,t)$ in an alternate form. We will place a $\pm$ superscript on $M_{j,n}$ whenever the underlying Brownian motion is so signed. Since the function

$$\varphi(x,y) = f\left(\frac{y+x}{2}\right)(y-x)$$

is skew symmetric, by Lemma 2.4 we have

$$V_n^{(1)}(f,t) = \sum_{j\in\mathbb{Z}} f(M_{j,n})\Delta X_{j,n}\big(U_{j,n}(t) - D_{j,n}(t)\big).$$

In light of Lemma 2.5, there will be three cases to consider, according to the sign of $j^*$. If $j^* = 0$, then $U_{j,n}(t) - D_{j,n}(t) = 0$ and, consequently, $V_n^{(1)}(f,t) = 0$. If $j^* > 0$, then $U_{j,n}(t) - D_{j,n}(t) = 1$ for $0 \leqslant j \leqslant j^* - 1$ and 0 otherwise; consequently,

$$V_n^{(1)}(f,t) = \sum_{j=0}^{j^*-1} f(M_{j,n}^+)\Delta X_{j,n}^+.$$

If, however, $j^* < 0$, then $U_{j,n}(t) - D_{j,n}(t) = -1$ for $j^* \leqslant j \leqslant -1$ and 0 otherwise; consequently,

$$\begin{aligned} V_n^{(1)}(f,t) &= -\sum_{j=j^*}^{-1} f\left(\frac{X_{j+1,n} + X_{j,n}}{2}\right)(X_{j+1,n} - X_{j,n}) \\ &= \sum_{j=j^*}^{-1} f\left(\frac{X_{-j-1,n}^- + X_{-j,n}^-}{2}\right)(X_{-j,n}^- - X_{-j-1,n}^-) \\ &= \sum_{j=0}^{|j^*|-1} f(M_{j,n}^-)\Delta X_{j,n}^-. \end{aligned}$$

In summary,

$$V_n^{(1)}(f,t) = \begin{cases} \sum_{j=0}^{j^*-1} f(M_{j,n}^+)\Delta X_{j,n}^+ & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ \sum_{j=0}^{|j^*|-1} f(M_{j,n}^-)\Delta X_{j,n}^- & \text{if } j^* < 0. \end{cases} \tag{2.10}$$

By combining (2.7), (2.9) and (2.10), we have

$$V_n^{(1)}(f,t) - \widetilde{V}_n^{(1)}(f,t) = A_n + B_n,$$

where

$$A_n = \begin{cases} \sum_{j=0}^{j^*-1} \left[f(M_{j,n}^+) - f(X_{j,n}^+) - \frac{1}{2}f'(X_{j,n}^+)\Delta X_{j,n}^+\right]\Delta X_{j,n}^+ & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ \sum_{j=0}^{|j^*|-1} \left[f(M_{j,n}^-) - f(X_{j,n}^-) - \frac{1}{2}f'(X_{j,n}^-)\Delta X_{j,n}^-\right]\Delta X_{j,n}^- & \text{if } j^* < 0, \end{cases}$$

and

$$B_n = \begin{cases} \frac{1}{2}\sum_{j=0}^{j^*-1} f'(X_{j,n}^+)\big((\Delta X_{j,n}^+)^2 - \Delta r_{j,n}\big) & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ \frac{1}{2}\sum_{j=0}^{|j^*|-1} f'(X_{j,n}^-)\big((\Delta X_{j,n}^-)^2 - \Delta r_{j,n}\big) & \text{if } j^* < 0. \end{cases}$$

Note that by Taylor's theorem

$$\left|f(M_{j,n}^\pm) - f(X_{j,n}^\pm) - \frac{1}{2}f'(X_{j,n}^\pm)\Delta X_{j,n}^\pm\right| \leqslant \frac{1}{8}\|f\|_{C_b^2(\mathbb{R})}|\Delta X_{j,n}^\pm|^2.$$

Hence,

$$|A_n| \leqslant \begin{cases} \frac{1}{8}\|f\|_{C_b^2(\mathbb{R})} \sum_{j=0}^{j^*-1} |\Delta X_{j,n}^+|^3 & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ \frac{1}{8}\|f\|_{C_b^2(\mathbb{R})} \sum_{j=0}^{|j^*|-1} |\Delta X_{j,n}^-|^3 & \text{if } j^* < 0. \end{cases}$$

However, for any integer $m$, we have, by the triangle inequality and Brownian scaling,

$$\begin{aligned} \left\| \sum_{j=0}^{|m|-1} |\Delta X_{j,n}^{\pm}|^3 \right\|_2 &\leqslant |m| \; \|(\Delta X_{0,n}^{\pm})^3\|_2 \\ &= |m| \; \mu_6^{1/2} 2^{-3n/4}. \end{aligned}$$

Since the random variable $j^*$ is independent of $X$, by conditioning on the value of $j^*$ and applying the above inequality we obtain

$$\|A_n\|_2 \leqslant \frac{1}{8}\|f\|_{C_b^2(\mathbb{R})} \mu_6^{1/2} 2^{-3n/4} \mathbb{E}(|j^*|).$$

Since $\{2^{n/2} Y(T_{k,n}), k \geqslant 0\}$ is a simple symmetric random walk on $\mathbb{Z}$, it follows that for each $t > 0$

$$\mathbb{E}(|j^*|) \leqslant \|j^*\|_2 = \sqrt{[2^n t]} = O(2^{n/2}). \tag{2.11}$$

Consequently

$$\|A_n\|_2 = O(2^{-n/4}). \tag{2.12}$$

Let us turn our attention to the analysis of $B_n$. For each $j \in \mathbb{Z}$, let

$$\varepsilon_{j,n}^{\pm} = (\Delta X_{j,n}^{\pm})^2 - \Delta r_{j,n}.$$

Observe that $\mathbb{E}(\varepsilon_{j,n}^{\pm}) = 0$ and $\mathrm{var}(\varepsilon_{j,n}^{\pm}) = 2^{-n}\mathrm{var}(X(1)^2)$. Let $m \in \mathbb{Z}$. Since the random variables $\{f'(X_{j,n}^{\pm})\varepsilon_{j,n}^{\pm}, 0 \leqslant j \leqslant |m|-1\}$ are pairwise uncorrelated and since $\varepsilon_{j,n}^{\pm}$ is independent of $f'(X_{j,n}^{\pm})$, it follows that

$$\begin{aligned} \mathrm{var}\left( \frac{1}{2} \sum_{j=0}^{|m|-1} f'(X_{j,n}^{\pm})\varepsilon_{j,n}^{\pm} \right) &= \frac{1}{4} \sum_{j=0}^{|m|-1} \mathbb{E}\big(f'(X_{j,n}^{\pm})^2\big)\mathrm{var}(\varepsilon_{j,n}^{\pm}) \\ &\leqslant C_1 |m| 2^{-n}, \end{aligned}$$

where $C_1 = \|f\|_{C_b^2(\mathbb{R})}^2 \mathrm{var}(X(1))^2/4$. Arguing as above, since $j^*$ is independent of $X$, it follows that

$$\begin{aligned} \|B_n\|_2^2 &\leqslant C_1 2^{-n}\mathbb{E}(|j^*|) \\ &= O(2^{-n/2}). \end{aligned}$$

We have used (2.11) to arrive at this last estimate. This estimate, in conjunction with (2.12), yields

$$\|V_n^{(1)}(f,t) - \widetilde{V}_n^{(1)}(f,t)\|_2 = O(2^{-n/4}). \tag{2.13}$$

Recalling that $r_{j^*,n} = Y(\tau_n)$, we have

$$|\widetilde{V}_n^{(1)}(f,t) - \widehat{V}_n^{(1)}(f,t)| \leqslant \left| \int_0^{r_{j^*,n}} \big(f(X_s) - f(\widetilde{X}_s)\big) dX_s \right| + \frac{1}{2} \left| \int_0^{r_{j^*,n}} \big(f(X_s) - f(\widetilde{X}_s)\big) ds \right|.$$

For $j > 0$ we have

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^{r_{j,n}} \big(f(X_s) - f(\widetilde{X}_s)\big)dX_s\right)^2\right] &= \int_0^{r_{j,n}} \mathbb{E}\left[|f(\widetilde{X}_s^+) - f(X_s^+)|^2\right] ds \\ &\leqslant \|f\|^2_{C_b^2(\mathbb{R})} \int_0^{r_{j,n}} \mathbb{E}\left[|X_s^+ - \widetilde{X}_s^+|^2\right] ds \\ &= \|f\|^2_{C_b^2(\mathbb{R})} \sum_{k=0}^{j-1} \int_{r_{k,n}}^{r_{k+1,n}} \mathbb{E}\left[|X^+(s) - X^+(r_{k,n})|^2\right] ds \\ &= \|f\|^2_{C_b^2(\mathbb{R})} j \int_0^{r_{1,n}} s ds \\ &= \frac{1}{2}\|f\|^2_{C_b^2(\mathbb{R})} j r_{1,n}^2.\end{aligned}$$

A similar argument handles the case $j < 0$, and in general

$$\mathbb{E}\left[\left(\int_0^{r_{j,n}} \big(f(X_s) - f(\widetilde{X}_s)\big)dX_s\right)^2\right] \leqslant \frac{1}{2}\|f\|^2_{C_b^2(\mathbb{R})}|j|2^{-n}. \tag{2.14}$$

Since $j^*$ is independent of the $X$, by conditioning on the value of $j^*$ and applying (2.14), we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^{r_{j^*,n}} \big(f(\widetilde{X}_s)dX_s - f(\widetilde{X}_s)\big)dX_s\right)^2\right] &= \frac{1}{2}\|f\|^2_{C_b^2(\mathbb{R})} 2^{-n}\mathbb{E}(|j^*|) \\ &= O(2^{-n/2}).\end{aligned} \tag{2.15}$$

We have used (2.11) to obtain this last estimate. Similarly, for any integer $j > 0$, we have

$$\begin{aligned}\left\|\int_0^{r_{j,n}} \big(f(X_s) - f(\widetilde{X}_s)\big)ds\right\|_2 &\leqslant \int_0^{r_{j,n}} \|f(X_s^+) - f(\widetilde{X}_s^+)\|_2 ds \\ &\leqslant \|f\|_{C_b^2(\mathbb{R})} \int_0^{r_{j,n}} \|X_s^+ - \widetilde{X}_s^+\|_2 ds \\ &= \|f\|_{C_b^2(\mathbb{R})} \sum_{k=0}^{j-1} \int_{r_{k,n}}^{r_{k+1,n}} \|X^+(s) - X^+(r_{k,n})\|_2 ds \\ &= \|f\|_{C_b^2(\mathbb{R})} j \int_0^{r_{1,n}} \sqrt{s} ds \\ &= \frac{2}{3}\|f\|_{C_b^2(\mathbb{R})} j r_{1,n}^{3/2}.\end{aligned}$$

A similar proof handles the case $j < 0$, and in general we have

$$\mathbb{E}\left[\left(\int_0^{r_{j,n}} \big(f(X_s)ds - f(\widetilde{X}_s)\big)ds\right)^2\right] \leqslant \frac{4}{9}\|f\|^2_{C_b^2(\mathbb{R})} j^2 2^{-3n/2}. \tag{2.16}$$

Since $j^*$ is independent of $X$, by conditioning on the value of $j^*$ and applying (2.16), we have

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^{r_{j^*,n}} \big(f(X_s)ds - f(\widetilde{X}_s)\big)ds\right)^2\right] &\leqslant \frac{4}{9}\|f\|^2_{C_b^2(\mathbb{R})} 2^{-3n/2}\mathbb{E}\big((j^*)^2\big) \\ &= O(2^{-n/2}).\end{aligned} \tag{2.17}$$

We have used (2.11) to obtain this last estimate. From (2.15) and (2.17), we have

$$\|\widetilde{V}_n^{(1)}(f,t) - \widehat{V}_n^{(1)}(f,t)\|_2 = O(2^{-n/4}). \tag{2.18}$$

Recalling that $r_{j^*,n} = Y(\tau_n)$, we have

$$|\widehat{V}_n^{(1)}(f,t) - V^{(1)}(f,t)| \leqslant \left| \int_{Y(\tau_n)}^{Y(t)} f(X_s)dX_s \right| + \left| \operatorname{sgn}(Y(t)) \int_0^{Y(t)} f'(X_s)ds - \operatorname{sgn}(Y(\tau_n)) \int_0^{Y(\tau_n)} f'(X_s)ds \right|.$$

Let $a, b \in \mathbb{R}$. Then by the Itô isometry

$$\mathbb{E}\left[ \left( \int_a^b f(X_s)dX_s \right)^2 \right] = \int_{a\wedge b}^{b\vee a} \mathbb{E}\big(f^2(X_s)\big)ds \leqslant \|f\|^2_{C_b^2(\mathbb{R})}|b-a|.$$

Since $X$ are $Y$ are independent, by item (a) of Lemma 2.3 we obtain

$$\mathbb{E}\left[ \left( \int_{Y(\tau_n)}^{Y(t)} f(X_s)dX_s \right)^2 \right] \leqslant \|f\|^2_{C_b^2(\mathbb{R})}\mathbb{E}\big(|Y(t) - Y(\tau_n)|\big) = O(2^{-n/8}). \tag{2.19}$$

By consideration of the cases,

$$\left| \operatorname{sgn}(Y(t)) \int_0^{Y(t)} f'(X_s)ds - \operatorname{sgn}(Y(\tau_n)) \int_0^{Y(\tau_n)} f'(X_s)ds \right|$$

is bounded by

$$\left| \int_{Y(\tau_n)}^{Y(t)} f'(X_s)ds \right| + \frac{1}{2}|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))| \left| \int_0^{Y(t)} f'(X_s)ds + \int_0^{Y(\tau_n)} f'(X_s)ds \right|.$$

However, by an elementary bound on the integral and item (a) of Lemma 2.3,

$$\left\| \int_{Y(\tau_n)}^{Y_t} f'(X_s)ds \right\|_2 \leqslant \|f\|_{C_b^2(\mathbb{R})}\|Y(t) - Y(\tau_n)\|_2 = O(2^{-n/8}). \tag{2.20}$$

Finally, note that

$$|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))| \left| \int_0^{Y(t)} f'(X_s)ds + \int_0^{Y(\tau_n)} f'(X_s)ds \right| \leqslant \|f\|_{C_b^2(\mathbb{R})}|\operatorname{sgn}(Y(t)) - \operatorname{sgn}(Y(\tau_n))|\big(|Y(t)| + |Y(\tau_n)|\big).$$

By Lemma 2.3(b),

$$\Big\| \big(\mathrm{sgn}(Y(t)) - \mathrm{sgn}(Y(\tau_n))\big) \Big( \int_0^{Y(t)} f'(X_s)ds + \int_0^{Y(\tau_n)} f'(X_s)ds \Big) \Big\|_2 = O(2^{-n/64}). \tag{2.21}$$

From (2.19), (2.20) and (2.21) we obtain

$$\|\widehat{V}_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2 = O(2^{-n/64}). \tag{2.22}$$

Combining (2.13), (2.18) and (2.22), it follows that

$$\|V_n^{(1)}(f,t) - V^{(1)}(f,t)\|_2 = O(2^{-n/64}),$$

which yields the $L^2(\mathbb{P})$ convergence in question. The almost sure convergence follows from applications of Markov's inequality and the Borel–Cantelli lemma. ◇

## 3. The Quadratic Variation of Iterated Brownian Motion

Given an integer $n \geqslant 0$ and a real number $t > 0$, let

$$V_n^{(2)}(t) = \sum_{k=0}^{[2^n t]-1} \big(Z(T_{k+1,n}) - Z(T_{k,n})\big)^2$$
$$V_n^{(2)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right) \big(Z(T_{k+1,n}) - Z(T_{k,n})\big)^2.$$

In this section, we will examine both strong and weak limit theorems associated with these quadratic variation processes. Our first result is the strong law of large numbers for $V_n^{(2)}(f,t)$.

**Theorem 3.1.** *Let $t > 0$ and $f \in C_b^2(\mathbb{R})$. Then,*

$$2^{-n/2} V_n^{(2)}(f,t) \to \int_0^t f(Z_s)ds,$$

*almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.*

As a corollary, we have $2^{-n/2}V_n^{(2)}(t) \to t$ almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. Our next result examines the deviations of the centered process $\big(2^{-n/2}V_n^{(2)}(t) - t\big)$ and was inspired by the connection between the quadratic variation of iterated Brownian motion and the stochastic process called Brownian motion in random scenery, first described and studied in [27]. Since the introduction of this model, various aspects of Brownian motion in random scenery have been studied in [7, 31, 32, 33, 34, 36],

We will use the following notation in the sequel. Let $D_{\mathbb{R}}[0,1]$ denote the space of real–valued functions on $[0,1]$ which are right continuous and have left–hand limits. Given random elements $\{T_n\}$ and $T$ in $D_{\mathbb{R}}[0,1]$, we will write $T_n \Longrightarrow T$ to denote the convergence in distribution of the $\{T_n\}$ to $T$ (see [6, Chapter 3]). Let $\{B_1(t), t \in \mathbb{R}\}$ be a two–sided Brownian motion and let $\{B_2(t), t \geqslant 0\}$ denote an independent standard Brownian motion. Let

$$\mathcal{G}(t) = \int_{\mathbb{R}} L_t^x(B_2)B_1(dx),$$

The process $\{\mathcal{G}(t), t \in \mathbb{R}\}$ is called a Brownian motion in random scenery. Our next result states that $V_n^{(2)}(t)$, suitably normalized, converges in $D_{\mathbb{R}}[0,1]$ to $\mathcal{G}(t)$.

**Theorem 3.2.** *As $n \to \infty$,*

$$\frac{2^{n/4}}{\sqrt{2}}\big(2^{-n/2}V_n^{(2)}(t) - t\big) \Longrightarrow \mathcal{G}(t).$$

We will prove these theorems in order, but first we will develop several lemmas pertaining to the local time of Brownian motion.

**Lemma 3.4.** *For real numbers $p, q > 0$,*

$$\sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_p^q = O(2^{n/2}).$$

**Proof.** We will use the following notation: given $x \in \mathbb{R}$, let

$$\tau_x = \inf\{s \geqslant 0 : Y_s = x\}.$$

Let $C = \mathbb{E}((L_1^0)^p)$. Then, from the strong Markov property, elementary properties of the local time process, the reflection principle, and a standard Gaussian estimate, it follows that

$$\begin{aligned}
\mathbb{E}((L_t^x(Y))^p) &= \int_0^t \mathbb{E}((L_t^x(Y))^p \mid \tau_x = s) d\mathbb{P}(\tau_x \leqslant s) \\
&= \int_0^t \mathbb{E}((L_{t-s}^0(Y))^p) d\mathbb{P}(\tau_x \leqslant s) \\
&\leqslant \mathbb{E}((L_t^0(Y))^p)\mathbb{P}(\tau_x \leqslant t) \\
&= 2Ct^{p/2}\mathbb{P}(Y_t \geqslant |x|) \\
&\leqslant 2Ct^{p/2}\exp\left(-x^2/(2t)\right).
\end{aligned}$$

Consequently, for real numbers $p, q > 0$,

$$\int_{\mathbb{R}} \|L_t^x(Y)\|_p^q dx < \infty.$$

Since the mapping $x \mapsto \|L_t^x(Y)\|_p^q$ is uniformly continuous,

$$\lim_{n\to\infty} \sum_{j\in Z} \|L_t^{r_{j,n}}(Y)\|_p^q 2^{-n/2} = \lim_{n\to\infty} \sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_p^q \Delta r_{j,n} = \int_{-\infty}^{\infty} \|L_t^x(Y)\|_p^q dx.$$

It follows that

$$\sum_{j\in Z} \|L_t^{r_{j,n}}(Y)\|_p^q = O(2^{n/2}),$$

which proves the lemma in question. ◇

**Lemma 3.5.** *Let $a, b \in \mathbb{R}$ with $ab \geqslant 0$. Then there exists a positive constant $\mu$, independent of $a, b$ and $t$, such that*

$$\|L_t^b(Y) - L_t^a(Y)\|_2 \leqslant \mu\sqrt{|b-a|}t^{1/4}\exp\big(-a^2/(4t)\big).$$

**Proof.** Let $c \in \mathbb{R}$ and $t \geqslant 0$. From [37, Theorem 1.7, p. 210] and its proof, there exists a constant $\gamma$, independent of $c$ and $t$, such that

$$\mathbb{E}\big((L_t^c(Y) - L_t^0(Y))^2\big) \leqslant \gamma |c| t^{1/2}. \tag{3.2}$$

By symmetry, it is enough to consider the case $0 \leqslant a < b$. By the strong Markov property, Brownian scaling, the reflection principle, item (3.2), and a standard estimate, we obtain

$$\begin{aligned}\mathbb{E}((L_t^b(Y) - L_t^a(Y))^2) &= \int_0^a \mathbb{E}((L_t^b(Y) - L_t^a(Y))^2 \mid \tau_a = s) d\mathbb{P}(\tau_a \leqslant s) \\ &= \int_0^a \mathbb{E}((L_{t-s}^{b-a}(Y) - L_{t-s}^0(Y))^2) d\mathbb{P}(\tau_a \leqslant s) \\ &\leqslant \gamma(b-a) t^{1/2} \mathbb{P}(\tau_a \leqslant t) \\ &\leqslant \gamma(b-a) t^{1/2} \exp\big(-a^2/(2t)\big).\end{aligned}$$

The desired result follows upon taking square roots and setting $\mu = \gamma^{1/2}$. ◇

What follows is an immediate application of the preceeding lemma.

**Lemma 3.6.** *Let $t > 0$. In the notation of (1.2),*

$$\sum_{j \in \mathbb{Z}} f(M_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \to \int_0^t f(Z_s) ds,$$

*almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.*

**Proof.** By the occupation times formula,

$$\begin{aligned}\int_0^t f(Z_s) ds &= \int_{-\infty}^{\infty} f(X_u) L_t^u(Y) du \\ &= \sum_{j \in \mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} f(X_u) L_t^u(Y) du.\end{aligned}$$

It follows that

$$\begin{aligned}&\left\| \int_0^t f(Z_s) - \sum_{j \in \mathbb{Z}} f(M_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \right\|_2 \\ &\qquad\qquad \leqslant \sum_{j \in \mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} \| f(X_u) L_t^u(Y) - f(M_{j,n}) L_t^{r_{j,n}}(Y) \|_2 \, du.\end{aligned} \tag{3.3}$$

Since $f \in C_b^2(\mathbb{R})$ and $X$ is independent of $Y$, we have

$$\begin{aligned}&\| f(X_u) L_t^u(Y) - f(M_{j,n}) L_t^{r_{j,n}}(Y) \|_2 \\ &\qquad \leqslant \| f \|_{C_b^2(\mathbb{R})} \left( \| L_t^u(Y) - L_t^{r_{j,n}}(Y) \|_2 + \| X_u - M_{j,n} \|_2 \| L_t^{r_{j,n}}(Y) \|_2 \right).\end{aligned} \tag{3.4}$$

However, by Lemma 3.5,

$$\left\| L_t^u(Y) - L_t^{r_{j,n}}(Y) \right\|_2 \leqslant C \sqrt{\Delta r_{j,n}} \exp\left( -\frac{(r_{j,n} \wedge r_{j+1,n})^2}{(4t)} \right),$$

where $C$ depends only upon $t$. By the integral test, the sums,

$$\sum_{j\in\mathbb{Z}} \exp\left(-\frac{(r_{j,n}\wedge r_{j+1,n})^2}{(4t)}\right)\Delta r_{j,n},$$

are bounded in $n$. Thus,

$$\begin{aligned}\sum_{j\in\mathbb{Z}}\int_{r_{j,n}}^{r_{j+1,n}} \left\|L_t^u(Y)-L_t^{r_{j,n}}(Y)\right\|_2 du &= O\big(\sqrt{\Delta r_{j,n}}\big)\\ &= O(2^{-n/4}).\end{aligned}\tag{3.5}$$

For $r_{j,n}\leqslant u\leqslant r_{j+1,n}$ we have $\|X_u-M_{j,n}\|_2=\sqrt{\Delta r_{j,n}}$. Thus, by Lemma 3.4, we have

$$\sum_{j\in\mathbb{Z}}\int_{r_{j,n}}^{r_{j+1,n}} \|X_u-M_{j,n}\|_2\|L_t^{r_{j,n}}(Y)\|_2 du = O(2^{-n/4}).\tag{3.6}$$

Combining (3.3), (3.4), (3.5) and (3.6) we see that

$$\left\|\int_0^t f(Z_s)ds-\sum_{j\in\mathbb{Z}} f(M_{j,n})L_t^{r_{j,n}}(Y)\Delta r_{j,n}\right\|_2 = O(2^{-n/4}).$$

This demonstrates the convergence in $L^2(\mathbb{P})$. By applications of Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well. $\diamond$

Our next result is from [28, Theorem 1.4] and its proof. See [3] for a related but slightly weaker version in $L^p(\mathbb{P})$.

**Lemma 3.7.** *There exists a positive random variable $K\in L^8(\mathbb{P})$ such that for all $j\in\mathbb{Z}$, $n\geqslant 0$, and $t\geqslant 0$,*

$$\left|U_{j,n}(t)-\frac{2^{n/2}}{2}L_t^{r_{j,n}}(Y)\right|\leqslant Kn2^{n/4}\sqrt{L_t^{r_{j,n}}(Y)}$$
$$\left|D_{j,n}(t)-\frac{2^{n/2}}{2}L_t^{r_{j,n}}(Y)\right|\leqslant Kn2^{n/4}\sqrt{L_t^{r_{j,n}}(Y)}.$$

**Proof of Theorem 3.1.** Since the mapping

$$\varphi(x,y)=f\left(\frac{y+x}{2}\right)(y-x)^2$$

is symmetric, by Lemma 2.4,

$$\begin{aligned}2^{-n/2}V_n^{(2)}(f,t) &= \sum_{j\in\mathbb{Z}} 2^{-n/2} f(M_{j,n})(\Delta X_{j,n})^2\big(U_{j,n}(t)+D_{j,n}(t)\big)\\ &= A_n+B_n+C_n,\end{aligned}$$

where

$$\begin{aligned}A_n &= \sum_{j\in\mathbb{Z}} 2^{-n/2} f(M_{j,n})(\Delta X_{j,n})^2\left(U_{j,n}(t)+D_{j,n}(t)-2^{n/2}L_t^{r_{j,n}}(Y)\right),\\ B_n &= \sum_{j\in\mathbb{Z}} f(M_{j,n})\left((\Delta X_{j,n})^2-\mathbb{E}\big((\Delta X_{j,n})^2\big)\right)L_t^{r_{j,n}}(Y),\\ C_n &= \sum_{j\in\mathbb{Z}} f(M_{j,n})L_t^{r_{j,n}}(Y)\Delta r_{j,n}.\end{aligned}$$

By Lemma 3.7, since $f \in C_b^2(\mathbb{R})$,

$$|A_n| \leqslant \|f\|_{C_b^2(\mathbb{R})} n 2^{-n/4} \sum_{j\in\mathbb{Z}} (\Delta X_{j,n})^2 K \sqrt{L_t^{r_{j,n}}(Y)}.$$

Since $X$ is independent of $Y$, by Hölder's inequality, for each $j \in \mathbb{Z}$,

$$\|(\Delta X_{j,n})^2 K \sqrt{L_t^{r_{j,n}}(Y)}\|_2 \leqslant \|(\Delta X_{1,n})^2\|_2 \|K\|_4 \|L_t^{r_{j,n}}(Y)\|_2^{1/2}.$$

By scaling, $\|(\Delta X_{j,n})^2\|_2 = 2^{-n/2}\sqrt{\mu_4}$. Hence, by the triangle inequality and Lemma 3.4,

$$\begin{aligned}\|A_n\|_2 &\leqslant n 2^{-3n/4} \|K\|_4 \sqrt{\mu_4} \sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_2^{1/2} \\ &= O(n 2^{-n/4}),\end{aligned}$$

which shows that $A_n \to 0$ in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, the convergence is almost sure, as well.

Let

$$X_{j,n}^* = \begin{cases} X_{j,n} & \text{if } j \geqslant 0 \\ X_{j+1,n} & \text{if } j < 0. \end{cases}$$

Then we may write $B_n = B_n^{(1)} + B_n^{(2)}$, where

$$\begin{aligned} B_n^{(1)} &= \sum_{j\in\mathbb{Z}} \big(f(M_{j,n}) - f(X_{j,n}^*)\big) \left((\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\right) L_t^{r_{j,n}}(Y), \\ B_n^{(2)} &= \sum_{j\in\mathbb{Z}} f(X_{j,n}^*) \left((\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\right) L_t^{r_{j,n}}(Y). \end{aligned}$$

By noting that $|M_{j,n} - X_{j^*,n}| = \frac{1}{2}|\Delta X_{j,n}|$, we see that

$$|B_n^{(1)}| \leqslant \frac{1}{2} \|f\|_{C_b^2(\mathbb{R})} \sum_{j\in\mathbb{Z}} |\Delta X_{j,n}| \left((\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\right) L_t^{r_{j,n}}.$$

Since $X$ and $Y$ are independent,

$$\begin{aligned}\|B_n^{(1)}\|_2 &\leqslant \frac{1}{2} \|f\|_{C_b^2(\mathbb{R})} \sum_{j\in\mathbb{Z}} \|\Delta X_{j,n}\|_4 \left\|(\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\right\|_4 \|L_t^{r_{j,n}}(Y)\|_2 \\ &= O(2^{-n/4}).\end{aligned}$$

We have used Brownian scaling and Lemma 3.4 to obtain this last estimate.

Observe that the collection $\{f(X_{j,n}^*) \left((\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\right), j \in \mathbb{Z}\}$ is centered and pairwise uncorrelated. Since $X$ and $Y$ are independent, we obtain

$$\operatorname{var}(B_n^{(2)}) = \sum_{j\in\mathbb{Z}} \|f(X_{j,n}^*)\|_2^2 \|L_t^{r_{j,n}}(Y)\|_2^2 \operatorname{var}\left((\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\right).$$

Since $f$ is bounded, by Brownian scaling,

$$\operatorname{var}\left((\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\right) = O(2^{-n}).$$

Therefore,

$$\|B_n^{(2)}\|_2 = O(2^{-n/4}).$$

In summary, $\|B_n\|_2 = O(2^{-n/4})$, which shows that $B_n \to 0$ in $L^2(\mathbb{P})$ as $n \to \infty$. By applications of Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well.

Finally, by Lemma 3.6, $C_n \to \int_0^t f(Z_s)ds$ almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$, which proves the theorem in question. ◇

We turn our attention to the proof of Theorem 3.2. In preparation for the proof of this result, we will prove several lemmas. For each integer $j$, each positive integer $n$ and each positive real number $t$, let

$$\mathfrak{L}_{j,n}(t) = 2^{-n/2}\big(U_{j,n}(t) + D_{j,n}(t)\big).$$

**Lemma 3.8.** *For each $t \geqslant 0$,*

$$\sum_{j\in\mathbb{Z}} \mathbb{E}\big(|\mathfrak{L}_{j,n}(t)|^3\big) = O(2^{n/2}).$$

**Proof.** By the triangle inequality and a standard convexity argument, it follows that

$$\mathbb{E}\big(|\mathfrak{L}_{j,n}(t)|^3\big) \leqslant 4\mathbb{E}\big(|L_t^{r_{j,n}}(Y)|^3\big) + 4\mathbb{E}\big(|\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)|^3\big).$$

By Lemma 3.4,

$$\sum_{j\in\mathbb{Z}} \mathbb{E}\big(|L_t^{r_{j,n}}(Y)|^3\big) = O(2^{n/2}).$$

By Lemma 3.7,

$$|\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)|^3 \leqslant K^3 n^3 2^{-3n/4}\big(L_t^{r_{j,n}}(Y)\big)^{3/2}.$$

By Hölder's inequality,

$$\mathbb{E}\big(|\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)|^3\big) \leqslant \|K\|_6^3 n^3 2^{-3n/4}\|L_t^{r_{j,n}}(Y)\|_3^{3/2}.$$

From Lemma 3.4, it follows that

$$\sum_{j\in\mathbb{Z}} \mathbb{E}\big(|\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)|^3\big) = O(n^3 2^{-n/4}).$$

This proves the lemma in question. ◇

**Lemma 3.9.** *For each pair of nonnegative real numbers $s$ and $t$ we have,*

$$\lim_{n\to\infty} \sum_{j\in\mathbb{Z}} \mathbb{E}\big(|\mathfrak{L}_{j,n}(s)\mathfrak{L}_{j,n}(t) - L_s^{r_{j,n}}(Y)L_t^{r_{j,n}}(Y)|\big)2^{-n/2} = 0.$$

**Proof.** We have the decomposition

$$\begin{aligned}|\mathfrak{L}_{j,n}(s)\mathfrak{L}_{j,n}(t) - L_s^{r_{j,n}}(Y)L_t^{r_{j,n}}(Y)| &\leqslant \big(\mathfrak{L}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)\big(\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big)\\ &\quad + \big(\mathfrak{L}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)L_t^{r_{j,n}}(Y) + \big(\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big)L_s^{r_{j,n}}(Y).\end{aligned}$$

By Lemma 3.7,

$$\big|\mathfrak{L}_{j,n}(s) - L_s^{r_{j,n}}(Y)\big)\big(\mathfrak{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)\big| \leqslant K^2 n^2 2^{-n/2}\sqrt{L_s^{r_{j,n}}(Y)}\sqrt{L_t^{r_{j,n}}(Y)}.$$

By Hölder's inequality, we have

$$\mathbb{E}\left(\left|\left(\mathcal{L}_{j,n}(s) - L_s^{r_{j,n}}(Y)\right)\left(\mathcal{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)\right)\right|\right) \leqslant \|K\|_6^2 n^2 2^{-n/2} \|L_s^{r_{j,n}}(Y)\|_{3/2}^{1/3} \|L_t^{r_{j,n}}(Y)\|_{3/2}^{1/3}.$$

By applications of the Cauchy–Schwarz inequality and Lemma 3.4, we obtain the following:

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \mathbb{E}\left(\left|\left(\mathcal{L}_{j,n}(s) - L_s^{r_{j,n}}(Y)\right)\left(\mathcal{L}_{j,n}(t) - L_t^{r_{j,n}}(Y)\right)\right|\right) 2^{-n/2} \\ \leqslant \|K\|_6^2 n^2 2^{-n} \sqrt{\sum_{j\in\mathbb{Z}} \|L_s^{r_{j,n}}(Y)\|_{3/2}^{2/3}} \sqrt{\sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_{3/2}^{2/3}} \\ = O(n^2 2^{-n/2}).\end{aligned}$$

The remaining terms can be handled similarly. ◇

**Lemma 3.10.** *Let* $0 \leqslant s \leqslant t \leqslant 1$. *Then,*

$$\sum_{j\in\mathbb{Z}} L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \to \int_{\mathbb{R}} L_s^x(Y) L_t^x(Y) dx,$$

*in* $L^1(\mathbb{P})$ *as* $n \to \infty$.

**Proof.** We have

$$\int_{\mathbb{R}} L_s^x(Y) L_t^x(Y) dx = \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} L_s^x(Y) L_t^x(Y) dx.$$

From this it follows that

$$\begin{aligned}\left\| \int_{\mathbb{R}} L_s^x(Y) L_t^x(Y) dx - \sum_{j\in\mathbb{Z}} L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \right\|_1 \\ \leqslant \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} \left\| L_s^x(Y) L_t^x(Y) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \right\|_1 dx.\end{aligned}$$

By the Hölder and Minkowski inequalities, we obtain

$$\begin{aligned}\left\| L_s^x(Y) L_t^x(Y) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \right\|_1 \leqslant \|L_t^x(Y) - L_t^{r_{j,n}}(Y)\|_2 \|L_s^x(Y)\|_2 \\ + \|L_s^x(Y) - L_s^{r_{j,n}}(Y)\|_2 \|L_t^{r_{j,n}}(Y)\|_2.\end{aligned}$$

Since $s, t \in [0,1]$, it follows that $\|L_s^x(Y)\|_2$ and $\|L_t^x(Y)\|_2$ are bounded by 1. Therefore, by Lemma 3.5 and Jensen's inequality, there exists a universal constant $C$ such that

$$\left\| L_s^x(Y) L_t^x(Y) - L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \right\|_1 \leqslant C \sqrt{\Delta r_{j,n}} \exp\left( -\frac{(r_{j,n} \wedge r_{j+1,n})^2}{4} \right).$$

By the integral test, the sums,

$$\sum_{j\in\mathbb{Z}} \exp\left( -\frac{(r_{j,n} \wedge r_{j+1,n})^2}{4} \right) \Delta r_{j,n},$$

are bounded in $n$. Since $\sqrt{\Delta r_{j,n}} = 2^{-n/4}$,

$$\left\| \int_{\mathbb{R}} L_s^x(Y) L_t^x(Y) dx - \sum_{j\in\mathbb{Z}} L_s^{r_{j,n}}(Y) L_t^{r_{j,n}}(Y) \Delta r_{j,n} \right\|_1 = O(2^{-n/4}).$$

This proves the lemma. ◇

Given a function $f$ defined on $[0,1]$ and $\delta > 0$, let

$$\omega(f,\delta) = \sup_{\substack{0\leqslant s,t\leqslant 1\\ |s-t|<\delta}} |f(t) - f(s)|.$$

**Lemma 3.11.** *There exists a universal $c \in (0,\infty)$ such that for all $a \in \mathbb{R}$ and $\delta > 0$,*

$$\|\omega(L^a(Y),\delta)\|_2^2 \leqslant c\Big((\delta \ln(1/\delta)) \wedge |a|\exp(-a^2/2)\Big).$$

**Proof.** Since local times are increasing in the time variable, we have $\omega(L^a(Y),\delta) \leqslant L_1^a(Y)$. Consequently, by Lemma 3.5,

$$\begin{aligned}\big\|\omega(L^a(Y),\delta)\big\|_2^2 &\leqslant \|L_1^a(Y)\|_2^2 \\ &\leqslant c|a|\exp(-a^2/4).\end{aligned}$$

However, by Tanaka's formula,

$$L_t^a(Y) = |Y_t - a| - |a| - \int_0^t \text{sgn }(Y_r - a) dY_r.$$

Hence, for $s < t$,

$$L_t^a(Y) - L_s^a(Y) \leqslant |Y_t - Y_s| - \int_s^t \text{sgn }(Y_r - a) dY_r.$$

By Lévy's representation theorem (see [37]), $t \mapsto \int_0^t \text{sgn }(Y_r - a) dY_r$ is a standard Brownian motion. Thus, there exists positive numbers $c$ and $\delta_0$ such that for all $0 \leqslant \delta \leqslant \delta_0$,

$$\|\omega(L^a(Y),\delta)\|_2^2 \leqslant c\delta \log(1/\delta).$$

We have used Lévy's theorem concerning the modulus of continuity of Brownian motion to obtain this last result; see [37] for details. ◇

**Proof of Theorem 3.2.** For each integer $j$ and each positive integer $n$, let

$$\varepsilon_{j,n} = \frac{2^{n/2}}{\sqrt{2}}\Big((\Delta X_{j,n})^2 - \mathbb{E}\big((\Delta X_{j,n})^2\big)\Big).$$

For each $n$, the random variables $\{\varepsilon_{j,n},\, j \in \mathbb{Z}\}$ are independent and identically distributed. A scaling argument shows that $\varepsilon_{j,n}$ is distributed as $\varepsilon = (X_1^2 - 1)/\sqrt{2}$ for all admissible integers $j$ and $n$. Let $\phi$ denote the characteristic function of $\varepsilon$. Since $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(\varepsilon^2) = 1$, we have, as $z \to 0$,

$$\log \phi(z) = -\frac{z^2}{2} + O(z^3).$$

Thus, there exist $\gamma > 0$ and $0 < \delta \leqslant 1$ such that

$$\left|\log \phi(z) + \frac{z^2}{2}\right| \leqslant \gamma |z|^3, \tag{3.7}$$

for all $|z| \leqslant \delta$.

By Lemma 2.4 and the definition of $\{\mathcal{L}_{j,n}(t), j \in \mathbb{Z}\}$,

$$2^{-n/2} V_n^{(2)}(t) = \sum_{j\in\mathbb{Z}} \big(\Delta X_{j,n}\big)^2 \mathcal{L}_{j,n}(t).$$

Noting that $\mathbb{E}\big((\Delta X_{j,n})^2\big) = \Delta r_{j,n} = 2^{-n/2}$, we arrive at the following:

$$2^{-n/2} V_n^{(2)}(t) = \sum_{j\in\mathbb{Z}} \sqrt{2}\varepsilon_{j,n}\mathcal{L}_{j,n}(t) 2^{-n/2} + 2^{-n}\sum_{j\in\mathbb{Z}} (U_{j,n}(t) + D_{j,n}(t)).$$

Concerning this last term on the right, we have

$$2^{-n}\sum_{j\in\mathbb{Z}} (U_{j,n}(t) + D_{j,n}(t)) = 2^{-n}[2^n t] = t + O(2^{-n}),$$

since the number of upcrossings and downcrossings of all the intervals $[r_{j,n}, r_{j+1,n}]$ by the random walk is equal to the number of steps taken by this same random walk. It follows that

$$\frac{2^{n/4}}{\sqrt{2}}\big(V_n^{(2)}(t) - t\big) = \sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{L}_{j,n}(t) 2^{-n/4} + O(2^{-3n/4}).$$

Letting

$$\mathcal{G}_n(t) = \sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{L}_{j,n}(t) 2^{-n/4},$$

it is enough to show that

$$\mathcal{G}_n(t) \Longrightarrow \mathcal{G}(t). \tag{3.8}$$

First we will demonstrate the convergence of the finite–dimensional distributions and then we will give the tightness argument.

Let $0 \leqslant t_1 < t_2 < \cdots < t_m \leqslant 1$ and let $\lambda_1, \lambda_2, \cdots, \lambda_m \in \mathbb{R}$. To demonstrate the convergence of the finite–dimensional distributions, it is enough to show that

$$\mathbb{E}\Big[\exp\big(i\sum_{k=1}^m \lambda_k \mathcal{G}_n(t_k)\big)\Big] \to \mathbb{E}\Big[\exp\big(i\sum_{k=1}^m \lambda_k \mathcal{G}(t_k)\big)\Big], \tag{3.9}$$

as $n \to \infty$. For simplicity, let,

$$a_{j,n} = 2^{-n/4}\sum_{k=1}^m \lambda_k \mathcal{L}_{j,n}(t_k)$$

$$\widetilde{a}_{j,n} = 2^{-n/4}\sum_{k=1}^m \lambda_k L_{t_k}^{r_{j,n}}(Y).$$

We have the following:

$$\left|\mathbb{E}\Big[\exp\big(i\sum_{k=1}^m \lambda_k \mathcal{G}_n(t_k)\big)\Big] - \mathbb{E}\Big[\exp\big(i\sum_{k=1}^m \lambda_k \mathcal{G}(t_k)\big)\Big]\ \right| \leqslant A_n + B_n + C_n,$$

where,

$$A_n = \Big|\mathbb{E}\Big[\exp\big(i\sum_{k=1}^m \lambda_k \mathcal{G}_n(t_k)\big)\Big] - \mathbb{E}\Big[\exp\big(-\frac{1}{2}\sum_{j\in\mathbb{Z}} a_{j,n}^2\big)\Big]\Big|,$$

$$B_n = \Big|\mathbb{E}\Big[\exp\big(-\frac{1}{2}\sum_{j\in\mathbb{Z}} a_{j,n}^2\big)\Big] - \mathbb{E}\Big[\exp\big(-\frac{1}{2}\sum_{j\in\mathbb{Z}} \widetilde{a}_{j,n}^2\big)\Big]\Big|,$$

$$C_n = \Big|\mathbb{E}\Big[\exp\big(-\frac{1}{2}\sum_{j\in\mathbb{Z}} \widetilde{a}_{j,n}^2\big)\Big] - \mathbb{E}\Big[\exp\big(i\sum_{k=1}^m \lambda_k \mathcal{G}(t_k)\big)\Big]\Big|.$$

We will estimate each term in turn.

Observe that

$$\sum_{k=1}^m \lambda_k \mathcal{G}_n(t_k) = \sum_{j\in\mathbb{Z}} \varepsilon_{j,n} a_{j,n}.$$

Let $\mathcal{Y}$ denote the $\sigma$–algebra generated by $\{Y_t,\, t\geqslant 0\}$ and observe that the random variables $\{a_{j,n},\, j\in\mathbb{Z}\}$ are $\mathcal{Y}$–measureable. Thus,

$$\begin{aligned}\mathbb{E}\Big[\exp\big(i\sum_{k=1}^m \lambda_k \mathcal{G}_n(t_k)\big)\Big] &= \mathbb{E}\Big\{\mathbb{E}\Big[\prod_{j\in\mathbb{Z}} e^{ia_{j,n}\varepsilon_{j,n}} \;\Big|\; \mathcal{Y}\Big]\Big\}\\ &= \mathbb{E}\Big[\prod_{j\in\mathbb{Z}} \phi(a_{j,n})\Big].\end{aligned}$$

Assuming that $\sum_{j\in\mathbb{Z}} |a_{j,n}|^3 \leqslant \delta^3$, we have, by (3.7),

$$\sum_{j\in\mathbb{Z}} \Big|\log\phi(a_{j,n}) - \frac{1}{2}a_{j,n}^2\Big| \leqslant \gamma \sum_{j\in\mathbb{Z}} |a_{j,n}|^3.$$

From this it follows that

$$\Big|\prod_{j\in\mathbb{Z}} \phi(a_{j,n}) - \exp\big(-\frac{1}{2}\sum_{j\in\mathbb{Z}} a_{j,n}^2\big)\Big| \leqslant \gamma e^{\gamma} \sum_{j\in\mathbb{Z}} |a_{j,n}|^3.$$

Since

$$\Big|\prod_{j\in\mathbb{Z}} \phi(a_{j,n}) - \exp\big(-\frac{1}{2}\sum_{j\in\mathbb{Z}} a_{j,n}^2\big)\Big| \leqslant 2,$$

we may conclude that

$$\Big|\prod_{j\in\mathbb{Z}} \phi(a_{j,n}) - \exp\big(-\frac{1}{2}\sum_{j\in\mathbb{Z}} a_{j,n}^2\big)\Big| \leqslant \gamma e^{\gamma} \sum_{j\in\mathbb{Z}} |a_{j,n}|^3 + 2\mathbb{I}\big(\sum_{j\in Z} |a_{j,n}|^3 > \delta^3\big).$$

Upon taking expectations and applying Markov's inequality, we obtain

$$A_n \leqslant C\sum_{j\in\mathbb{Z}} \mathbb{E}(|a_{j,n}|^3),$$

where $C = (\gamma e^{\gamma} + 2\delta^{-3})$. However, by a convexity argument and Lemma 3.8, we have

$$\begin{aligned}\sum_{j\in\mathbb{Z}} E(|a_{j,n}|^3) &\leqslant m^2 2^{-3n/4} \sum_{k=1}^m |\lambda_k|^3 \sum_{j\in\mathbb{Z}} \mathbb{E}\big(\mathcal{L}_{j,n}^3(t_k)\big)\\ &= O(2^{-n/4}),\end{aligned}$$

which shows that $A_n \to 0$ as $n \to \infty$.

Note that,

$$B_n \leqslant \frac{1}{2} \sum_{j\in\mathbb{Z}} \mathbb{E}\big(|a_{j,n}^2 - \widetilde{a}_{j,n}^2|\big)$$
$$\leqslant \frac{1}{2} \sum_{k=1}^m \sum_{\ell=1}^m |\lambda_k||\lambda_\ell| \sum_{j\in\mathbb{Z}} \mathbb{E}\big(|\mathcal{L}_{j,n}(t_k)\mathcal{L}_{j,n}(t_\ell) - L_{t_k}^{r_{j,n}}(Y) L_{t_\ell}^{r_{j,n}}(Y)|\big) 2^{-n/2}.$$

By Lemma 3.9, we see that $B_n \to 0$.

Finally, observe that

$$\mathbb{E}\Big[\exp\big(i \sum_{k=1}^m \lambda_k \mathcal{G}(t_k)\big)\Big] = \mathbb{E}\Big[\exp\big(-\frac{1}{2}\int_{\mathbb{R}} \big(\sum_{k=1}^m \lambda_k L_{t_k}^x(Y)\big)^2 dx\big)\Big].$$

Thus,

$$C_n \leqslant \frac{1}{2}\mathbb{E}\Big[\big|\sum_{j\in\mathbb{Z}} \widetilde{a}_{j,n}^2 - \int_{\mathbb{R}} \big(\sum_{k=1}^m \lambda_k L_{t_k}^x(Y)\big)^2 dx\big|\Big]$$
$$\leqslant \frac{1}{2} \sum_{k=1}^m \sum_{\ell=1}^m |\lambda_k|\ |\lambda_\ell| \ \Big\| \sum_{j\in\mathbb{Z}} \mathcal{L}_{j,n}(t_k)\mathcal{L}_{j,n}(t_\ell)\Delta r_{j,n} - \int_{\mathbb{R}} L_{t_k}^x(Y) L_{t_\ell}^x(Y) dx \Big\|_1.$$

By Lemma 3.10, $C_n \to 0$, which, in conjunction with the above, verifies (3.9).

To demonstrate tightness, observe that

$$\omega(\mathcal{G}_n, \delta) \leqslant \sum_{j\in\mathbb{Z}} 2^{-n/4} \varepsilon_{j,n} \omega(\mathcal{L}_{j,n}, \delta).$$

It follows that

$$\operatorname{var}\big(\omega(\mathcal{G}_n, \delta)\big) = \sum_{j\in\mathbb{Z}} 2^{-n/2} \|\omega(\mathcal{L}_{j,n}, \delta)\|_2^2. \tag{3.10}$$

By Lemma 3.7, the triangle inequality and the fact that the local times are increasing in the time variable, we have

$$\omega(\mathcal{L}_{j,n}, \delta) \leqslant 2Kn2^{-n/4}\sqrt{L_1^{r_{j,n}}(Y)} + \omega(L^{r_{j,n}}(Y), \delta).$$

Thus, by a simple convexity inequality,

$$\|\omega(\mathcal{L}_{j,n}, \delta)\|_2^2 \leqslant 8n^2 2^{-n/2} \Big\| K\sqrt{L_1^{r_{j,n}}(Y)} \Big\|_2^2 + 2\|\omega(L^{r_{j,n}}(Y), \delta)\|_2^2$$
$$= A(n) + 2\|\omega(L^{r_{j,n}}(Y), \delta)\|_2^2,$$

with obvious notation.

By Hölder's inequality and some algebra,

$$\Big\| K\sqrt{L_1^{r_{j,n}}(Y)} \Big\|_2^2 \leqslant \|K\|_4^2 \|L_1^{r_{j,n}}(Y)\|_2.$$

Thus, by Lemma 3.4, we have

$$\sum_{j\in\mathbb{Z}} 2^{-n/2} A_n = O(n^2 2^{-n/2}). \tag{3.11}$$

Given $\delta > 0$, let us divide the integers into two classes $J_1$ and $J_2$, where

$$J_1 = \{j \in \mathbb{Z} : |j| \leqslant \delta^{-1/2} 2^{n/2}\}$$
$$J_2 = J_1^c.$$

Then by Lemma 3.11,

$$\sum_{j\in J_1} 2^{-n/2} \|\omega(L^{r_{j,n}}, \delta)\|_2^2 \leqslant c|J_1| 2^{-n/2} \delta \log(\delta^{-1}) \leqslant 2c\sqrt{\delta} \log(\delta^{-1}). \tag{3.12}$$

However, recalling that $\Delta r_{j,n} = 2^{-n/2}$ and applying Lemma 3.11,

$$\sum_{j\in J_2} 2^{-n/2} \|\omega(L^{r_{j,n}}, \delta)\|_2^2 \leqslant c \sum_{j\in J_2} |r_{j,n}| \exp\big(-r_{j,n}^2/2\big) \Delta r_{j,n} \sim 2c \int_{\delta^{-1/2}}^{\infty} |x| \exp\big(-x^2/2\big)\, dx. \tag{3.13}$$

Combining (3.10), (3.11), (3.12) and (3.13) gives the requisite tightness. This demonstrates (3.8) and the theorem is proved. ◇

# 4. Higher Order Variation

In this section, we will examine strong and weak limit theorems for the tertiary and quartic variation of iterated Brownian motion. Let us begin by recalling a theorem, essentially due to [8].

**Proposition 4.1.** *Let $t \geqslant 0$ and $p > 0$. The following hold in $L^p(\mathbb{P})$ :*

*(a)* $\displaystyle\sum_{k=0}^{[2^{n/2}t]} \big(Z(r_{k+1,n}) - Z(r_{k,n})\big)^3 \to 0;$

*(b)* $\displaystyle\sum_{k=0}^{[2^{n/2}t]} \big(Z(r_{k+1,n}) - Z(r_{k,n})\big)^4 \to 3t.$

Our next two theorems generalize the above along our random partitions. Given an integer $n \geqslant 0$ and a real number $t > 0$, let

$$V_n^{(3)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right) \big(Z(T_{k+1,n}) - Z(T_{k,n})\big)^3,$$
$$V_n^{(4)}(f,t) = \sum_{k=0}^{[2^n t]-1} f\left(\frac{Z(T_{k+1,n}) + Z(T_{k,n})}{2}\right) \big(Z(T_{k+1,n}) - Z(T_{k,n})\big)^4.$$

Whenever $f \equiv 1$, we will write $V_n^{(3)}(t)$ and $V_n^{(4)}(t)$ in place of $V_n^{(3)}(f,t)$ and $V_n^{(4)}(f,t)$, respectively. Our first result is a strong limit theorem for the tertiary variation of iterated Brownian motion and is related to Theorem 2.1 and Proposition 4.1(a).

**Theorem 4.2.** *Let $t > 0$ and let $f \in C_b^2(\mathbb{R})$. Then*

$$V_n^{(3)}(f,t) \to 0,$$

*almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.*

Our next result is a strong limit theorem for the quartic variation of iterated Brownian motion and is related to Theorem 3.1 and Proposition 4.1(b).

**Theorem 4.3.** *Let $t > 0$ and let $f \in C_b^2(\mathbb{R})$. Then*

$$V_n^{(4)}(f,t) \to 3\int_0^t f(Z_s)ds,$$

*almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.*

As corollaries to Theorem 4.2 and Theorem 4.3, we have $V_n^{(3)}(t) \to 0$ and $V_n^{(4)}(t) \to 3t$ almost surely and in $L^2(\mathbb{P})$. Our next two results concern the deviations of $V_n^{(3)}(t)$ and $V_n^{(4)}(t) - 3t$ : we will demonstrate that $V_n^{(3)}(t)$ and $V_n^{(4)}(t) - 3t$, suitably normalized, converge in distribution to an iterated Brownian motion and to Brownian motion in random scenery, respectively. As in §3, let $\{B_1(t), t \in \mathbb{R}\}$ denote a standard two–sided Brownian motion and let $\{B_2(t), t \geqslant 0\}$ denote an independent standard Brownian motion. Observe that $\{B_1 \circ B_2(t), t \geqslant 0\}$ is an iterated Brownian motion and that

$$\mathcal{G}(t) = \int_{\mathbb{R}} L_t^x(B_2)B_1(dx),$$

is a Brownian motion in random scenery.

**Theorem 4.4.** *As $n \to \infty$,*

$$\frac{2^{n/2}}{\sqrt{15}}V_n^{(3)}(t) \Longrightarrow B_1 \circ B_2(t).$$

**Theorem 4.5.** *As $n \to \infty$,*

$$\frac{2^{n/4}}{\sqrt{96}}\big(V_n^{(4)}(t) - 3t\big) \Longrightarrow \mathcal{G}(t).$$

We will prove these theorems in order.

**Proof of Theorem 4.2.** Since the mapping

$$\varphi(x,y) = f\left(\frac{y+x}{2}\right)(y-x)^3$$

is skew symmetric, by Lemma 2.4 we have

$$V_n^{(3)}(f,t) = \sum_{j\in\mathbb{Z}} f(M_{j,n})(\Delta X_{j,n})^3\big(U_{j,n}(t) - D_{j,n}(t)\big).$$

By Lemma 2.5 and by following the argument preceeding (2.10), we obtain

$$V_n^{(3)}(f,t) = \begin{cases} \sum_{j=0}^{j^*-1} f(M_{j,n}^+)(\Delta X_{j,n}^+)^3 & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ \sum_{j=0}^{|j^*|-1} f(M_{j,n}^-)(\Delta X_{j,n}^-)^3 & \text{if } j^* < 0. \end{cases}$$

However, for any integer $m$, by the triangle inequality, the boundedness of $f$ and Brownian scaling, we have

$$\begin{aligned} \left\| \sum_{j=0}^{|m|-1} f(M_{j,n}^{\pm})\big(\Delta X_{j,n}^{\pm}\big)^3 \right\|_2 &\leqslant \|f\|_{C_b^2(\mathbb{R})} \|(\Delta X_{0,n}^{\pm})^3\|_2 |m| \\ &= \|f\|_{C_b^2(\mathbb{R})} |m| \mu_6^{1/2} 2^{-3n/4}. \end{aligned}$$

Since the random variable $j^*$ is independent of $X$, by conditioning on the value of $j^*$ and applying the above inequality we obtain

$$\begin{aligned} \mathbb{E}\Big(\big(V_n^{(3)}(f,t)\big)^2\Big) &\leqslant \|f\|_{C_b^2(\mathbb{R})}^2 \mu_6 2^{-3n/2} \mathbb{E}\big((j^*)^2\big) \\ &= O(2^{-n/2}). \end{aligned}$$

We have used (2.11) to obtain this last estimate. This demonstrates the $L^2(\mathbb{P})$–convergence in question. By applications of Markov's inequality and the Borel–Cantelli lemma, the convergence is almost sure, as well. ◇

**Proof of Theorem 4.3.** Since the mapping

$$\varphi(x,y) = f\left(\frac{y+x}{2}\right)(y-x)^4$$

is symmetric, by Lemma 2.4 we have

$$\begin{aligned} V_n^{(4)}(f,t) &= \sum_{j\in\mathbb{Z}} f(M_{j,n})(\Delta X_{j,n})^4 \big(U_{j,n}(t) + D_{j,n}(t)\big) \\ &= A_n + B_n + C_n, \end{aligned}$$

where

$$\begin{aligned} A_n &= \sum_{j\in\mathbb{Z}} f(M_{j,n})(\Delta X_{j,n})^4 \left(U_{j,n}(t) + D_{j,n}(t) - 2^{n/2} L_t^{r_{j,n}}(Y)\right), \\ B_n &= \sum_{j\in\mathbb{Z}} f(M_{j,n}) \left((\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big)\right) 2^{n/2} L_t^{r_{j,n}}(Y), \\ C_n &= \sum_{j\in\mathbb{Z}} 3 f(M_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n}. \end{aligned}$$

Since $f \in C_b^2(\mathbb{R})$, by Lemma 3.7 we have

$$|A_n| \leqslant \|f\|_{C_b^2(\mathbb{R})} n 2^{n/4} \sum_{j\in\mathbb{Z}} (\Delta X_{j,n})^4 K \sqrt{L_t^{r_{j,n}}(Y)}.$$

Since $X$ is independent of $Y$, by Hölder's inequality we have, for each $j \in \mathbb{Z}$,

$$\|(\Delta X_{j,n})^4 K \sqrt{L_t^{r_{j,n}}(Y)}\|_2 \leqslant \|(\Delta X_{1,n})^4\|_2 \|K\|_4 \|L_t^{r_{j,n}}(Y)\|_2^{1/2}.$$

By scaling, $\|(\Delta X_{j,n})^4\|_2 = 2^{-n}\sqrt{\mu_8}$. Hence, by the triangle inequality and Lemma 3.4,

$$\begin{aligned}\|A_n\|_2 &\leqslant \|f\|_{C_b^2(\mathbb{R})} n 2^{-3n/4} \|K\|_4 \sqrt{\mu_8} \sum_{j\in\mathbb{Z}} \|L_t^{r_{j,n}}(Y)\|_2^{1/2} \\ &= O(n2^{-n/4}),\end{aligned}$$

which shows that $A_n \to 0$ in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, the convergence is almost sure, as well.

Let

$$X_{j,n}^* = \begin{cases} X_{j,n} & \text{if } j \geqslant 0 \\ X_{j+1,n} & \text{if } j < 0. \end{cases}$$

Then we may write $B_n = B_n^{(1)} + B_n^{(2)}$, where

$$\begin{aligned} B_n^{(1)} &= \sum_{j\in\mathbb{Z}} \big(f(M_{j,n}) - f(X_{j,n}^*)\big) \left((\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big)\right) 2^{n/2} L_t^{r_{j,n}}(Y), \\ B_n^{(2)} &= \sum_{j\in\mathbb{Z}} f(X_{j,n}^*) \left((\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big)\right) 2^{n/2} L_t^{r_{j,n}}(Y). \end{aligned}$$

Noting that $|M_{j,n} - X_{j,n}^*| = \frac{1}{2}|\Delta X_{j,n}|$, we have

$$|B_n^{(1)}| \leqslant \frac{\|f\|_{C_b^2(\mathbb{R})}}{2} 2^{n/2} \sum_{j\in\mathbb{Z}} |\Delta X_{j,n}| \left((\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big)\right) L_t^{r_{j,n}}(Y).$$

Since $X$ and $Y$ are independent,

$$\begin{aligned}\|B_n^{(1)}\|_2 &\leqslant \frac{\|f\|_{C_b^2(\mathbb{R})}}{2} 2^{n/2} \sum_{j\in\mathbb{Z}} \|\Delta X_{j,n}\|_4 \left\| \big(\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big) \right\|_4 \|L_t^{r_{j,n}}(Y)\|_2 \\ &= O(2^{-n/4}).\end{aligned}$$

We have used Brownian scaling and Lemma 3.4 to obtain this last estimate.

Observe that the collection $\{f(X_{j,n}^*)\left((\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big)\right), j \in \mathbb{Z}\}$ is centered and pairwise uncorrelated. Since $X$ and $Y$ are independent,

$$\operatorname{var}(B_n^{(2)}) = 2^n \sum_{j\in\mathbb{Z}} \|f(X_{j,n}^*)\|_2^2 \|L_t^{r_{j,n}}(Y)\|_2^2 \operatorname{var}\left((\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big)\right).$$

By Brownian scaling,

$$\operatorname{var}\left((\Delta X_{j,n})^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big)\right) = O(2^{-2n}).$$

Therefore,

$$\|B_n^{(2)}\|_2 = O(2^{-n/4}).$$

In summary, $\|B_n\|_2 = O(2^{-n/4})$, which shows that $B_n \to 0$ in $L^2(\mathbb{P})$ as $n \to \infty$. By applications of Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well.

Finally, by Lemma 3.6, $C_n \to 3\int_0^t f(Z_s)ds$ almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. This proves Theorem 4.3. ◇

**Proof of Theorem 4.4.** Since the mapping

$$\varphi(x,y) = (y-x)^3$$

is skew–symmetric, by Lemma 2.4 we have,

$$V_n^{(3)}(t) = \sum_{j\in\mathbb{Z}} \big(\Delta X_{j,n}\big)^3 \big(U_{j,n}(t) - D_{j,n}(t)\big).$$

From Lemma 2.5 and some algebra, it follows that

$$V_n^{(3)}(t) = \begin{cases} \sum_{j=0}^{j^*-1} \big(\Delta X_{j,n}^{+}\big)^3 & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ \sum_{j=0}^{|j^*|-1} \big(\Delta X_{j,n}^{-}\big)^3 & \text{if } j^* < 0. \end{cases} \tag{4.1}$$

For each $j \in \mathbb{Z}$ and each integer $n \geqslant 0$, we have

$$\mathrm{var}\Big(\big(\Delta X_{j,n}\big)^3\Big) = 15 \cdot 2^{-3n/2}.$$

Let

$$\varepsilon_{j,n}^{\pm} = \frac{1}{\sqrt{15}} 2^{3n/4} \big(\Delta X_{j,n}^{\pm}\big)^3. \tag{4.2}$$

A scaling argument shows that, for each $n$, the random variables $\{\varepsilon_{j,n}^{\pm}, j \geqslant 0\}$ are independent and identically distributed as $\varepsilon = X_1^3/\sqrt{15}$. For future reference, let us note that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(\varepsilon^2) = 1$. For each $t \geqslant 0$, let

$$X_n^{\pm}(t) = \begin{cases} 2^{-n/4} \sum_{j=0}^{[2^{n/2}t]-1} \varepsilon_{j,n}^{\pm} & \text{if } t \geqslant 2^{-n/2} \\ 0 & \text{if } 0 \leqslant t < 2^{-n/2}. \end{cases}$$

For $t \in \mathbb{R}$, let

$$X_n(t) = \begin{cases} X_n^{+}(t) & \text{if } t \geqslant 0 \\ X_n^{-}(t) & \text{if } t < 0. \end{cases}$$

In order that we may emphasize their dependence upon $n$ and $t$, recall that

$$\tau_n = \tau(n,t) = T_{[2^n t],n}$$
$$j^* = j^*(n,t) = 2^{n/2} Y(\tau_n).$$

For $t \in [0,1]$, let

$$Y_n(t) = Y(\tau(n,t)).$$

We observe that

$$\frac{1}{\sqrt{15}} 2^{n/2} V_n^{(3)}(t) = X_n \circ Y_n(t). \tag{4.3}$$

Let $D_{\mathbb{R}}[0,\infty)$ denote the space of all real–valued functions on $[0,\infty)$ which are right continuous and have left limits. Given a function $g : \mathbb{R} \to \mathbb{R}$, let us define $g^+, g^- : [0,\infty) \to \mathbb{R}$ accordingly: for each $t \geqslant 0$, let $g^+(t) = g(t)$ and let $g^-(t) = g(-t)$. Let

$$D^*_{\mathbb{R}}(\mathbb{R}) = \{g : \mathbb{R} \to \mathbb{R} \ : \ g^+ \in D_{\mathbb{R}}[0,\infty) \, \text{and} \, g^- \in D_{\mathbb{R}}[0,\infty)\}.$$

Let $q$ denote the usual Skorohod metric on $D_{\mathbb{R}}[0,\infty)$ (cf. [18, p. 117]). Then we can define a metric $q^*$ on $D^*_{\mathbb{R}}(\mathbb{R})$ as follows: given $f, g \in D^*_{\mathbb{R}}(\mathbb{R})$, let

$$q^*(f,g) = q(f^+, g^+) + q(f^-, g^-).$$

So defined, $\big(D^*_{\mathbb{R}}(\mathbb{R}),\ q^*\big)$ is a complete separable metric space. Moreover, $\{g_n\}$ converges to $g$ in $D^*_{\mathbb{R}}(\mathbb{R})$ if and only if $\{g_n^+\}$ and $\{g_n^-\}$ converge to $g^+$ and $g^-$ in $D_{\mathbb{R}}[0,\infty)$, respectively. By Donsker's theorem, $X_n^+ \Longrightarrow B_1^+$ and $X_n^- \Longrightarrow B_1^-$ in $D_{\mathbb{R}}[0,\infty)$ consequently,

$$X_n \Longrightarrow B_1 \quad \text{in} \quad D^*_{\mathbb{R}}(\mathbb{R}). \tag{4.4}$$

By another application of Donsker's theorem,

$$Y_n \Longrightarrow B_2 \quad \text{in} \quad D_{\mathbb{R}}([0,1]). \tag{4.5}$$

From (4.4) and (4.5), the independence of $X$ and $Y$ and the independence of $B_1$ and $B_2$, it follows that

$$(X_n, Y_n) \Longrightarrow (B_1, B_2) \quad \text{in} \quad D^*_{\mathbb{R}}(\mathbb{R}) \times D_{\mathbb{R}}([0,1]).$$

Since $(x,y) \in D^*_{\mathbb{R}}(\mathbb{R}) \times D_{\mathbb{R}}([0,1]) \mapsto D_{\mathbb{R}}([0,1]) \ni x \circ y$ is measurable and since $B_1 \circ B_2$ is continuous, it follows that

$$X_n \circ Y \Longrightarrow B_1 \circ B_2 \quad \text{in} \quad D_{\mathbb{R}}([0,1]).$$

Recalling (4.3), this proves the theorem. ◇

**Proof of Theorem 4.5.** For each integer $j$ and each positive integer $n$, let

$$\varepsilon_{j,n} = \frac{2^n}{\sqrt{96}} \Big( \big(\Delta X_{j,n}\big)^4 - \mathbb{E}\big((\Delta X_{j,n})^4\big) \Big).$$

For each $n$, the random variables $\{\varepsilon_{j,n}, j \in \mathbb{Z}\}$ are independent and identically distributed. A scaling argument shows that $\varepsilon_{j,n}$ is distributed as $\varepsilon = (X_1^4 - 3)/\sqrt{96}$ for all admissible integers $j$ and $n$. For future reference, we note that $\mathbb{E}(\varepsilon) = 0$ and $\mathbb{E}(\varepsilon^2) = 1$.

By Lemmas 2.4 and 2.5,

$$\begin{aligned} V_n^{(4)}(t) &= \sum_{j \in \mathbb{Z}} \big(\Delta X_{j,n}\big)^4 \big(U_{j,n}(t) + D_{j,n}(t)\big) \\ &= \sum_{j \in \mathbb{Z}} \sqrt{96}\, 2^{-n/2} \varepsilon_{j,n} \mathcal{L}_{j,n}(t) + 3 \cdot 2^{-n} \sum_{j \in \mathbb{Z}} \big(U_{j,n}(t) + D_{j,n}(t)\big). \end{aligned}$$

Arguing as in the proof of Theorem 3.2, we have

$$3 \cdot 2^{-n} \sum_{j \in \mathbb{Z}} \big(U_{j,n}(t) + D_{j,n}(t)\big) = 3t + O(2^{-n}).$$

From this it follows that

$$2^{-n/4}\big(V_n^{(4)}(t) - 3t\big) = O(2^{-3n/4}) + \sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{L}_{j,n}(t)2^{-n/4}.$$

As was shown in the proof of Theorem 3.2,

$$\sum_{j\in\mathbb{Z}} \varepsilon_{j,n}\mathcal{L}_{j,n}(t)2^{-n/4} \Longrightarrow \mathcal{G}(t).$$

This finishes the proof. ◇

# 5. An Excursion–Theoretic Construction of the Itô Integral

In this section we show that for $f \in C_b^2(\mathbb{R})$, the Itô integral process $\int_0^t f(Y_r)dY_r$ can be defined by means of the random partitions defined in §2. For each integer $n \geqslant 0$ and $k \in \mathbb{Z}$, let

$$Y_{k,n} = Y(T_{k,n}).$$

We offer the following theorem:

**Theorem 5.1.** *Let $t > 0$ and let $f \in C_b^2(\mathbb{R})$. Then*

$$\sum_{k=0}^{[2^n t]-1} f(Y_{k,n})\Delta Y_{k,n} \to \int_0^t f(Y_r)dY_r.$$

*almost surely and in $L^2(\mathbb{P})$, as $n \to \infty$.*

We will need the following lemma, which is a simple consequence of the mean value theorem for integrals.

**Lemma 5.2.** *Let $a, b \in \mathbb{R}$, $a < b$, and let $f \in C_b^2(\mathbb{R})$. Let*

$$a = u_0 < u_1 < \cdots < u_{n-1} < u_n = b$$

*be a partition of $[a, b]$. Then*

$$\left| \int_a^b f(s)ds - \sum_{k=0}^{n-1} f(u_k)\Delta u_k \right| \leqslant \|f\|_{C_b^2(\mathbb{R})}|b-a| \max_{0\leqslant k\leqslant n-1}\{|\Delta u_k|\}.$$

**Proof of Theorem 5.1.** By the proof of Lemma 2.4,

$$\begin{aligned}\sum_{k=0}^{[2^n t]-1} f(Y_{k,n})\Delta Y_{k,n} &= \sum_{j\in\mathbb{Z}} [f(r_{j,n})\Delta r_{j,n}U_{j,n}(t) - f(r_{j+1,n})\Delta r_{j,n}D_{j,n}(t)] \\ &= \sum_{j\in\mathbb{Z}} f(r_{j,n})\Delta r_{j,n}\big(U_{j,n}(t) - D_{j,n}(t)\big) \\ &\qquad - \sum_{j\in\mathbb{Z}} \big(f(r_{j+1,n}) - f(r_{j,n})\big)\Delta r_{j,n}D_{j,n}(t) \\ &= I_n - II_n,\end{aligned}$$

in the obvious notation. We will next show that

$$I_n \to \int_0^{Y_t} f(u)du, \tag{5.1}$$

$$II_n \to \frac{1}{2}\int_0^t f'(Y_u)du, \tag{5.2}$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$.

First observe that

$$\begin{aligned}\left|I_n - \int_0^{Y_t} f(u)du\right| &\leqslant \left|\int_0^{Y(\tau_n)} f(u)du - \int_0^{Y_t} f(u)du\right| + \left|I_n - \int_0^{Y(\tau_n)} f(u)du\right| \\ &= A_n + B_n,\end{aligned}$$

in the obvious notation.

Together with an elementary bound, Lemma 2.3 implies,

$$\begin{aligned}\|A_n\|_2 &\leqslant \|f\|_{C_b^2(\mathbb{R})}\|Y(t) - Y(\tau_n)\|_2 \\ &= O(2^{-n/8}).\end{aligned} \tag{5.3}$$

By Lemma 2.5,

$$I_n = \begin{cases} \sum_{j=0}^{j^*-1} f(r_{j,n})\Delta r_{j,n} & \text{if } j^* > 0 \\ 0 & \text{if } j^* = 0 \\ -\sum_{j=j^*}^{-1} f(r_{j,n})\Delta r_{j,n} & \text{if } j^* < 0. \end{cases}$$

Thus, by Lemma 5.2,

$$B_n \leqslant \|f\|_{C_b^2(\mathbb{R})}|Y(\tau_n)|2^{-n/2}.$$

In the proof of Lemma 2.3, it was shown that $\{\|Y(\tau_n)\|,\ n \geqslant 0\}$ is bounded in $n$. It follows that

$$\|B_n\|_2 = O(2^{-n/2}). \tag{5.4}$$

Combining (5.3) and (5.4), we have the following:

$$I_n \to \int_0^{Y_t} f(u)du,$$

in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well. This verifies (5.1).

Observe that

$$\begin{aligned}II_n &= \sum_{j\in\mathbb{Z}} \big(f(r_{j+1,n}) - f(r_{j,n}) - f'(r_{j,n})\Delta r_{j,n}\big)\Delta r_{j,n} D_{j,n}(t) \\ &\qquad + \sum_{j\in\mathbb{Z}} f'(r_{j,n})\big(\Delta r_{j,n}\big)^2 D_{j,n}(t) \\ &= A_n + B_n,\end{aligned}$$

using obvious notation. By Taylor's theorem,

$$\begin{aligned}|A_n| &\leqslant \frac{1}{2}\|f\|_{C_b^2(\mathbb{R})}2^{-3n/2}\sum_{j\in Z} D_{j,n}(t) \\ &= O(2^{-n/2}).\end{aligned} \tag{5.5}$$

We have used the the fact that $\sum_{j\in\mathbb{Z}} D_{j,n}(t) \leqslant [2^n t]$ to obtain this last bound. Observe that,

$$\left| B_n - \frac{1}{2}\int_{\mathbb{R}} f'(u) L_t^u(Y)du \right| \leqslant B_n^{(1)} + \frac{1}{2} B_n^{(2)},$$

where,

$$B_n^{(1)} = \|f\|_{C_b^2(\mathbb{R})} 2^{-n} \sum_{j\in\mathbb{Z}} \left| D_{j,n}(t) - \frac{2^{n/2}}{2} L_t^{r_{j,n}}(Y) \right|,$$

and

$$B_n^{(2)} = \left| \sum_{j\in\mathbb{Z}} f'(r_{j,n}) L_t^{r_{j,n}}(Y) \Delta r_{j,n} - \int_{\mathbb{R}} f'(u) L_t^u(Y) du \right|.$$

By Lemma 3.7,

$$A_n \leqslant \|f\|_{C_b^2(\mathbb{R})} n 2^{-3n/4} \sum_{j\in\mathbb{Z}} K \sqrt{L_t^{r_{j,n}}(Y)}.$$

Consequently, by the Minkowski and Hölder inequalities,

$$\begin{aligned} \|B_n^{(1)}\|_2 &\leqslant \|f\|_{C_b^2(\mathbb{R})} n 2^{-3n/4} \sum_{j\in\mathbb{Z}} \|K\|_4 \|L_t^{r_{j,n}}(Y)\|_2^{1/2} \\ &= O(n 2^{-n/4}). \end{aligned} \tag{5.6}$$

We have used Lemma 3.4 to obtain this last bound.

As in the proof of Proposition 3.3, we have

$$B_n^{(2)} \leqslant \|f\|_{C_b^2(\mathbb{R})} \sum_{j\in\mathbb{Z}} \int_{r_{j,n}}^{r_{j+1,n}} |L_t^u(Y) - L_t^{r_{j,n}}(Y)| du.$$

Consequently, by symmetry,

$$\|B_n^{(2)}\|_2 \leqslant 2 \|f\|_{C_b^2(\mathbb{R})} \sum_{j\geqslant 0} \int_{r_{j,n}}^{r_{j+1,n}} \|L_t^u(Y) - L_t^{r_{j,n}}(Y)\|_2 du.$$

By Lemma 3.5,

$$\|L_t^u(Y) - L_t^{r_{j,n}}(Y)\|_2 \leqslant C \sqrt{\Delta r_{j,n}} \exp\left(-r_{j,n}^2/2\right).$$

Thus,

$$\begin{aligned} \|B_n^{(2)}\|_2 &\leqslant 2\|f\|_{C_b^2(\mathbb{R})} C 2^{-n/4} \sum_{j\geqslant 0} \exp\left(-r_{j,n}^2/2\right) \Delta r_{j,n} \\ &= O(2^{-n/4}). \end{aligned} \tag{5.7}$$

Combining (5.6) and (5.7), we see that,

$$II_n \to \frac{1}{2} \int_{\mathbb{R}} f'(u) L_t^u(Y) du,$$

in $L^2(\mathbb{P})$ as $n \to \infty$. By Markov's inequality and the Borel–Cantelli lemma, this convergence is almost sure, as well. By the occupation times formula, this verifies (5.2).

We can now finish the proof. By (5.1) and (5.2),

$$\sum_{k=0}^{[2^n t]} f(Y_{k,n}) \Delta Y_{k,n} \to \int_0^{Y_t} f(u) du - \frac{1}{2} \int_0^t f'(Y_u) du, \tag{5.8}$$

almost surely and in $L^2(\mathbb{P})$ as $n \to \infty$. Let $F(t) = \int_0^t f(u)du$ and apply Itô's formula to $F(Y_t)$ to see that the right hand side of (5.8) is another way to write $\int_0^t f(Y_s) dY_s$. This proves the theorem. $\diamond$

## References


[1] M.A. Arcones (1995), On the law of the iterated logarithm for Gaussian processes, *J. Theoret. Probab.*, **8**, 877–903

[2] M.T. Barlow (1992), Harmonic analysis on fractal spaces, *Sém. de Bourbaki*, Astérisque, 44éme année, 1991–92, n° 755, 345–368

[3] R.F. Bass (1987), $L_p$–inequalities for functionals of Brownian motion, *Sém. de Prob. XXI*, 206–217, Lecture Notes in Math., **1247**, Springer–Verlag, Berlin–Heidelberg–New York

[4] J. Bertoin (1996), Iterated Brownian motion and stable (1/4) subordinator, *Stat. Probab. Lett.*, **27**(2), 111–114

[5] J. Bertoin and Z. Shi (1996), Hirsch's integral test for iterated Brownian motion. *Séminaire de Probabilités, XXX*, Lecture Notes in Mathematics, #1626, 361–368 (ed's: J. Azéma, M. Emery, P.A. Meyer and M. Yor)

[6] P. Billingsley (1979), *Convergence of Probability Measures*, Wiley, New York

[7] E. Bolthausen (1989), A central limit theorem for two-dimensional random walks in random sceneries, *Ann. Prob.* **17** (1), 108–115.

[8] K. Burdzy (1993), Some path properties of iterated Brownian motion. *Seminar in Stochastic Processes 1992*, 67–87, Birkhäuser, Boston (ed's: E. Çinlar, K.L. Chung and M. Sharpe)

[9] K. Burdzy (1993), Variation of iterated Brownian motion, *Workshop and Conference on Measure–Valued Processes, Stochastic Partial Differential Equations and Interacting Particle Systems*. CRM Proceedings and Lecture Notes

[10] K. Burdzy and D. Khoshnevisan (1995), The level sets of iterated Brownian motion. *Séminaire de Probabilités, XXIX*, Lecture Notes in Mathematics, #1613, 231–236 (ed's: J. Azéma, M. Emery, P.A. Meyer and M. Yor)

[11] K. Burdzy and D. Khoshnevisan (1996), Brownian motion in a Brownian crack. Preprint

[12] E. Csáki, M. Csörgő, A. Földes and P. Révész (1995), Global Strassen type theorems for iterated Brownian motion. *Stoch. Proc. Appl.*, **59**, 321–341

[13] E. Csáki, M. Csörgő, A. Földes and R. Révész (1996), The local time of iterated Brownian motion. *J. Theor. Prob.*, **9**(3), 717–743

[14] D.S. Dean and K.M. Jansons (1993), Brownian excursions on combs, *J. Stat. Phys.*, **70**, 1313–1331

[15] P. Deheuvels and D.M. Mason (1992), A functional LIL approach to pointwise Bahadur–Kiefer theorems, *Prob. in Banach Spaces*, **8**, 255–266 (ed's: R.M. Dudley, M.G. Hahn and J. Kuelbs)

[16] K.D. Elworthy (1982), *Stochastic Differential Equations on Manifolds*, Cambridge University Press, Cambridge-New York

[17] R. Engelman and Z. Jaeger (1986), *Fragmentation, Form and Flow in Fractured Media*, Proc. of the 1986 $F^3$ conference held at Neve Ilan, Israel, Isr. Phys. Soc. and Amer. Inst. Phys., Bristol, England

[18] S.N. Ethier and T.G. Kurtz (1986), *Markov Processes: Characterization and Convergence*, Wiley, New York

[19] T. Funaki (1979), A probabilistic construction of the solution of some higher order parabolic differential equations, *Proc. Japan Acad.*, **55**, 176–179

[20] S. Havlin and D. Ben-Avraham (1987) Diffusion in disordered media, *Adv. Phys.*, **36**(6), 695–798

[21] S. Havlin, J.E. Kiefer and G.H. Weiss (1987), Anomolous diffusion on a random comb–like structure, *Phys. Rev. A*, **36**, Aug., Ser. 3, 1403–1408

[22] S. Havlin and G.H. Weiss (1986), Some properties of a random walk on a comb structure, *Physica A*, **134**, 474–482

[23] Y. Hu (1997), Hausdorff and packing functions of the level sets of iterated Brownian motion. Preprint

[24] Y. Hu, D. Pierre-Lotti-Viaud and Z. Shi (1995), Laws of the iterated logarithm for iterated Wiener processes, *J. Theor. Prob.*, **8**(2), 303–319.
[25] Y. Hu and Z. Shi (1995), The Csörgő–Révész modulus of non–differentiability of iterated Brownian motion, *Stochastic Processes and their Applications*, **58**, 267–279.
[26] B. Kahng and S. Redner (1989), Scaling of the first–passage time and the survival probability on exact and quasi–exact self–similar structures, *J. Phys. A: Math. Gen.*, **22**(7), 887–902
[27] H. Kesten and F. Spitzer (1979), A limit theorem related to a new class of self–similar process, *Z. Wahrsch. Verw. Gebiete* **50**, 327–340.
[28] D. Khoshnevisan (1994), Exact rates of convergence to Brownian local times, *Ann. Prob.*, **22**(3), 1295–1330
[29] D. Khoshnevisan and T.M. Lewis (1996), The uniform modulus of continuity for iterated Brownian motion, *J. Theoret. Probab.* **9**(2), 317–333.
[30] D. Khoshnevisan and T.M. Lewis (1996), Chung's law of the iterated logarithm for iterated Brownian motion, *Ann. Inst. Henri Poin.: Prob. et Stat.* **32**(3), 349–359.
[31] R. Lang and X. Nguyen (1983), Strongly correlated random fields as observed by a random walker. *Z. Wahrsch. Verw. Gebiete* **64** (3), 327–340.
[32] T. M. Lewis (1992), A self-normalized law of the iterated logarithm for random walk in random scenery. *J. Theoret. Probab.* **5**(4), 629–659.
[33] T. M. Lewis (1993), A law of the iterated logarithm for random walk in random scenery with deterministic normalizers. *J. Theoret. Probab.* **6**(2), 209–230.
[34] J. H. Lou (1985) Some properties of a special class of self-similar processes. *Z. Wahrsch. Verw. Gebiete* **68**(4), 493–502.
[35] T. Lyons (1997), Differential Equations driven by rough signals, To appear in *Revista Math. Iberioamericana*
[36] B. Rémillard and D. A. Dawson (1991), A limit theorem for Brownian motion in a random scenery. *Canad. Math. Bull.* **34**(3), 385–391.
[37] D. Revuz and M. Yor (1991), *Continuous Martingales and Brownian Motion*, Springer–Verlag, Berlin–Heidelberg–New York
[38] Z. Shi (1995), Lower limits of iterated Wiener processes, *Stat. Prob. Lett.*, **23**, 259–270
[39] G.H. Weiss and S. Havlin (1987), Use of comb–like models to mimic anamolous diffusion on fractal structures, *Philosophical Magazine B: Physics of Condensed Matter*, **56**(6), 941–947
[40] Y. Xiao (1996), Local Times and Related Properties of Multi-dimensional iterated Brownian Motion. To appear in *J. Theoret. Probab.*

Davar Khoshnevisan
Department of Mathematics
University of Utah
Salt Lake City, UT. 84112
davar@math.utah.edu

Thomas M. Lewis
Department of Mathematics
Furman University
Greenville, SC. 29613
tom.lewis@furman.edu