THE ANNALS OF PROBABILITY, **22**(3), 1295-1330 (1994)

Exact rates of convergence to Brownian local time

By

Davar Khoshnevisan

Department of Mathematics
University of Washington
Seattle, WA. 98195
U.S.A.

Abstract. In this paper we are mainly concerned with proving lower bounds on some uniform approximation schemes for the local times of one–dimensional Brownian motion. Consequently, this leads to many exact limit theorems for Brownian local times. The latter results are Paul Lévy's occupation time approximation, the downcrossing theorem and an intrinsic construction.

A.M.S. Subject Classification. (1990) Primary. 60J55; Secondary. 60G17.
Keywords and Phrases. Local Times, excursion theory, uniform large deviations.
Running Title. Asymptotics for Local Times.

THE ANNALS OF PROBABILITY, **22**(3), 1295-1330 (1994)

1. Introduction and main results.

Suppose $\{Z(s); s \geq 0\}$ is a standard one dimensional Brownian motion starting at zero. Let $\{L_t^x; x \in \mathbb{R}^1, t \geq 0\}$ denote its process of local times. In other words, (see Chapter VI of Revuz and Yor (1991)) with probability one,

$$(1.1) \qquad \int_{\mathbb{R}^1} f(x) L_t^x dx = \int_0^t f(Z(s))ds, \qquad \text{for all } t \geq 0 \text{ and measurable } f : \mathbb{R}^1 \mapsto \mathbb{R}^1.$$

That the null set in (1.1) can be chosen to be independent of the choice of f is a consequence of Trotter's theorem which states: $(t,x) \mapsto L_t^x$ is almost surely continuous. (Up to a modification which we shall take for granted.) For the latter theorem, see Trotter (1958). A modern account appears in Theorem VI.1.7 of Revuz and Yor (1991). For instance, let $\{\Psi_\varepsilon(\cdot); \varepsilon > 0\}$ be an approximation to the identity, i.e., $\Psi_1(x) \geq 0$, for all $x \in \mathbb{R}^1$, $\int_{\mathbb{R}^1} \Psi_1(x)dx = 1$ and $\Psi_\varepsilon(x) \stackrel{df}{=} \varepsilon^{-1}\Psi_1(x\varepsilon^{-1})$. Then almost surely, $\lim_{\varepsilon \to 0} \int_0^t \Psi_\varepsilon(Z(s) - x)ds = L_t^x$, uniformly over all $x \in \mathbb{R}^1$. An important example is Lévy's occupation time approximation which is obtained by taking $\Psi_\varepsilon(x) \stackrel{df}{=} \varepsilon^{-1} 1_{[0,\varepsilon]}(x)$. Borodin (1986) has sharpened this by observing that

$$(1.2) \qquad \limsup_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^1} \frac{|\frac{1}{\varepsilon}\int_0^t 1_{[x,x+\varepsilon]}(Z(s))ds - L_t^x|}{\sqrt{\varepsilon \log(1/\varepsilon)}} \leq 2(L_t^*)^{1/2}, \qquad \text{a.s.}$$

where here and throughout, $L_t^* \stackrel{df}{=} \sup_{x \in \mathbb{R}^1} L_t^x$ and $\log x$ denotes the logarithm of x in base e. The proof of (1.2) is simple: by (1.1), $\varepsilon^{-1}\int_0^t 1_{[x,x+\varepsilon]}(Z(s))ds = \varepsilon^{-1}\int_x^{x+\varepsilon} L_t^u du$. So (1.2) follows immediately from Ray's modulus of continuity (see Ray (1963) and McKean (1962)), viz.,

$$(1.3) \qquad \limsup_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^1} \frac{|L_t^{x+\varepsilon} - L_t^x|}{\sqrt{\varepsilon \log(1/\varepsilon)}} = 2(L_t^*)^{1/2}, \qquad \text{a.s.}$$

Our first theorem improves (1.2) by obtaining an exact asymptotic lower bound as well as a better upper bound. More precisely, we have the following result:

Theorem 1.1. *For every $t > 0$, with probability one,*

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^1} \frac{|\varepsilon^{-1}\int_0^t 1_{[x,x+\varepsilon]}(Z(s))ds - L_t^x|}{\sqrt{\varepsilon \log(1/\varepsilon)}} = ((4/3)L_t^*)^{1/2}.$$

Remark 1.1.1. A little reflection shows that while the upper bound in Theorem 1.1 is sharp, it is the lower bound that is difficult to prove. This is due to the fact that the level sets are highly dependent. So any proof would have to exploit this dependence structure. We shall do so by using D. Williams' version of the Ray–Knight theorem. See Section 3.

The local version of Theorem 1.1 (i.e., without the supremum) is somewhat different. We state it as a formal theorem but omit the proof as it is similar to the proof of Theorem 1.1.

Theorem 1.2. *For every $t > 0$ and $x \in \mathbb{R}^1$, with probability one,*

$$\limsup_{\varepsilon \downarrow 0} \frac{|\varepsilon^{-1} \int_0^t 1_{[x,x+\varepsilon]}(Z(r))dr - L_t^x|}{\sqrt{\varepsilon \log\log(1/\varepsilon)}} = ((4/3)L_t^x)^{1/2}.$$

It is worth mentioning that the above is a genuine law of the iterated logarithm in the sense that the liminf of the left quantity in Theorem 1.2 is zero and hence no limit could possibly exist.

To state and motivate the next theorem, define $u_\varepsilon(x,t)$ to be the total number of times before time t that Z has upcrossed the interval $[x, x+\varepsilon]$. It is a well–known result of P. Lévy that with probability one, $\lim_{\varepsilon \to 0} 2\varepsilon u_\varepsilon(x,t) = L_t^x$. See Exercise XII.2.10 of Revuz and Yor (1991). For other proofs and variants see also Williams (1977), Chung and Durrett (1976) and Maisonneuve (1981). Chacon et al. (1981) proved that this convergence is uniform over all $x \in \mathbb{R}^1$. This fact is also hinted at in Williams (1979), Exercise II.62.7. In particular, the null set in question can be chosen to be independent of the "level set", x. Borodin (1986) has proven the following refinement of this development:

$$\sup_{x \in \mathbb{R}^1} |2\varepsilon u_\varepsilon(x,t) - L_t^x| = O\big(\varepsilon^{1/2} \log(1/\varepsilon)\big), \qquad \text{a.s.} \tag{1.4}$$

as $\varepsilon \downarrow 0$. In light of Theorem 1.1, the above theorem is surprising in that the power of the logarithm has changed. Indeed, one would guess the convergence rate to be of the order, $O\big((\varepsilon \log(1/\varepsilon))^{1/2}\big)$. Theorem 1.4 below shows that this is the case, while Theorem 1.3 provides the corresponding local version.

Theorem 1.3. *For every $t > 0$ and $x \in \mathbb{R}^1$, with probability one,*

$$\limsup_{\varepsilon \downarrow 0} \frac{|2\varepsilon u_\varepsilon(x,t) - L_t^x|}{\sqrt{\varepsilon \log\log(1/\varepsilon)}} = 2(L_t^x)^{1/2}.$$

Remark 1.3.1. As is the case with the rest of the results in this paper, the more important aspect of this theorem is the lower bound. The upper bound has been anticipated to some degree in the existing literature. See Williams (1979), page 91.

Theorem 1.3 (as well as all of the subsequent theorems) are proved by means of excursion theory. However, it is interesting that we have not found truely excursion–theoretic proofs for Theorems 1.1 and 1.2.

Theorem 1.4. *For every $t > 0$, with probability one,*

$$\lim_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^1} \frac{|2\varepsilon u_\varepsilon(x,t) - L_t^x|}{\sqrt{\varepsilon \log(1/\varepsilon)}} = 2(L_t^*)^{1/2}.$$

To state the final pair of theorems, let $e_\varepsilon(x,t)$ denote the total number of times before time t that Z makes excursions away from x with lifetimes greater than ε. It is well–known that $\lim_{\varepsilon \to 0} (\pi\varepsilon/2)^{1/2} e_\varepsilon(x,t) = L_t^x$. For example, see Maisonneuve (1981) or Proposition XII.2.9 of Revuz and Yor (1991). Theorem 1.1 of Perkins (1981) states that the convergence is uniform over all $x \in \mathbb{R}^1$. (Due to his choice of scale function, Perkins' local time is half of ours thus accounting for the extra factor of 2. See (1.2) of Perkins (1981).) More recently, Csörgő and Révész (1986) have proved the analogue of (1.4) for the process e_ε. Namely, as $\varepsilon \downarrow 0$,

$$\sup_{x \in \mathbb{R}^1} |(\pi\varepsilon/2)^{1/2} e_\varepsilon(x,t) - L_t^x| = O\big(\varepsilon^{1/4}\log(1/\varepsilon)\big), \qquad \text{a.s.} \tag{1.5}$$

Our next two theorems provide the exact rate of convergence to (1.5) and its local version.

Theorem 1.5. *For every $t > 0$ and $x \in \mathbb{R}^1$, with probability one,*

$$\limsup_{\varepsilon \downarrow 0} \frac{|(\pi\varepsilon/2)^{1/2} e_\varepsilon(x,t) - L_t^x|}{\varepsilon^{1/4}\sqrt{\log\log(1/\varepsilon)}} = (2\pi)^{1/4}(L_t^x)^{1/2}.$$

Theorem 1.5 is a special case of Theorem 1 of Horváth (1990).

Theorem 1.6. *For every $t > 0$, with probability one,*

$$(\pi/2)^{1/4}(L_t^*)^{1/2} \le \liminf_{\varepsilon\downarrow 0} \sup_{x\in\mathbb{R}^1} \frac{|(\pi\varepsilon/2)^{1/2}e_\varepsilon(x,t) - L_t^x|}{\varepsilon^{1/4}\sqrt{\log(1/\varepsilon)}}$$
$$\le \limsup_{\varepsilon\downarrow 0} \sup_{x\in\mathbb{R}^1} \frac{|(\pi\varepsilon/2)^{1/2}e_\varepsilon(x,t) - L_t^x|}{\varepsilon^{1/4}\sqrt{\log(1/\varepsilon)}} \le (2\pi)^{1/4}(L_t^*)^{1/2}.$$

We have not been able to identify the exact constant in Theorem 1.6 (or the existence of a limit, for that matter.) However, the proof leads us to believe the following:

Conjecture 1.7. *For each $t > 0$, with probability one,*

$$\lim_{\varepsilon\downarrow 0} \sup_{x\in\mathbb{R}^1} \frac{|(\pi\varepsilon/2)^{1/2}e_\varepsilon(x,t) - L_t^x|}{\varepsilon^{1/4}\sqrt{\log(1/\varepsilon)}} = (2\pi)^{1/4}(L_t^*)^{1/2}.$$

Remark 1.7.1. Many of the approximation theorems for local times are known to hold uniformly in $t \in T$, where T is an arbitrary nonrandom compact subset of $[0,\infty)$. These include (1.2) through (1.5). Our proofs can be modified to show that Theorems 1.4 and 1.6 hold uniformly over t–compacts. However, the proof of Theorem 1.1 does not extend to handle t–uniform results.

Much is now known about convergence theorems for local times at a fixed level. An umbrella method of doing this for Markov processes appears in Fristedt and Taylor (1983). Various uniform approximation theorems are also known for some Lévy processes. See Barlow et al. (1986a,b) and the references therein. A host of strong limit theorems related to Brownian local time can be found in Knight (1981).

In Section 2, we demonstrate some modulus of continuity results for "smoothed" Brownian motion. These results will then be used in Section 3 to prove Theorem 1.1. Sections 4 through 7 contain proofs for Theorems 1.3 through 1.6, respectively.

Finally some notation is in order here. Throughout, we shall think of Z as the coordinate functions in the space of continuous functions on the positive half–line, $\mathfrak{C}([0,\infty))$. This means that $Z(t)(\omega) \stackrel{df}{=} \omega(t)$ for all $\omega \in \mathfrak{C}([0,\infty))$. Define $T_x \stackrel{df}{=} \inf\{s : Z(s) = x\}$ and let (ϑ_t) be the shifts on the paths of Z. In other words, $\vartheta_t(Z)(s) = Z(t+s)$. More generally, if $\Lambda = \Lambda\big(Z(r); a \le r \le b\big)$ is a measurable functional of Z, $\vartheta_t(\Lambda) = \Lambda\big(Z(r); t+a \le r \le t+b\big)$.

Constants will be denoted by C_i and for the sake of rigor, their dependence on any other variable will be explicitly stated.

Acknowledgement. Some of the ideas in this paper developed in conversations with Rich Bass, Chris Burdzy and Ellen Toby on different occasions. Many thanks are also due to an anonymous referee for some very useful suggestions to improve the presentation of the paper.

2. Modulus results for Brownian processes.

Throughout this section, $\{Z(t); t \geq 0\}$ denotes a one–dimensional Brownian motion and $\{\beta_\alpha(t); t \geq 0\}$ is a Bessel process of dimension $\alpha > 0$. In other words, (β_α) is a non–negative diffusion with infinitesimal generator given by the following:

$$\mathcal{L}_\alpha(f)(x) = \frac{1}{2} f''(x) + \frac{\alpha - 1}{2x} f'(x),$$

for all twice continuously differentiable functions, $f : (0, \infty) \mapsto \mathbb{R}^1$ which satisfy

$$\lim_{\varepsilon \to 0} \varepsilon^{\alpha - 2} \big(f(\varepsilon) - f(0) \big) = 0.$$

We next recall Lévy's modulus of continuity for the processes (β_α) and (Z), respectively:

$$\lim_{h \downarrow 0} \sup_{t \in T} \frac{|\beta_\alpha(t+h) - \beta_\alpha(t)|}{\varphi(h)} = \sqrt{2}, \qquad \text{a.s.} \tag{2.1a}$$

$$\lim_{h \downarrow 0} \sup_{t \in T} \frac{|Z(t+h) - Z(t)|}{\varphi(h)} = \sqrt{2}, \qquad \text{a.s.} \tag{2.1b}$$

where $\varphi(t) \stackrel{df}{=} \sqrt{t(\ 1 \vee \log(1/t)\ )}$ for $t > 0$, and $T \subseteq [0, \infty)$ is any (nonrandom) compact set. The proof of (2.1) with limsup instead of the limit can be found, for example, within the results of Revuz and Yor (1991), Ch. XI. Minor adjustments to the proof establish the existence of the limit, see for example Csörgő and Révész (1981).

Of particular interest to us is that as $h \downarrow 0$,

$$\begin{aligned} \sup_{t \in T} \left| h^{-1} \int_t^{t+h} Z(s) ds - Z(t) \right| &\leq \sup_{t \in T} h^{-1} \int_t^{t+h} |Z(s) - Z(t)| ds \\ &\leq \sqrt{2} \varphi(h)(1 + o(1)), \qquad \text{a.s.} \end{aligned} \tag{2.1c}$$

with a similar estimate holding for (β_α). The goal of this section is to prove the exact version of (2.1c) for the processes (Z), (β_α) and (β_α^2) (the latter process will be of use in the next section on local times). More precisely, we have the following:

Theorem 2.1.

(a) Fix any $x \in \mathbb{R}^1$ and any compact $T \subseteq [0, \infty)$. Then, given $Z(0) = x$,

$$\lim_{h \downarrow 0} \sup_{t \in T} \frac{|h^{-1} \int_t^{t+h} Z(s)ds - Z(t)|}{\varphi(h)} = \sqrt{\frac{2}{3}}, \qquad \text{a.s.}$$

(b) Fix $\alpha \geq 2$, $x \in \mathbb{R}^1$ and compact $T \subseteq (0, \infty)$. Then, given $\beta_\alpha(0) = x$,

$$\lim_{h \downarrow 0} \sup_{t \in T} \frac{|h^{-1} \int_t^{t+h} \beta_\alpha(s)ds - \beta_\alpha(t)|}{\varphi(h)} = \sqrt{\frac{2}{3}}, \qquad \text{a.s.}$$

(c) For all $\alpha \geq 2$, $x \in \mathbb{R}^1_+$ and all compact $T \subseteq [0, \infty)$, given $\beta_\alpha(0) = x$,

$$\lim_{h \downarrow 0} \sup_{t \in T} \frac{|h^{-1} \int_t^{t+h} \beta_\alpha^2(s)ds - \beta_\alpha^2(t)|}{\varphi(h)} = \sqrt{\frac{8}{3}} \sup_{t \in T} \beta_\alpha(t) \qquad \text{a.s.}$$

Remark 2.1.1. (i) It is clear from part (a) of Theorem 2.1 that the estimate in (2.1c) is not sharp.

(ii) With extra effort, one can improve part (b) to all compacts $T \subseteq [0, \infty)$. However, as this extension is unnecessary for our purposes, we shall not present a proof here.

(iii) One can extend this theorem to the α–dimensional Bessel processes with $\alpha < 2$, by adapting the proof of part (b) using a localization argument.

(iv) Theorem 2.1 (and the corresponding Theorems 1.1 and 1.2) have recently been extended by Marcus and Rosen (1993) to a class of symmetric (nearly stable) Lévy processes using a striking isomorphism theorem of Dynkin (1984).

There is also a local version of Theorem 2.1. To this end, let us define:

$$\psi(h) \stackrel{df}{=} \sqrt{h(\ 1 \vee \log\log(1/h)\ )},$$

for all $h > 0$. The local version of Theorem 2.1 is as follows:

Theorem 2.2.

(a) Fix $t > 0$ *and* $x \in \mathbb{R}^1$. *Then given* $Z(0) = x$,

$$\limsup_{h \downarrow 0} \frac{|h^{-1}\int_t^{t+h} Z(r)dr - Z(t)|}{\psi(h)} = \sqrt{\frac{2}{3}}, \qquad \text{a.s.}$$

(b) For all $\alpha \geq 2$, $t > 0$ *and* $x \in \mathbb{R}^1$, *given* $\beta_\alpha(0) = x$,

$$\limsup_{h \downarrow 0} \frac{|h^{-1}\int_t^{t+h} \beta_\alpha^2(r)dr - \beta_\alpha^2(t)|}{\psi(h)} = \sqrt{\frac{8}{3}}\beta_\alpha(t), \qquad \text{a.s.}$$

The remainder of this section contains a proof of Theorem 2.1. This is patterned after Paul Lévy's proof of the uniform modulus of continuity of Brownian motion mixed together with some interpolation ideas. We omit the proof of Theorem 2.2 since it contains no new ideas.

Proof of Theorem 2.1(a). Without loss of generality, suppose $T = [0, 1]$ and $x = 0$; the necessary modifications in the general case are easy to make. For any h and $t > 0$, define

$$I(h;t) \overset{df}{=} h^{-1}\int_t^{t+h} Z(s)ds - Z(t) = h^{-1}\int_t^{t+h}(Z(s) - Z(t))ds.$$

Therefore, $I(h;t)$ is a Gaussian random variable with mean zero and variance given by,

$$\begin{aligned}\mathbb{E}(\ I(h;t)\ )^2 &= 2h^{-2}\int_t^{t+h}\int_t^s \mathbb{E}\Big((\ Z(s) - Z(t)\ )(\ Z(u) - Z(t)\ )\Big)duds \\ &= 2h^{-2}\int_t^{t+h}\int_t^s (u-t)duds = \frac{h}{3}.\end{aligned}$$

In particular, by elementary facts about Gaussian distributions,

$$\lim_{x \to \infty} x^{-2}\log \mathbb{P}(|I(h;t)| \geq x\sqrt{h/3}) = -1/2.$$

Moreover, since the distribution of $h^{-1/2}I(h;t)$ is independent of h and t, the limit is uniform over all h and $t > 0$. In particular, fixing $\theta > 0$ and $\varepsilon \in (0, 1)$, it follows that there exist finite positive constants, $C_1(\theta, \varepsilon) \leq C_2(\theta, \varepsilon)$ such that for all $t, h > 0$,

$$C_1(\theta,\varepsilon)h^{\theta(1+\varepsilon)} \leq \mathbb{P}(|I(h;t)| \geq \varphi(h)\sqrt{2\theta/3}) \leq C_2(\theta,\varepsilon)h^{\theta(1-\varepsilon)}. \tag{2.2}$$

Now a Borel–Cantelli argument is in order. Indeed, by (2.2), for any fixed $\rho > 1$,

$$\mathbb{P}(|I(\rho^{-n}; k\rho^{-n})| \geq \varphi(\rho^{-n})\sqrt{2\theta/3}) \geq C_1(\theta,\varepsilon)\rho^{-n\theta(1+\varepsilon)},$$

for all $0 \leq k \leq [\rho^n]$ and all $n \geq 1$. Since $\{I(\rho^{-n}, k\rho^{-n}); 0 \leq k \leq [\rho^n]\}$ is an independent sequence, for all n large enough,

$$\begin{aligned}\mathbb{P}\Big(\max_{k \leq [\rho^n]} |I(\rho^{-n}, k\rho^{-n})| \leq \varphi(\rho^{-n})\sqrt{2\theta/3}\Big) &\leq \Big(1 - C_1(\theta,\varepsilon)\rho^{-n\theta(1+\varepsilon)}\Big)^{[\rho^n]} \\ &\leq \exp\Big(-\frac{1}{2}C_1(\theta,\varepsilon)\rho^{-n\theta(1+\varepsilon)+n}\Big),\end{aligned}$$

which sums if $0 < \theta < (1+\varepsilon)^{-1}$. Therefore by the Borel–Cantelli lemma,

$$\liminf_{n\to\infty} \max_{k \leq [\rho^n]} \frac{|I(\rho^{-n}; k\rho^{-n})|}{\varphi(\rho^{-n})} \geq \sqrt{\frac{2\theta}{3}}, \qquad \text{a.s.}$$

for all $\theta \in (0, (1+\varepsilon)^{-1})$. Since $\varepsilon > 0$ and $\theta < (1+\varepsilon)^{-1}$ are arbitrary, letting $\varepsilon \downarrow 0$ and $\theta \uparrow 1$ along a countable sequence, it follows that

$$\liminf_{n\to\infty} \sup_{0 \leq t \leq 1-\rho^{-n}} \frac{|I(\rho^{-n}; t)|}{\varphi(\rho^{-n})} \geq \sqrt{\frac{2}{3}}, \qquad \text{a.s.}$$

Now for each $h > 0$ there exists a unique integer, N_h, such that $\rho^{-(N_h+1)} \leq h < \rho^{-N_h}$. Writing

$$I(h;t) = (\rho^{N_h+1}h)^{-1} I(\rho^{-(N_h+1)}; t) + h^{-1}\int_{t+\rho^{-(N_h+1)}}^{t+h} (\ Z(s) - Z(t)\ ) ds,$$

it follows that,

$$\begin{aligned}\liminf_{h\downarrow 0} \sup_{0 \leq t \leq 1-h} \frac{|I(h;t)|}{\varphi(h)} &\geq \liminf_{h\downarrow 0} \sup_{0 \leq t \leq 1-h} (\rho^{N_h+1}h)^{-1} \frac{|I(\rho^{-(N_h+1)}; t)|}{\varphi(h)} - \\ &\quad - \limsup_{h\downarrow 0} \sup_{0 \leq t \leq 1-h} (h\varphi(h))^{-1} \int_{t+\rho^{-(N_h+1)}}^{t+h} |Z(s) - Z(t)| ds \\ &\geq \rho^{-3/2}\sqrt{\frac{2}{3}} - \limsup_{h\downarrow 0} \sup_{\substack{0 \leq s,t \leq 1 \\ |s-t| \leq \rho^{-N_h}}} \Big(\frac{h - \rho^{-(N_h+1)}}{h}\Big) \frac{|Z(s) - Z(t)|}{\varphi(\rho^{-N_h})} \\ &\geq \rho^{-3/2}\sqrt{\frac{2}{3}} - \sqrt{2}(\rho - 1),\end{aligned}$$

by (2.1b). In the above, we have used the elementary inequality:

$$\varphi(h) \le \varphi(\rho^{-N_h}) = (1 + o(1))\sqrt{\rho}\varphi(\rho^{-(N_h+1)}),$$

as $h \downarrow 0$. Since $\rho > 1$ is arbitrary, the lower bound follows by taking $\rho \uparrow 1$ along a countable sequence, i.e., we have

$$\liminf_{h\downarrow 0} \sup_{0\le t\le 1-h} \frac{|I(h;t)|}{\varphi(h)} \ge \sqrt{\frac{2}{3}}, \qquad \text{a.s.}$$

The proof of the upper bound uses an entropy argument together with the upper estimate in (2.2). To this end, fix $\rho > 1$ and define

$$T_n \stackrel{df}{=} T_n(\theta, \varepsilon, \rho) = \{\ t \ge 0 :\ t = j\rho^{-n\theta(1-\varepsilon)} n^2 \text{ for some integer } 0 \le j \le \rho^{n\theta(1-\varepsilon)} n^{-2}\ \}.$$

Notice that $\#T_n \le C_3(\theta,\varepsilon) n^{-2} \rho^{n\theta(1-\varepsilon)}$, for some finite positive constant, $C_3(\theta,\varepsilon)$. Therefore by (2.2),

$$\begin{aligned}\mathbb{P}\left(\max_{t\in T_n} |I(\rho^{-n};t)| \ge \varphi(\rho^{-n})\sqrt{2\theta/3}\right) &\le C_2(\theta,\varepsilon)\#T_n \rho^{-n\theta(1-\varepsilon)} \\ &\le C_3(\theta,\varepsilon)C_2(\theta,\varepsilon)n^{-2},\end{aligned}$$

which sums in n. So by the Borel–Cantelli lemma, for each $\rho > 1$, $\varepsilon \in (0,1)$ and all $\theta > 0$,

$$\text{(2.3)} \qquad \limsup_{n\to\infty} \max_{t\in T_n} \frac{|I(\rho^{-n};t)|}{\varphi(\rho^{-n})} \le \sqrt{\frac{2\theta}{3}}.$$

Now,

$$\text{(2.4)} \qquad \sup_{0\le t\le 1} |I(\rho^{-n};t)| \le \max_{t\in T_n} |I(\rho^{-n};t)| + \sup_{0\le t\le 1} |I(\rho^{-n};t) - I(\rho^{-n}; f_n(t))|,$$

where $f_n(t)$ is the point in T_n nearest to t (with some convention about breaking ties.) We note that,

$$\sup_{t\in T_n} |t - f_n(t)| = \rho^{-n\theta(1-\varepsilon)} n^2.$$

Therefore, by (2.1b),

$$\begin{aligned}\sup_{0\le t\le 1} |I(\rho^{-n};t) - I(\rho^{-n}; f_n(t))| &\le 2 \sup_{0\le t\le 2} |Z(t) - Z(f_n(t))| \\ &= 2^{3/2}(1+o(1))\sqrt{\rho^{-n\theta(1-\varepsilon)} n^2 \log(\rho^{n\theta(1-\varepsilon)} n^{-2})} \\ &= 2^{3/2}(1+o(1))(\theta(1-\varepsilon)\log\rho)^{1/2} \rho^{-n\theta(1-\varepsilon)/2} n^{3/2} \\ &= o(\varphi(\rho^{-n})),\end{aligned}$$

if $\theta > (1-\varepsilon)^{-1}$. Therefore, for all $\theta > (1-\varepsilon)^{-1}$, by (2.3) and (2.4), almost surely,

$$\limsup_{n\to\infty} \sup_{0\le t\le 1} \frac{|I(\rho^{-n};t)|}{\varphi(\rho^{-n})} \le \sqrt{\frac{2\theta}{3}}.$$

Since $\theta > (1-\varepsilon)^{-1}$ and $\varepsilon \in (0,1)$ are arbitrary, we have

$$\limsup_{n\to\infty} \sup_{0\le t\le 1} \frac{|I(\rho^{-n};t)|}{\varphi(\rho^{-n})} \le \sqrt{\frac{2}{3}}, \qquad \text{a.s.} \tag{2.5}$$

It remains to show that the behavior of the limsup in question is the same along any subsequence. To this end, fix $\rho > 1$ as above. For each $h > 0$ (small), there exists a unique integer, $N_h \ge 1$, such that $\rho^{-(N_h+1)} \le h < \rho^{-N_h}$. Proceeding as in the proof of the lower bound,

$$\begin{aligned}
I(h;t) &= h^{-1} \int_t^{t+h} (\ Z(s) - Z(t)\ ) ds \\
&= h^{-1} \int_t^{t+\rho^{-(N_h+1)}} (\ Z(s) - Z(t)\ ) ds + h^{-1} \int_{t+\rho^{-(N_h+1)}}^{t+h} (\ Z(s) - Z(t)\ ) ds \\
&= (\rho^{N_h+1} h)^{-1} I(\rho^{-(N_h+1)};t) + h^{-1} \int_{t+\rho^{-(N_h+1)}}^{t+h} (\ Z(s) - Z(t)\ ) ds.
\end{aligned}$$

Therefore by (2.1b), as $h \downarrow 0$ (and hence $N_h \uparrow \infty$), the following holds uniformly in $0 \le t \le 1$:

$$\begin{aligned}
|I(h;t)| &\le |I(\rho^{-(N_h+1)};t)| + \rho(\rho-1) \sup_{\substack{|s-t|\le \rho^{-N_h} \\ 0\le s,t\le 1}} |Z(s) - Z(t)| \\
&\le \sqrt{\frac{2}{3}} (1+o(1)) \varphi(\rho^{-(N_h+1)}) + \sqrt{2}\rho(\rho-1)(1+o(1))\varphi(\rho^{-N_h}).
\end{aligned}$$

Note that as $h \downarrow 0$, $\varphi(h) \ge \varphi(\rho^{-(N_h+1)}) = (1+o(1))\rho^{-1/2}\varphi(\rho^{-N_h})$. This implies that for all $\rho > 1$,

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \frac{|I(h;t)|}{\varphi(h)} \le \sqrt{\frac{2}{3}} + \sqrt{2}\rho^{3/2}(\rho-1).$$

Since $\rho > 1$ is arbitrary, the proof of Theorem 2.1(a) is completed upon taking $\rho \downarrow 1$ along a countable sequence. $\square$

Proof of Theorem 2.1(b). Fix $\alpha \ge 2$. Without loss of much generality, suppose that $T = [\varepsilon, 1]$ for some $\varepsilon > 0$. For any $E \subseteq \mathbb{R}^1_+$, let $\mathfrak{C}(E)$ denote the collection of all continuous functions from E in to $\mathbb{R}^1$. As usual, we endow $\mathfrak{C}(E)$ with the compact open topology. An application of Girsanov's theorem (see Revuz and Yor (1991), p. 419, exercise (1.22)) reveals that for any $s, x > 0$ and for all measurable $A \subseteq \mathfrak{C}([0, s])$,

$$\mathbb{P}(\pi_s(\beta_\alpha) \in A | \beta_\alpha(0) = x) = x^{-(\alpha-1)/2}\mathbb{E}(D_s 1_A(\pi_s(X))|Z(0) = x), \tag{2.6}$$

where $T_0 \stackrel{df}{=} \inf\{s : Z(s) = 0\}$,

$$D_s \stackrel{df}{=} Z(s \wedge T_0)^{(\alpha-1)/2} \exp\bigg(-\frac{1}{8}(\alpha-1)(\alpha-3)\int_0^s Z(r)^{-2} dr\bigg),$$

and for any $s > 0$ and $f \in \mathfrak{C}(E)$, $\pi_s(f)$ denotes the element of $\mathfrak{C}([0, s])$ given by

$$\pi_s(f)(r) \stackrel{df}{=} f(r) \qquad \text{for all } 0 \le r \le s.$$

Taking A^c to be the collection of all elements, f, of $\mathfrak{C}([0, 1])$ which satisfy:

$$\lim_{h\downarrow 0} \sup_{0\le t\le 1-h} \frac{|h^{-1}\int_t^{t+h} f(r)dr - f(t)|}{\varphi(h)} = \sqrt{\frac{2}{3}},$$

part (b) follows for $x > 0$ as a consequence of (2.6). It is therefore enough to prove part (b) when $x = 0$.

By the Markov property and the case $x > 0$ proved above, for any $0 < \varepsilon < 1$,

$$\lim_{h\downarrow 0} \sup_{\varepsilon\le t\le 1-h} \frac{|h^{-1}\int_t^{t+h} \beta_\alpha(r)dr - \beta_\alpha(t)|}{\varphi(h)} = \sqrt{\frac{2}{3}}, \qquad \text{a.s.}$$

This proves part (b). $\square$

Proof of Theorem 2.1(c). Fix $\alpha \ge 2$. Suppose, without loss of much generality that $T = [0, 1]$ and $x = 0$. Define

$$J(h; t) \stackrel{df}{=} h^{-1}\int_t^{t+h} \beta_\alpha^2(r)dr,$$
$$\widetilde{J}(h; t) \stackrel{df}{=} h^{-1}\int_t^{t+h} (\ \beta_\alpha(r) - \beta_\alpha(t)\ )dr.$$

Then for all $t \in [0, 1]$ and all $h > 0$,

$$\frac{|J(h; t) - \beta_\alpha^2(t)|}{\varphi(h)} = 2\beta_\alpha(t)\frac{|\widetilde{J}(h; t)|}{\varphi(h)} + R(h; t),$$

where by (2.1a), as $h \downarrow 0$,

$$\sup_{0\le t\le 1-h} |R(h;t)| \le \sup_{0\le t\le 1-h} \frac{h^{-1}\int_t^{t+h} |\beta_\alpha(r)-\beta_\alpha(t)|^2 dr}{\varphi(h)}$$
$$\le 2(1+o(1))\varphi(h) \to 0, \qquad \text{a.s.}$$

It is therefore enough to prove:

$$\lim_{h\downarrow 0} \sup_{0\le t\le 1-h} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} = \sqrt{\frac{2}{3}} \sup_{0\le t\le 1} \beta_\alpha(t), \tag{2.7}$$

almost surely. Suppose instead, that we could prove that for any $\varepsilon > 0$, with probability one,

$$\lim_{h\downarrow 0} \sup_{\varepsilon\le t\le 1-h} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} = \sqrt{\frac{2}{3}} \sup_{\varepsilon\le t\le 1} \beta_\alpha(t). \tag{2.8}$$

It is not hard to see that (2.8) implies (2.7). Indeed, (2.8) would imply that for all $\varepsilon > 0$,

$$\liminf_{h\downarrow 0} \sup_{0\le t\le 1-h} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} \ge \sqrt{\frac{2}{3}} \sup_{\varepsilon\le t\le 1} \beta_\alpha(t), \qquad \text{a.s.}$$

which proves the lower bound in (2.7) by taking $\varepsilon \to 0$ along a countable sequence. (Recall that $t \mapsto \beta_\alpha(t)$ is almost surely continuous.) To see the upper bound in (2.7), note that

$$\sup_{0\le t\le 1-h} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} \le \Big(\sup_{0\le t\le \varepsilon} + \sup_{\varepsilon\le t\le 1-h}\Big) \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)}.$$

Therefore by (2.8),

$$\limsup_{h\downarrow 0} \sup_{0\le t\le 1-h} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} \le \limsup_{h\downarrow 0} \sup_{0\le t\le \varepsilon} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} + \sqrt{\frac{2}{3}} \sup_{\varepsilon\le t\le 1} \beta_\alpha(t)$$
$$\le \sqrt{2} \sup_{0\le t\le \varepsilon} \beta_\alpha(t) + \sqrt{\frac{2}{3}} \sup_{0\le t\le 1} \beta_\alpha(t),$$

by (2.1a). Since the left hand side of the above inequalities is independent of ε and $\varepsilon > 0$ is arbitrary, (2.7) follows by sample path continuity of $t \mapsto \beta_\alpha(t)$. Hence it is sufficient to prove (2.8) for all $\varepsilon > 0$.

Recall that the compact set mentioned in part (b) is an arbitrary compact subset of $[\varepsilon, 1]$. As a result, by part (b) of Theorem 2.1, with probability one,

$$\lim_{h\downarrow 0} \sup_{a\le t\le b} \frac{|\widetilde{J}(h;t)|}{\varphi(h)} = \sqrt{\frac{2}{3}} \qquad \text{for all rational } a, b:\ \varepsilon \le a < b \le 1.$$

By a condensation argument, this implies that the (random) set, $\widetilde{T}$, defined by

$$\widetilde{T} \stackrel{df}{=} \left\{ \varepsilon \le t \le 1 : \lim_{h \downarrow 0} \frac{|\widetilde{J}(h;t)|}{\varphi(h)} = \sqrt{\frac{2}{3}} \right\},$$

has a dense trace on $[\varepsilon, 1]$, with probability one. (In other words, $\overline{\widetilde{T} \cap [\varepsilon, 1]} = [\varepsilon, 1]$, a.s.) Evidently, with probability one,

$$\begin{aligned} \liminf_{h \downarrow 0} \sup_{\varepsilon \le t \le 1-h} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} &\ge \liminf_{h \downarrow 0} \beta_\alpha(t_0) \frac{|\widetilde{J}(h;t_0)|}{\varphi(h)} \\ &= \sqrt{\frac{2}{3}} \beta_\alpha(t_0), \end{aligned}$$

for all $t_0 \in \widetilde{T} \cap [\varepsilon, 1]$. Here we have only used the definition of $\widetilde{T}$. Therefore by the (a.s.) density of $\widetilde{T} \cap [\varepsilon, 1]$ in $[\varepsilon, 1]$ and the (a.s.) path continuity for $t \mapsto \beta_\alpha(t)$, the lower bound in (2.8) holds. The upper bound in (2.8) is even simpler, for

$$\begin{aligned} \limsup_{h \downarrow 0} \sup_{\varepsilon \le t \le 1} \beta_\alpha(t) \frac{|\widetilde{J}(h;t)|}{\varphi(h)} &\le \limsup_{h \downarrow 0} \sup_{\varepsilon \le t \le 1} \beta_\alpha(t) \sup_{\varepsilon \le t \le 1} \frac{|\widetilde{J}(h;t)|}{\varphi(h)} \\ &= \sqrt{\frac{2}{3}} \sup_{\varepsilon \le t \le 1} \beta_\alpha(t), \end{aligned}$$

by part (b). This finishes proof of (2.8) and hence part (c) of Theorem 2.1. $\square$

3. Proof of Theorem 1.1.

Let us enlarge the probability space so that it includes an exponential random variable, λ, which is independent of Z and whose mean is 2. Define,

$$m \stackrel{df}{=} \inf_{0 \le r \le \lambda} Z(r) \qquad \text{and} \qquad M \stackrel{df}{=} \sup_{0 \le r \le \lambda} Z(r).$$

Let $x_0 > 0$ be fixed. Moreover, suppose we have enlarged the probability space even further so that it includes independent Bessel processes, β_4, $\widetilde{\beta}_4$ and β_2, with dimensions: 4, 4 and 2, respectively. The point is that these processes are totally independent of Z and λ as well as each other, except that they are conditioned to satisfy:

$$\begin{aligned} \widetilde{\beta}_4(0) &= \beta_4(0) = 0 \\ \beta_2(1) &= \beta_4(1 - e^{2m}) \\ \widetilde{\beta}_4(e^{-2x_0} - e^{-2M}) &= \beta_2(e^{2x_0}). \end{aligned}$$

Then according to Ray's description of $L^{\cdot}_\lambda$, (see Itô and McKean (1965) and Williams (1974)), one can construct a version of $L^{\cdot}_\lambda$ (which we shall continue to write as $L^{\cdot}_\lambda$), with the following realization:

$$
(3.1) \qquad 2L^x_\lambda = \begin{cases} 0 & \text{if } x \le m \text{ or } x \ge M \\ a(x) \overset{df}{=} e^{-2x}\beta_4^2(e^{2x} - e^{2m}) & \text{if } m \le x \le 0 \\ b(x) \overset{df}{=} e^{-2x}\beta_2^2(e^{2x}) & \text{if } 0 \le x \le x_0 \\ c(x) \overset{df}{=} e^{2x}\widetilde{\beta}_4^2(e^{-2x} - e^{-2M}) & \text{if } x_0 \le x \le M \end{cases},
$$

conditioned on the event $\{Z(0) = x_0\}$. (We note here that our speed measure is half of that of Itô and McKean and hence their $4L^{\cdot}_\lambda$ is replaced by our $2L^{\cdot}_\lambda$.) Applying Theorem 2.1(c) to this version of $L^{\cdot}_\lambda$, we see that for any compact set $T \subseteq \mathbb{R}^1$, conditioned on $Z(0) = x_0 > 0$,

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\sup_{x\in T}\frac{|\varepsilon^{-1}\int_x^{x+\varepsilon} 2L^x_\lambda dx - 2L^x_\lambda|}{\varphi(\varepsilon)} \\
&\qquad = \sqrt{\frac{8}{3}}\max\left\{\sup_{m\le z\le 0} a^{1/2}(z),\ \sup_{0\le z\le x_0} b^{1/2}(z),\ \sup_{x_0\le z\le M} c^{1/2}(z)\right\} \\
&\qquad = \sqrt{\frac{16}{3}}(\sup_{x\in T} L^x_\lambda)^{1/2}.
\end{aligned}
$$

By considering $-Z(\cdot)$ instead, since $\mathbb{P}(Z(\lambda) = 0) = 0$, we see from (1.3) that for all compact $T \subseteq \mathbb{R}^1$, almost surely,

$$
\lim_{\varepsilon\downarrow 0}\sup_{x\in T}\frac{|\varepsilon^{-1}\int_0^\lambda 1_{[x,x+\varepsilon]}(Z(r))dr - L^x_\lambda|}{\varphi(\varepsilon)} = ((4/3)\sup_{x\in T} L^x_\lambda)^{1/2}.
$$

Therefore,

$$
\mathbb{P}\left(\lim_{\varepsilon\downarrow 0}\sup_{|x|\le n}\frac{|\varepsilon^{-1}\int_0^\lambda 1_{[x,x+\varepsilon]}(Z(r))dr - L^x_\lambda|}{\varphi(\varepsilon)} = ((4/3)\sup_{|x|\le n} L^x_\lambda)^{1/2},\ \text{for all } n \ge 1\right) = 1.
$$

Taking $n = n(\omega) \overset{df}{=} M(\omega) \vee (-m(\omega))$, it follows from (3.1) that,

$$
\sup_{|x|>n} L^x_\lambda = \sup_{|x|>n}\int_0^\lambda 1_{[x,x+\varepsilon]}(Z(r))dr = 0,
$$

almost surely. Therefore, $\mathbb{P}(A(\lambda)) = 1$, where for all $t \geq 0$,

$$A(t) \stackrel{df}{=} \left\{ \omega : \limsup_{\varepsilon \downarrow 0} \sup_{x \in \mathbb{R}^1} \frac{|\varepsilon^{-1} \int_0^t 1_{[x,x+\varepsilon]}(Z(r)(\omega))dr - L_t^x(\omega)|}{\varphi(\varepsilon)} = ((4/3)L_t^*(\omega))^{1/2} \right\}.$$

By independence,

$$\begin{aligned} 1 = \mathbb{P}(A(\lambda)) &= \int_0^\infty \mathbb{P}(A(t))\mathbb{P}(\lambda \in dt) \\ &= \frac{1}{2} \int_0^\infty \mathbb{P}(A(t)) e^{-t/2} dt. \end{aligned}$$

Let $u > 0$ be any positive real number. By considering $u\lambda$ instead of λ, the above analysis together with Brownian scaling show that for all $u > 0$,

$$1 = \frac{1}{2u} \int_0^\infty \mathbb{P}(A(t)) e^{-t/(2u)} dt.$$

Therefore, by the inversion theorem for Laplace transforms, $\mathbb{P}(A(t)) = 1$ for all $t > 0$. This concludes the proof of Theorem 1.1. $\square$

4. Proof of Theorem 1.3.

By Brownian scaling, it suffices to consider only the case $t = 1$. Furthermore, since for all $\varepsilon > 0$:

$$L_t^x = u_\varepsilon(x,t) = 0, \qquad \text{on } \{t \leq T_x\},$$

by considering the Brownian motion, $\{\vartheta_{T_x}(Z)(t); t \geq 0\}$, it suffices to prove the theorem when $x = 0$.

For every $w > 0$, define $\mathbb{Q}_w(\cdots)$ to be a nice version of the regular conditional probability, $\mathbb{P}(\cdots | L_1^0 = w)$. In light of the above discussion, it is evidently sufficient to prove that for all $w > 0$:

$$\limsup_{\varepsilon \downarrow 0} \frac{|2\varepsilon u_\varepsilon(0,1) - w|}{\psi(\varepsilon)} = 2w^{1/2}, \qquad \mathbb{Q}_w - \text{a.s.} \tag{4.1}$$

where $\psi(h) \stackrel{df}{=} \sqrt{h(\ 1 \vee \log\log(1/h)\ )}$, as in Section 3.

Define the inverse local times by:

$$\tau_t^x \stackrel{df}{=} \inf\{s > 0 : L_s^x = t\}, \tag{4.2a}$$

$$\tau(t) \stackrel{df}{=} \tau_t^0. \tag{4.2b}$$

Also define the excursion process, $\{e_s^x(\cdot); s \geq 0\}$ of (Z) from x as follows:

$$e_s^x(r) \stackrel{df}{=} \begin{cases} Z(\tau_{s-}^x + r) & \text{, if } 0 < r \leq \tau_s^x - \tau_{s-}^x \\ x & \text{, if } r = 0 \\ \delta & \text{, otherwise} \end{cases},$$

where δ is the "coffin state". The reader will note that $\{e_s^x(\cdot); s \geq 0\}$ is a stochastic process whose state space is $U_x \cup \{\delta\}$, where

$$U_x \stackrel{df}{=} \left\{ f \in \mathfrak{C}([0,\infty)) : f(0) = x, \text{ and } f(u) = x, \text{ for all } u \geq R_x^f, \text{ for some } R_x^f \right\}.$$

The σ–field on U_x is the one inherited from $\mathfrak{C}([0,\infty))$. For any $t \geq 0$ and all measurable $A \subseteq U_x$ define, $N_t^x(A) \stackrel{df}{=} \sum_{s \leq t} 1_A(e_s^x)$. We note that $N_t^x(A)$ is at most a countable sum and hence well–defined. Define the corresponding Itô measure, n^x, by $n^x(A) \stackrel{df}{=} \mathbb{E} N_1^x(A)$. It is not hard to see that $\{N_t^x(A); t \geq 0\}$ is a Poisson process with intensity $n^x(A)$, if the latter is finite. In this case, $\{N_t^x(A) - n^x(A)t; t \geq 0\}$ is a mean zero martingale and hence a monotone class argument implies the following "exit system formula" for all $t > 0$:

$$\mathbb{E} \sum_{s \leq t} Y(s) f(e_s^0) = \mathbb{E} \int_0^t Y(s) n^{Z(s)}(f) L_{ds}^0,$$

for all bounded (say) predictable processes, (Y), and all bounded measurable functions, $f : \cup_x U_x \cup \{\delta\} \mapsto \mathbb{R}^1$, such that $f(\delta) = 0$. Here, as is customary, $n^x(f) \stackrel{df}{=} \int_{U_x \cup \{\delta\}} f(e) n^x(de)$. One can write a similar equation for excursions from any other $x \neq 0$. Moreover, by the exit system formula, under $\mathbb{Q}_w$, $N_w^0(A)$ is a Poisson random variable with mean $n^0(A)w$.

Fix $w > 0$ throughout and define,

$$A(\varepsilon) \stackrel{df}{=} \{f \in U_0 : \sup_{0 \leq r \leq R_0^f} f(r) \geq \varepsilon\}.$$

Since $|u_\varepsilon(0,1) - N_{L_1^0}^0(A(\varepsilon))| \leq 1$, it is sufficient (by (4.1) and the preceding discussion) to prove that:

$$\limsup_{\varepsilon \downarrow 0} \frac{|2\varepsilon N_w^0(A(\varepsilon)) - w|}{\psi(w)} = 2w^{1/2}, \qquad \mathbb{P} - \text{a.s.} \tag{4.3}$$

It is (4.3) that we shall prove; but first, we next need two technical lemmas.

Lemma 4.1. *Suppose $\{X(\alpha); \alpha \geq 0\}$ is a $\mathbb{R}^1_+$–valued stochastic process satisfying the following for every $\alpha \geq 0$:*

(a) $\mathbb{E}X(\alpha) \stackrel{df}{=} \mu(\alpha) < \infty$;

(b) $\mathrm{Var}X(\alpha) \stackrel{df}{=} \sigma^2(\alpha) < \infty$;

(c) for all $t \geq 0$, $\mathbb{E}\exp(itX(\alpha)) = (f(t))^\alpha$, where f is a characteristic function (independent of α).

Then uniformly over $x = o(\alpha^{1/6})$,

$$\mathbb{P}(|X(\alpha) - \mu(\alpha)| \geq x\sigma(\alpha)) = \sqrt{\frac{2}{\pi}}\; x^{-1} \exp(-x^2/2)\; (1 + o(1)).$$

Proof. This lemma is a trivial modification of a large deviations theorem of Cramér. See Theorem 8.1.1 of Ibragimov and Linnik (1971) for Petrov's refinement of this result. $\square$

Lemma 4.2. *Fix $w > 0$ and $\theta > 0$. Then as $\varepsilon \downarrow 0$,*

$$\mathbb{P}\Big(|2\varepsilon N^0_w(A(\varepsilon)) - w| \geq 2\sqrt{\theta w}\psi(\varepsilon) \Big) = \frac{\sqrt{2/\pi}}{(2\theta \log\log(1/\varepsilon))^{1/2} (\log(1/\varepsilon))^\theta}\; (1 + o(1)).$$

Proof. Fix $\theta > 0$ and let $X(\varepsilon) \stackrel{df}{=} N^0_w(A(\varepsilon))$. Then by the discussion preceding Lemma 4.1, $\{X(\varepsilon); \varepsilon > 0\}$ satisfies the assumptions of Lemma 4.1 with $\mu(\varepsilon) = \sigma^2(\varepsilon) = n^0(A(\varepsilon))$, which is equal to $(2\varepsilon)^{-1}$ by Proposition XII.3.6. of Revuz and Yor (1991). Hence by Lemma 4.1, uniformly over all $x = o(\varepsilon^{-1/12})$,

$$\mathbb{P}\Big(|N^0_w(A(\varepsilon)) - (w/2\varepsilon)| \geq \sqrt{w/(2\varepsilon)}x \Big) = \sqrt{\frac{2}{\pi}} x^{-1} \exp(-x^2/2)\; (1 + o(1)),$$

as $\varepsilon \downarrow 0$. The lemma follows upon letting $x \stackrel{df}{=} \sqrt{2\theta \log\log(1/\varepsilon)}$. $\square$

Now we can finish the proof of (4.3) and hence Theorem 1.3. As in the proof of the law of the iterated logarithm for Brownian motion, the proof consists of establishing an upper bound as well as a lower bound. We shall start by proving the former:

Fix $\rho \in (0,1)$ and let $\rho(n) \stackrel{df}{=} \rho^n$. Then by Lemma 4.2, for any $\theta > 0$,

$$\mathbb{P}\Big(|2\rho(n) N^0_w(A(\rho(n))) - w| \geq 2\sqrt{\theta w}\psi(\rho(n)) \Big)$$
$$= (\theta\pi \log n)^{-1/2} (n \log(1/\rho))^{-\theta}\; (1 + o(1)),$$

as $n \to \infty$. Therefore by the Borel–Cantelli lemma, for all $\theta > 1$,

$$\limsup_{n\to\infty} \frac{|2\rho(n)N_w^0(A(\rho(n))) - w|}{\psi(\rho(n))} \le 2\sqrt{\theta w}, \qquad \text{a.s.}$$

Taking $\theta \downarrow 1$ along a countable sequence, it follows that

$$\limsup_{n\to\infty} \frac{|2\rho(n)N_w^0(A(\rho(n))) - w|}{\psi(\rho(n))} \le 2\sqrt{w}, \qquad \text{a.s.} \tag{4.4}$$

Now for every $\varepsilon \in (0,1)$, there exists an integer N_ε such that $\rho(N_\varepsilon + 1) \le \varepsilon \le \rho(N_\varepsilon)$. By sample path continuity, everytime Z upcrosses $[0,\varepsilon]$, it has also upcrossed $[0, \rho(N_\varepsilon + 1)]$. Therefore, $N_w^0(A(\varepsilon)) \le N_w^0(A(\rho(N_\varepsilon + 1)))$. Likewise, $N_w^0(A(\varepsilon)) \ge N_w^0(A(\rho(N_\varepsilon)))$. Hence by (4.4) and some arithmetic,

$$\limsup_{\varepsilon\downarrow 0} \frac{|2\varepsilon N_w^0(A(\varepsilon)) - w|}{\psi(\varepsilon)} \le 2\sqrt{w/\rho}, \qquad \text{a.s.}$$

Since $\rho \in (0,1)$ is arbitrary, we can take $\rho \uparrow 1$ along a countable sequence to arrive at the desired upper bound in (4.3):

$$\limsup_{\varepsilon\downarrow 0} \frac{|2\varepsilon N_w^0(A(\varepsilon)) - w|}{\psi(\varepsilon)} \le 2\sqrt{w}, \qquad \text{a.s.}$$

Next we prove the lower bound, thus completing the proof of Theorem 1.3. Throughout the rest of this proof, let us redefine $\rho(n)$ by $\rho(n) \stackrel{df}{=} n^{-n}$. The key lemma is the following:

Lemma 4.3. *With probability one,*

$$\limsup_{n\to\infty} \frac{|\rho(n+1)N_w^0(A(\rho(n+1))) - \rho(n)N_w^0(A(\rho(n)))|}{\psi(\rho(n))} = \sqrt{w}.$$

Assuming the truth of this lemma for the time being, (4.3) and hence Theorem 1.3 follow from the following inequalities:

$$\begin{aligned}
\limsup_{\varepsilon\downarrow 0} \frac{|2\varepsilon N_w^0(A(\varepsilon)) - w|}{\psi(\varepsilon)} &\ge \limsup_{n\to\infty} \frac{|2\rho(n)N_w^0(A(\rho(n))) - w|}{\psi(\rho(n))} \\
&\ge 2\sqrt{w} - 2\limsup_{n\to\infty} \frac{|\rho(n+1)N_w^0(A(\rho(n+1))) - w|}{\psi(\rho(n))} \\
&= 2\sqrt{w}.
\end{aligned} \tag{4.6}$$

In proving (4.6), we have used Lemma 4.3 together with the proven upper bound in (4.3) as well as the fact that $\psi(\rho(n+1))/\psi(\rho(n)) = (en)^{-1/2}(1+o(1)) \to 0$, as $n \to \infty$.

Proof of Lemma 4.3. Recall that $\rho(n) = n^{-n}$ and $w > 0$ is fixed. To simplify the notation, let us define $N(n) \stackrel{df}{=} N_w^0(A(\rho(n)))$ and $\psi_n \stackrel{df}{=} \psi(\rho(n))$. From the exit system formula, it follows that conditioned on $\{N(n) = k\}$,

$$N(n+1) = \sum_{j=1}^{k} \Delta_{j,n},$$

where $\{\Delta_{j,n}; j \geq 1\}$ are i.i.d. geometric random variables with distribution given by,

$$\mathbb{P}(\Delta_{1,n} = j) = \begin{cases} \left(1 - \frac{\rho(n+1)}{\rho(n)}\right)^{j-1} \left(\frac{\rho(n+1)}{\rho(n)}\right) & \text{if } j = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, in particular,

$$\mathbb{E}\Delta_{1,n} = \frac{\rho(n)}{\rho(n+1)}, \tag{4.6a}$$

$$\text{Var } \Delta_{1,n} = \frac{(\rho(n) - \rho(n+1))\rho(n)}{\rho^2(n+1)}. \tag{4.6b}$$

Let $\widetilde{\Delta}_{j,n} \stackrel{df}{=} \Delta_{j,n} - \mathbb{E}\Delta_{j,n}$. For $\theta > 0$, define,

$$E_\theta(n) \stackrel{df}{=} \left\{\omega : \frac{|\rho(n+1)N(n+1) - \rho(n)N(n)|(\omega)}{\psi_n} \geq \theta\sqrt{w}\right\}, \tag{4.7a}$$

$$\widetilde{E}(n) \stackrel{df}{=} \left\{k \in \mathbb{Z}_+^1 : |2k\rho(n) - w| \geq 2\sqrt{2w\rho(n)\log n}\right\}. \tag{4.7b}$$

Since $N(n)$ is Poisson with mean $w(2\rho(n))^{-1}$, Lemma 4.2 implies that as $n \to \infty$,

$$\begin{aligned} \mathbb{P}(N(n) \notin \widetilde{E}(n)) &= \sqrt{\frac{1}{2\pi}}(\log n)^{-1/2}n^{-2}\ (1+o(1)) \\ &= o(n^{-2}). \end{aligned} \tag{4.7c}$$

Therefore,

$$\begin{aligned} \mathbb{P}(E_\theta(n)) &= \sum_{k=0}^{\infty} \mathbb{P}(E_\theta(n) \mid N(n) = k)\mathbb{P}(N(n) = k) \\ &= \sum_{k \notin \widetilde{E}(n)} \mathbb{P}\Big(|\sum_{i=1}^{k} \widetilde{\Delta}_{j,n}| \geq \theta\sqrt{w}\psi_n/\rho(n+1)\Big)\mathbb{P}(N(n) = k) + o(n^{-2}). \end{aligned}$$

Now fix $\theta \in (1, \sqrt{2})$. Next we will prove that uniformly over all $k \notin \widetilde{E}(n)$:

$$\mathbb{P}\Big(|\sum_{i=1}^{k} \widetilde{\Delta}_{j,n}| \geq \theta\sqrt{w}/\rho(n+1) \Big) = \frac{1}{\sqrt{\pi}\theta} (\log n)^{-1/2} n^{-\theta^2} \ (1+o(1)). \tag{4.8}$$

Indeed, by (4.6b), for all $k \notin \widetilde{E}(n)$,

$$\text{Var } (\sum_{i=1}^{k} \Delta_{j,n}) = \frac{w\rho(n)}{2\rho^2(n+1)} \ (1+o(\alpha_n)),$$

where $\alpha_n \stackrel{df}{=} n^{-(n-1)/2}(\log n)^{1/2}$. Therefore,

$$\begin{aligned}
\mathbb{P}\Big(|\sum_{j=1}^{k} \widetilde{\Delta}_{j,n}| \geq \theta\sqrt{w}\psi_n/\rho(n+1) \Big) &= \mathbb{P}\Big(|S_{k,n}| \geq \theta\sqrt{2(1+o(\alpha_n))\log n} \Big) \\
&= \mathbb{P}\Big(|S_{k,n}| \geq \theta\sqrt{2}\frac{\psi_n}{\sqrt{\rho(n)}} \Big) \\
&= \sqrt{\frac{2}{\pi\theta}} (\log n)^{-1/2} n^{-\theta^2} \ (1+o(1)),
\end{aligned}$$

proving (4.8). In the above, we have applied Lemma 4.1 to the normalized sum:

$$S_{k,n} \stackrel{df}{=} (\text{Var } \sum_{i=1}^{k} \widetilde{\Delta}_{i,n})^{-1/2} \sum_{i=1}^{k} \widetilde{\Delta}_{j,n}.$$

As a result, (4.7c) and (4.9) together imply the following:

$$\mathbb{P}(E_\theta(n)) = \frac{1}{\sqrt{\pi}\theta} (\log n)^{-1/2} n^{-\theta^2} \ (1+o(1)). \tag{4.9}$$

In particular, picking $\theta \in (1, \sqrt{2})$, we see that $\sum_n \mathbb{P}(E_\theta(n)) < \infty$, and hence by the Borel–Cantelli lemma — letting $\theta \downarrow 1$ along a countable sequence — it follows that with probability one,

$$\limsup_{n\to\infty} \frac{|\rho(n+1)N(n+1) - \rho(n)N(n)|}{\psi_n} \leq \sqrt{w},$$

which proves the upper half of Lemma 4.3.

For the lower half, fix any $\theta \in (0,1)$. Then by (4.9),

$$\sum_{n=1}^{\infty} \mathbb{P}(E_\theta(n)) = \infty. \tag{4.10}$$

By the strong Markov property (or by using the exit system formula), conditioned on $N(n+m)$, $E_\theta(n+m)$ and $E_\theta(n)$ are independent. Therefore,

$$\begin{aligned}
&\mathbb{P}(E_\theta(n+m)\cap E_\theta(n)) = \\
&\quad = \sum_{k=0}^{\infty} \mathbb{P}(E_\theta(n+m) \mid N(n+m)=k)\mathbb{P}(E_\theta(n) \mid N(n+m)=k)\mathbb{P}(N(n+m)=k) \\
&\quad = \sum_{k\notin \widetilde{E}(n+m)} (\cdots) + o((n+m)^{-2}),
\end{aligned} \tag{4.11}$$

by (4.7c). By (4.8), however, uniformly over all $k \notin \widetilde{E}(n+m)$,

$$\mathbb{P}(E_\theta(n+m) \mid N(n+m)=k) = \frac{1}{\sqrt{\pi}\theta}(\log(n+m))^{-1/2}(n+m)^{-\theta^2}\ (1+o(1)).$$

Therefore by (4.11), (4.9) and (4.7c):

$$\begin{aligned}
\mathbb{P}(E_\theta(n+m)\cap E_\theta(n)) &= \frac{1}{\sqrt{\pi}\theta}(\log(n+m))^{-1/2}(n+m)^{-\theta^2}\times \\
&\qquad \times \mathbb{P}(E_\theta(n); N(n+m)\notin \widetilde{E}(n+m))\ (1+o(1)) \\
&= (1+o(1))\ \mathbb{P}(E_\theta(n+m))\mathbb{P}(E_\theta(n)).
\end{aligned} \tag{4.12}$$

Hence, by Kochen and Stone (1964), (4.10) and (4.12) imply that $\mathbb{P}(E_\theta(n), \text{ i.o.}) = 1$. Taking $\theta \uparrow 1$ along a countable sequence, we arrive at the lower bound in Lemma 4.3. This finishes the proof. □

5. Proof of Theorem 1.4.

Let λ be an exponential holding time — independent of the process Z — such that $\mathbb{E}\lambda = 1$. We shall start with some technical lemmas.

Lemma 5.1. *For all $x \in \mathbb{R}^1$, $\mathbb{P}(T_x < \lambda) = \exp(-\sqrt{2}|x|)$.*

Proof. Since $\mathbb{P}(T_0 = 0) = 1$, it suffices to prove the lemma for $x > 0$. By independence of T_x and λ:

$$\mathbb{P}(\lambda > T_x) = \int_0^\infty \mathbb{P}(\lambda > a)\mathbb{P}(T_x \in da) = \mathbb{E}\exp(-T_x).$$

Now from Itô's formula (see Theorem IV.3.3 of Revuz and Yor (1991)),

$$\{\exp(\sqrt{2}Z(t\wedge T_x) - T_x \wedge t); t \geq 0\},$$

is a positive bounded martingale. Therefore, by the Doob's optional stopping theorem: $\mathbb{E}\exp(-T_x) = \exp(-\sqrt{2}x)$. The result follows. □

Lemma 5.2. *For all* $x \in \mathbb{R}^1$,

$$\mathbb{P}(L_\lambda^x > a) = \exp(-\sqrt{2}(|x| + a))1_{(0,\infty)}(a) + (\ 1 - \exp(-\sqrt{2}|x|)\ )1_{\{0\}}(a).$$

Proof. We first write,

$$\mathbb{P}(L_\lambda^x > a) = \mathbb{E}\bigg(\ 1_{[0,\lambda)}(T_x)\mathbb{P}(L_\lambda^x > a \mid \mathcal{F}_{T_x})\bigg). \tag{5.1}$$

However, on $\{\omega : T_x(\omega) < \lambda(\omega)\}$: $\lambda = T_x + \vartheta_{T_x}(\lambda)$, almost surely. (Notation being obvious.) Since $t \mapsto L_t^x$ is an additive functional, it follows that on $\{\omega : T_x(\omega) < \lambda(\omega)\}$: $L_\lambda^x = L_{T_x}^x + \vartheta_{T_x}(L_{\lambda - T_x}^x) = \vartheta_{T_x}(L_\lambda^x)$. Therefore if $a > 0$, by (5.1),

$$\mathbb{P}(L_\lambda^x > a) = \mathbb{P}(T_x < \lambda)\mathbb{P}(L_\lambda^0 > a). \tag{5.2}$$

It remains to compute $\mathbb{P}(L_\lambda^0 > a)$. First off, for any $a, b > 0$,

$$\mathbb{P}(L_\lambda^0 > a + b) = \mathbb{P}(\tau(b) \le \lambda, L_\lambda^0 > a + b),$$

where $\tau(\cdot)$ is the inverse local time at zero as defined in (4.2b). But by independence, on $\{\omega : \tau(b)(\omega) \le \lambda\}$: $\lambda = \tau(b) + \vartheta_{\tau(b)}(\lambda)$, almost surely. Therefore, on $\{\tau(b) \le \lambda\}$: $L_\lambda^0 = L_{\tau(b)}^0 + \vartheta_{\tau(b)}(L_\lambda^0) = b + \vartheta_{\tau(b)}(L_\lambda^0)$. Since $Z(\tau(b)) = 0$, a.s., we see from the strong Markov property that,

$$\begin{aligned}\mathbb{P}(L_\lambda^0 > a + b) &= \mathbb{P}(\tau(b) \le \lambda, \vartheta_{\tau(b)}(L_\lambda^0) > a) = \mathbb{P}(\tau(b) \le \lambda)\mathbb{P}(L_\lambda^0 > a) \\ &= \mathbb{P}(L_\lambda^0 \ge b)\mathbb{P}(L_\lambda^0 > a).\end{aligned}$$

Evidently, $\mathbb{P}(L_\lambda^0 = b) = 0$. From this fact, it follows that L_λ^0 has an exponential distribution. To compute its mean, we recall that by Tanaka's formula (see Theorem VI.1.2 of Revuz and Yor (1991)),

$$|Z(\lambda)| = \int_0^\lambda \text{sgn }(Z(s))dZ(s) + L_\lambda^0.$$

Since the stochastic integral in question has mean zero,

$$\mathbb{E}L_\lambda^0 = \mathbb{E}|Z(\lambda)| = \sqrt{\frac{2}{\pi}}\mathbb{E}\sqrt{\lambda} = 2^{-1/2}.$$

Therefore, $\mathbb{P}(L_\lambda^0 > a) = \exp(-\sqrt{2}a)1_{[0,\infty)}(a)$. If $x \ne 0$ and $a > 0$, by (5.2) and Lemma 5.1,

$$\mathbb{P}(L_\lambda^x > a) = \exp(-\sqrt{2}(a + |x|)).$$

If $a = 0$: $\mathbb{P}(L_\lambda^x = 0) = \mathbb{P}(T_x \ge \lambda) = 1 - \exp(-\sqrt{2}|x|)$. This finishes the proof of the lemma. $\square$

Lemma 5.3. *Let $\theta > 0$ be fixed. Then as $\varepsilon \downarrow 0$,*

$$\mathbb{P}\Big(|2\varepsilon u_\varepsilon(x,\lambda) - L_\lambda^x| \geq 2\varphi(\varepsilon)\sqrt{\theta L_\lambda^x}, T_x < \lambda \Big) = (\theta\pi \log(1/\varepsilon))^{-1/2}\varepsilon^\theta \exp(-\sqrt{2}|x|)\,(1+o(1)),$$

where the $o(1)$ term is independent of $x \in \mathbb{R}^1$.

Proof. Using the excursion theory notation (cf. the paragraph following (4.2)), let us define,

$$A(\varepsilon) \stackrel{df}{=} \bigcup_{x\in\mathbb{R}^1} \Big\{ f \in U_x : \sup_{0\leq r\leq R_x^f} f(r) \geq x+\varepsilon \Big\}. \tag{5.3}$$

(One can alternatively take the union over the rationals to avoid measurability problems.) As noted earlier, almost surely, $\sup_{x\in\mathbb{R}^1} |N^x_{L_\lambda^x}(A(\varepsilon)) - u_\varepsilon(x,\lambda)| \leq 1$. Therefore by increasing θ a little bit we see that it suffices to prove the lemma with $N^x_{L_\lambda^x}(A(\varepsilon))$ replacing $u_\varepsilon(x,\lambda)$. Since, $N^x_{L_\lambda^x}(A(\varepsilon)) = \vartheta_{T_x}(N^x_{L_\lambda^x}(A(\varepsilon)))$, and $L_\lambda^x = \vartheta_{T_x}(L_\lambda^x)$, an application of the strong Markov property shows that,

$$\begin{aligned}
&\mathbb{P}\Big(|2\varepsilon N^x_{L_\lambda^x}(A(\varepsilon)) - L_\lambda^x| \geq 2\varphi(\varepsilon)\sqrt{\theta L_\lambda^x}, T_x < \lambda \Big) \\
&\quad = \mathbb{P}(T_x < \lambda)\mathbb{P}\Big(|2\varepsilon N^0_{L_\lambda^0}(A(\varepsilon)) - L_\lambda^0| \geq 2\varphi(\varepsilon)\sqrt{\theta L_\lambda^0} \Big) \\
&\quad = e^{-\sqrt{2}|x|} \int_0^\infty \mathbb{P}\Big(|2\varepsilon N^0_w(A(\varepsilon)) - w| \geq 2\varphi(\varepsilon)\sqrt{w\theta} \Big) \mathbb{P}(L_\lambda^0 \in dw) \qquad \text{(by Lemma 5.1)} \\
&\quad = e^{-\sqrt{2}|x|} \Big(\int_0^{\theta\log(1/\varepsilon)} + \int_{\theta\log(1/\varepsilon)}^\infty \Big) \cdots \\
&\qquad\qquad \cdots\ \mathbb{P}\Big(|2\varepsilon N^0_w(A(\varepsilon)) - w| \geq 2\varphi(\varepsilon)\sqrt{w\theta} \Big) \sqrt{2} e^{-\sqrt{2}w} dw \qquad \text{(by Lemma 5.2)} \\
&\quad \stackrel{df}{=} \sqrt{2} e^{-\sqrt{2}|x|} \int_0^{\theta\log(1/\varepsilon)} \mathbb{P}(\cdots) e^{-\sqrt{2}w} dw + R(x,\varepsilon),
\end{aligned}$$

where $R(x,\varepsilon) \geq 0$ for all $x \in \mathbb{R}^1$ and $\varepsilon > 0$ and satisfies:

$$\sup_{x\in\mathbb{R}^1} R(x,\varepsilon) \leq \sqrt{2} \int_{\theta\log(1/\varepsilon)}^\infty \exp(-\sqrt{2}w)dw = \varepsilon^{\sqrt{2}\theta}. \tag{5.4}$$

On the other hand, as pointed out in Section 4, $N^0_w(A(\varepsilon))$ is a Poisson random variable with mean $(w/2\varepsilon)$. Therefore proceeding as in Lemma 4.2, by Lemma 4.1, uniformly over all $w \in (0, \theta\log(1/\varepsilon))$:

$$\mathbb{P}\Big(|2\varepsilon N^0_w(A(\varepsilon)) - w| \geq 2\varphi(\varepsilon)\sqrt{w\theta} \Big) = (\theta\pi\log(1/\varepsilon))^{-1/2}\varepsilon^\theta\ (1+o(1)),$$

as $\varepsilon \downarrow 0$. This and (5.4) together finish the proof of the lemma. $\square$

We are now ready to prove the upper bound in Theorem 1.5. Fix $R, \eta, \theta > 0$ and define:

$$K(\varepsilon, \eta, R) \stackrel{df}{=} \Big\{ s \in \mathbb{R}^1 : \ s \in \varepsilon^\eta \big(\mathbb{Z}^1 \cap [-R, R] \big) \Big\}$$

$$G(x; \theta, \varepsilon) \stackrel{df}{=} \Big\{ \omega : \ |2\varepsilon u_\varepsilon(x, \lambda)(\omega) - L_\lambda^x(\omega)| \geq 2\varphi(\varepsilon)\sqrt{\theta L_\lambda^x(\omega)} \Big\}.$$

It is clear that for all $\varepsilon \in (0, 1/10)$, $\# K(\varepsilon, \eta, R) \leq C_4(\eta, R)\varepsilon^{-\eta}$, where $C_4(\eta, R)$ is a positive finite constant. Lemma 5.3 guarantees the existence of a positive finite constant $C_5(\theta, \eta, R)$, such that for all $\varepsilon \in (0, 1/10)$:

$$\begin{aligned} \mathbb{P}\Big(\bigcup_{x \in K(\varepsilon,\eta,R)} G(x;\theta,\varepsilon) \cap \{T_x < \lambda\} \Big) &\leq \# K(\varepsilon, \eta, R) \max_{x \in K(\varepsilon,\eta,R)} \mathbb{P}\Big(G(x;\theta,\varepsilon) \cap \{T_x < \lambda\} \Big) \\ &\leq C_5(\theta, \eta, R)\big(\log(1/\varepsilon)\big)^{-1/2} \varepsilon^{\theta - \eta}. \end{aligned}$$

Fix $\rho \in (0, 1)$, $\eta > 1$ and $\theta > \eta$. Replacing ε by $\rho(n) \stackrel{df}{=} \rho^n$ in the above, it follows that

$$\sum_{n=1}^\infty \mathbb{P}\Big(\bigcup_{x \in K(\rho(n),\eta,R)} G(x;\theta,\rho(n)) \cap \{T_x < \lambda\} \Big) \leq C_6(\theta,\eta,R,\rho) \sum_{n=1}^\infty \rho^{n(\theta-\eta)} (\log n)^{-1/2} < \infty,$$

for some positive finite constant, $C_6(\theta, \eta, R, \rho)$. However, on the set $\{T_x \geq \lambda\}$, $e_\varepsilon(x, \lambda) = L_\lambda^x = 0$ for all $\varepsilon > 0$, the null set being independent of $x \in \mathbb{R}^1$. Moreover, it is trivial that for all ω: $\sup_{x \in K(\varepsilon,\eta,R)} L_\lambda^x(\omega) \leq L_\lambda^*(\omega)$. Therefore, by the Borel–Cantelli lemma, the following holds with probability one:

$$\limsup_{n\to\infty} \sup_{x \in K(\rho(n),\eta,R)} \frac{|2\rho(n) u_{\rho(n)}(x,\lambda) - L_\lambda^x|}{\varphi(\rho(n))} \leq 2\big(\theta \sup_{x\in\mathbb{R}^1} L_\lambda^x\big)^{1/2}. \tag{5.5}$$

Let $x \in [-R, R]$. By $f_n(x)$ we shall mean the element of $K(\rho(n), \eta, R)$ which is the closest to x. To make this choice unique, we shall require that $f_n(x) \leq x$. We futher point out that, $\sup_{x\in[-R,R]} |x - f_n(x)| = \rho^{\eta n}$. By (1.3), the uniform modulus of continuity of $x \mapsto L_\lambda^x$ implies that with probability one,

$$\begin{aligned} \sup_{x \in [-R,R]} |L_\lambda^x - L_\lambda^{f_n(x)}| &= O\big(\sqrt{|x - f_n(x)| \log |x - f_n(x)|^{-1}}\big) \\ &= O(\rho^{n\eta/2}\sqrt{\log n}), \\ &= o(\varphi(\rho(n))), \end{aligned} \tag{5.6}$$

as $n \to \infty$. On the other hand, since $K(\varepsilon, \rho(n), R) \subseteq K(\varepsilon, \rho(n+1), R)$ and $t \mapsto Z(t)$ is continuous (a.s.),

$$u_{\rho(n)}(f_n(x), \lambda) \le u_{\rho(n+1)}(x, \lambda) \le u_{\rho(n+1)}(f_n(x + \rho(n)^\eta), \lambda).$$

The above, together with (5.5) and (5.6) imply that almost surely:

$$\limsup_{n\to\infty} \sup_{x\in[-R,R]} \frac{|2\rho(n)u_{\rho(n)}(x,\lambda) - L_\lambda^x|}{\varphi(\rho(n))} \le 2\big(\theta L_\lambda^*\big)^{1/2}.$$

We recall that $\eta, R > 0$, $\theta > \eta$ and $\rho \in (0,1)$ were arbitrary. Arguing as in the previous section, it follows that

$$\limsup_{\varepsilon\downarrow 0} \sup_{x\in\mathbb{R}^1} \frac{|2\varepsilon u_\varepsilon(x,\lambda) - L_\lambda^x|}{\varphi(\varepsilon)} \le 2\big(L_\lambda^*\big)^{1/2}. \tag{5.7}$$

By scaling, the above still holds, if λ is an exponential holding time with mean u, for any $u > 0$. Therefore, by considering inverse Laplace transforms — as in the end of Section 3 — the upper bound follows, i.e., (5.7) holds with λ replaced by any fixed $t > 0$.

To conclude, we next provide a proof for the lower bound corresponding to (5.7). Throughout, fix $M > 1$ and for all $x \in \mathbb{R}^1$ and $\varepsilon > 0$ define,

$$A(x,\varepsilon) \stackrel{df}{=} \Big\{ f \in U_x : \sup_{0\le r\le R_x^f} f(r) \in [x+\varepsilon, x+M\varepsilon) \Big\}.$$

Thus the excursions in $A(x,\varepsilon)$ are exactly those which have upcrossed $[x, x+\varepsilon]$ but not $[x, x+M\varepsilon)$. Let $N(x,\varepsilon)$ denote the corresponding counting measure to $A(x,\varepsilon)$, i.e.,

$$N(x,\varepsilon) \stackrel{df}{=} N^x_{L_\lambda^x}(A(x,\varepsilon)).$$

For all $x \in \mathbb{R}^1$ and all $\varepsilon, \theta > 0$, define

$$\widetilde{G}(x;\theta,\varepsilon) \stackrel{df}{=} \Big\{ \omega : |2\varepsilon N(x,\varepsilon)(\omega) - (1-M^{-1})L_\lambda^x(\omega)| \le 2\varphi(\varepsilon)\sqrt{\theta\big(1-M^{-1}\big)L_\lambda^x(\omega)} \Big\}.$$

Note that:

(R1) When M is large, the collection, $\widetilde{G}$, is approximately the same as the complement of G where the latter was defined earlier in this section.

(R2) For each $x \in \mathbb{R}^1$ and $\varepsilon, \theta > 0$, on the set, $\{T_x \le \lambda\}$, $\widetilde{G}(x;\theta,\varepsilon) = \vartheta_{T_x}(\widetilde{G}(x;\theta,\varepsilon))$, almost surely.

(R3) Fix $w > 0$. Then given $\{L_\lambda^x = w\}$, we have $T_x \le \lambda$, almost surely, since the support of $t \mapsto L_t^x$ is with probability one equal to $\{s : Z(s) = x\}$.

(R4) Fix $w > 0$ and $x \ge M\varepsilon$. Then given $\{L_\lambda^x = w\}$, $N(x,\varepsilon)$ is independent of L_λ^0 and $N(0,\varepsilon)$. This is a consequence of the exit system formula and the fact that $A(x,\varepsilon) \cap A(0,\varepsilon) = \emptyset$.

Next we state and prove two lemmas which shall be of use in the proof of the lower bound.

Lemma 5.4. *Fix $w, x > 0$. Then there exist an i.i.d. sequence of random variables, $\{\delta_1, \delta_2, \cdots\}$ which are exponentially distributed with mean $2x$ and an independent random variable, N, such that $(N-1)$ is Poisson with mean $w/2x$, and that given $\{L_\lambda^x = w\}$,*

$$\sum_{j=0}^{N} \delta_j \le L_\lambda^0 \le \sum_{j=0}^{N+1} \delta_j,$$

almost surely.

Proof. Let $U_0 \stackrel{df}{=} 0$ and iteratively define,

$$U_{2j+1} \stackrel{df}{=} U_{2j} + \vartheta_{U_{2j}}(T_x) \qquad (j \ge 0)$$

$$U_{2j} \stackrel{df}{=} U_{2j-1} + \vartheta_{U_{2j-1}}(T_0), \qquad (j \ge 1).$$

Then the U_i's are the uppcrossing times of the interval $[0,x]$. Since the support of the increasing function: $t \mapsto L_\lambda^0$ is almost surely the set $\{s : Z(s) = 0\}$,

$$\begin{aligned} L_\lambda^0 &= \sum_{j=0}^{N} \left(L_{U_{2j+1}}^0 - L_{U_{2j}}^0\right) + \left(L_\lambda^0 - L_{U_{2N+1}}^0\right) \\ &\stackrel{df}{=} \sum_{j=0}^{N} \delta_j + \left(L_\lambda^0 - L_{U_{2N+1}}^0\right), \end{aligned}$$

where $N \stackrel{df}{=} \max\{j : U_{2j+1} \le \lambda\}$. Note that $N-1$ is the number of downcrossings of $[0,x]$ before time λ. Therefore, by the exit system formula, given $\{L_\lambda^x = w\}$, $(N-1)$ is Poisson with mean $w/2x$. Moreover, given $\{L_\lambda^x = w\}$,

$$\sum_{j=0}^{N} \delta_j \le L_\lambda^0 \le \sum_{j=0}^{N+1} \delta_j,$$

and the δ_i's are i.i.d. exponential and are independent of N. It remains to compute the mean of δ_1. By Tanaka's formula (Theorem VI.1.2 of Revuz and Yor (1991)),

$$Z(t)^+ = \int_0^t 1_{(0,\infty)}(Z(s))dZ(s) + \frac{1}{2}L_t^0.$$

The above and Doob's optional sampling theorem together imply that $\mathbb{E}\delta_1 = \mathbb{E}L_{T_x}^0 = 2x$. The lemma follows. $\square$

Lemma 5.5. *With the notation of Lemma 5.4, for all $\xi \in \big(0,(2x)^{-1}\big)$,*

$$\mathbb{E}\exp\left(\xi\sum_{j=0}^{N+1}\delta_j\right) = (1-2x\xi)^{-3}\exp\left(\frac{w\xi}{1-2x\xi}\right).$$

The proof of Lemma 5.5 is omitted as it only involves basic calculations with Gamma distributions.

We are ready to proceed with the proof of the lower bound. Fix for now $\varepsilon \in \big(0, M^{-1}\big)$, some $x \in [M\varepsilon, 1]$ and an arbitrary $\theta \in (0,1)$. For typographical ease, we shall suppress the θ and ε in the definition of $\widetilde{G}$, i.e., we write $\widetilde{G}(x)$ for $\widetilde{G}(x;\theta,\varepsilon)$. We first estimate $\mathbb{P}(\widetilde{G}(0)\cap\widetilde{G}(x)\cap\{T_x \le \lambda\})$. Indeed,

$$\begin{aligned}
\mathbb{P}(\widetilde{G}(0)\cap\widetilde{G}(x)\cap\{T_x\le\lambda\}) = \\
&= \sqrt{2}\int_0^\infty \mathbb{P}\big(\widetilde{G}(0)\cap\widetilde{G}(x)\mid L_\lambda^x = w\big)\exp\big(-\sqrt{2}(x+w)\big)dw \qquad \text{(by Lemma 5.2)}\\
&= \sqrt{2}e^{-\sqrt{2}x}\int_0^\infty \mathbb{P}\big(\widetilde{G}(x)\mid L_\lambda^x = w\big)\mathbb{P}\big(\widetilde{G}(0)\mid L_\lambda^x = w\big)\exp(-\sqrt{2}w)dw \quad \text{(by (R4))}\\
&\le \sqrt{2}\Big(\int_0^\varepsilon + \int_\varepsilon^{2\log(1/\varepsilon)} + \int_{2\log(1/\varepsilon)}^\infty\Big)\cdots\\
&\qquad \cdots\mathbb{P}\big(\widetilde{G}(x)\mid L_\lambda^x = w\big)\mathbb{P}\big(\widetilde{G}(0)\mid L_\lambda^x = w\big)\exp(-\sqrt{2}w)dw\\
&= \sqrt{2}\Big(\int_0^\varepsilon + \int_\varepsilon^{2\log(1/\varepsilon)} + \int_{2\log(1/\varepsilon)}^\infty\Big)\cdots\\
&\qquad \cdots\mathbb{P}\big(\widetilde{G}(0)\mid L_\lambda^0 = w\big)\mathbb{P}\big(\widetilde{G}(0)\mid L_\lambda^x = w\big)\exp(-\sqrt{2}w)dw.
\end{aligned}$$

The last statement follows from the strong Markov property and (R2). Thus, it suffices to estimate seperately the following terms:

$$(5.8a)\quad E_1(\varepsilon) \stackrel{df}{=} \sqrt{2}\int_0^\varepsilon \mathbb{P}\big(\widetilde{G}(0)\mid L_\lambda^0 = w\big)\mathbb{P}\big(\widetilde{G}(0)\mid L_\lambda^x = w\big)\exp(-\sqrt{2}w)dw$$

$$E_2(\varepsilon) \stackrel{df}{=} \sqrt{2} \int_{\varepsilon}^{2\log(1/\varepsilon)} \mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^0 = w\big)\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^x = w\big) \exp(-\sqrt{2}w)dw \tag{5.8b}$$

$$E_3(\varepsilon) \stackrel{df}{=} \sqrt{2} \int_{2\log(1/\varepsilon)}^{\infty} \mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^0 = w\big)\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^x = w\big) \exp(-\sqrt{2}w)dw. \tag{5.8c}$$

Estimating E_1 and E_3 is easy. Indeed,

$$E_1(\varepsilon) \le \sqrt{2}\varepsilon \tag{5.9a}$$

$$E_3(\varepsilon) \le \sqrt{2} \int_{2\log(1/\varepsilon)}^{\infty} \exp(-\sqrt{2}w)dw = \varepsilon^{2\sqrt{2}} \le \varepsilon. \tag{5.9b}$$

We shall next estimate the terms in E_2. First we note that,

$$\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^0 = w\big) = \mathbb{P}\Big(|2\varepsilon N_w^0(A(0,\varepsilon)) - w| \le 2\varphi(\varepsilon)\sqrt{\theta(1-M^{-1})w}\Big),$$

and $N_w^0(A(0,\varepsilon))$ is Poisson with mean, $n^0(A(0,\varepsilon)) = n^0(A(\varepsilon)) - n^0(A(M\varepsilon)) = (w/2\varepsilon)(1 - M^{-1})$, where $A(\varepsilon)$ is given by (5.3). Therefore, as in Lemma 4.2,

$$\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^0 = w\big) = 1 - \big(\theta\pi\log(1/\varepsilon)\big)^{-1/2}\varepsilon^\theta\ (1+o(1)), \tag{5.10}$$

uniformly over all $w \in (0, 2\log(1/\varepsilon))$. To estimate the other terms in $E_2(\varepsilon)$, we note that for all $w \in \big(0, 2\log(1/\varepsilon)\big)$ and all $x \ge M\varepsilon$:

$$\begin{aligned}\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^x = w\big) &= \int_0^\infty \mathbb{P}\big(\widetilde{G}(0), L_\lambda^0 \in dz \mid L_\lambda^x = w\big)\\ &= \int_0^\infty \mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^0 = z\big)\mathbb{P}\big(L_\lambda^0 \in dz \mid L_\lambda^x = w\big),\end{aligned}$$

since by the exit system formula, $\widetilde{G}(0)$ is independent of L_λ^x, conditioned on L_λ^0. Hence,

$$\begin{aligned}\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^x = w\big) &= \bigg(\int_0^{4\log(1/\varepsilon)} + \int_{4\log(1/\varepsilon)}^\infty\bigg)\cdots\\ &\qquad \cdots\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^0 = z\big)\mathbb{P}\big(L_\lambda^0 \in dz \mid L_\lambda^x = w\big)\\ &\stackrel{df}{=} E_4(\varepsilon, w) + E_5(\varepsilon, w).\end{aligned} \tag{5.11}$$

By Lemma 5.4, for all $\xi \in \big(0, (2x)^{-1}\big)$,

$$\begin{aligned}E_5(\varepsilon, w) &\le \mathbb{P}\bigg(\sum_{j=0}^{N+1} \delta_j \ge 4\log(1/\varepsilon)\bigg)\\ &\le (1-2x\xi)^{-3}\exp\bigg(\frac{w\xi}{1-2x\xi} - 4\xi\log(1/\varepsilon)\bigg).\end{aligned}$$

Pick $\xi \overset{df}{=} (2x)^{-1}\big(1 - \sqrt{w/4\log(1/\varepsilon)}\big)$, we get the following estimate for E_5:

$$E_5(\varepsilon, w) \le \Big(\frac{4\log(1/\varepsilon)}{w}\Big)^{3/2} \exp\Big(- \frac{1}{2x}\big(\sqrt{w} - \sqrt{4\log(1/\varepsilon)}\big)^2\Big).$$

Since $\theta \in (0,1)$, uniformly over all $x \in [M\varepsilon, 1]$,

$$\begin{aligned} \sup_{\varepsilon \le w \le 2\log(1/\varepsilon)} E_5(\varepsilon, w) &\le \big(4\varepsilon^{-1}\log(1/\varepsilon)\big)^{3/2} \exp\big(- (2x\varepsilon)^{-1}\big) \\ &\le C_7(M)\varepsilon. \end{aligned} \tag{5.12}$$

for some finite positive constant $C_7(M)$. Furthermore, as in Lemma 4.2, uniformly over all $z \in \big(0, 4\log(1/\varepsilon)\big)$:

$$\mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^0 = z\big) = 1 - \big(\theta\pi\log(1/\varepsilon)\big)^{-1/2}\varepsilon^\theta\ (1 + o(1)),$$

where the $o(1)$ term is independent of $x \ge M\varepsilon$. Hence uniformly over all w and all $x \ge M\varepsilon$,

$$\begin{aligned} E_4(\varepsilon, w) &\le 1 - \big(\theta\pi\log(1/\varepsilon)\big)^{-1/2}\varepsilon^\theta\ (1 + o(1)) \\ &\le \exp\big(- C_8(\theta, M)\log^{-1/2}(1/\varepsilon)\varepsilon^\theta\big), \end{aligned}$$

for some finite positive constant $C_8(\theta, M)$. Therefore, by (5.11) and (5.12),

$$\begin{aligned} \mathbb{P}\big(\widetilde{G}(0) \mid L_\lambda^x = w\big) &\le \exp\big(- C_8(\theta, M)\log^{-1/2}(1/\varepsilon)\varepsilon^\theta\big) + C_7(M)\varepsilon \\ &\le C_9(\theta, M)\exp\big(- C_8(\theta, M)\log^{-1/2}(1/\varepsilon)\varepsilon^\theta\big). \end{aligned}$$

uniformly over all $x \in [M\varepsilon, 1]$ and every $w \in [M\varepsilon, 2\log(1/\varepsilon)]$. Here $C_9(\theta, M)$ is some finite positive constant. Therefore by (5.8b) and (5.10), there exist finite positive constants $C_{10}(\theta, M)$ and $C_{11}(\theta, M)$ such that,

$$\begin{aligned} \sup_{x \in [M\varepsilon, 1]} E_2(\varepsilon) \le C_{10}(\theta, M) &\exp\big(- C_8(\theta, M)\log^{-1/2}(1/\varepsilon)\varepsilon^\theta\big) \times \\ &\times \Big(1 - \big(\theta\pi\log(1/\varepsilon)\big)^{-1/2}\varepsilon^\theta\ (1 + o(1))\Big) \\ \le C_{10}(\theta, M) &\exp\Big(- 2C_{11}(\theta, M)\log^{-1/2}(1/\varepsilon)\varepsilon^\theta\Big). \end{aligned}$$

Hence, from (5.9) it follows that for some finite constant $C_{12}(\theta, M)$,

$$\mathbb{P}\big(\widetilde{G}(0) \cap \widetilde{G}(x) \cap \{T_x \le \lambda\}\big) \le C_{12}(\theta, M) \exp\big(- 2C_{11}(\theta, M) \log^{-1/2}(1/\varepsilon)\varepsilon^{\theta}\big). \tag{5.13}$$

By induction, the proof of (5.13) shows the existence of positive finite constants, $C_{13} = C_{13}(\theta, M)$ and $C_{14} = C_{14}(\theta, M)$, such that for any integer $\nu \in \big[2, (M\varepsilon)^{-1}\big]$ and all $x_1, x_2, \cdots, x_\nu \in [M\varepsilon, 1]$, which satisfy $x_i \ge x_{i-1} + M\varepsilon$,

$$\begin{aligned}\mathbb{P}\Big(\widetilde{G}(0) \cap \widetilde{G}(x_1) \cap \cdots \widetilde{G}(x_\nu) \cap \{T_{x_\nu} \le \lambda\}\Big) &\le C_{12}\nu \exp\big(- C_{11} \log^{-1/2}(1/\varepsilon)\nu\varepsilon^{\theta}\big) \\ &\le C_{13} \exp\Big(- C_{14}\nu\varepsilon^{\theta}\Big).\end{aligned}$$

We shall let $x_j \overset{df}{=} jM\varepsilon$, $j = 0, \ldots, \big[(M\varepsilon)^{-1}\big]$. It follows that $\nu = \big[(M\varepsilon)^{-1}\big]$ in the above. Moreover,

$$\begin{aligned}&\mathbb{P}\Big(\widetilde{G}(x;\theta,\varepsilon) \cap \{T_x \le \lambda\} \text{ for all } x \in [0,1]\Big) \\ &\qquad \le \mathbb{P}\Big(\widetilde{G}(0) \cap \widetilde{G}(x_1) \cap \cdots \widetilde{G}(x_\nu) \cap \{T_{x_\nu} \le \lambda\}\Big) \\ &\qquad \le C_{13}(\theta, M) \exp\Big(- C_{14}(\theta, M) M^{-1} [\varepsilon]^{\theta - 1}\Big).\end{aligned} \tag{5.14}$$

With no essential changes, the proof of (5.14) can be extended to show that for all real–valued $a \le b$, there exist positive finite constants, $C_{15}(\theta, M, a, b)$ and $C_{16}(\theta, M, a, b)$ such that

$$\begin{aligned}&\mathbb{P}\Big(\widetilde{G}(x;\theta,\varepsilon) \cap \{T_x \le \lambda\} \text{ for all } x \in [a,b]\Big) \\ &\qquad \le C_{16}(\theta, M, a, b) \exp\Big(- C_{15}(\theta, M, a, b) [\varepsilon]^{\theta - 1}\Big).\end{aligned}$$

Fix $\rho \in (0,1)$ and define $\rho(n) \overset{df}{=} \rho^n$. Since we had fixed $\theta \in (0,1)$, it follows at once that for all $a \le b$:

$$\sum_n \mathbb{P}\Big(\widetilde{G}\big(x;\theta,\rho(n)\big) \cap \{T_x \le \lambda\} \text{ for all } x \in [a,b]\Big) < \infty.$$

The Borel–Cantelli lemma and a sample path argument together show that

$$\liminf_{n\to\infty} \sup_{x \in \mathbb{R}^1} \frac{\big|2\rho(n) N\big(x, \rho(n)\big) - \big(1 - M^{-1}\big) L_\lambda^x\big|}{\varphi\big(\rho(n)\big)} \ge 2 \inf_{a \le x \le b} \sqrt{\theta(1 - M^{-1})\ L_\lambda^x}.$$

Letting $\theta \uparrow 1$ along a countable sequence, since $a \le b$ is arbitrary and $x \mapsto L_\lambda^x$ is continuous,

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}^1} \frac{\left|2\rho(n)N\big(x,\rho(n)\big) - \big(1-M^{-1}\big)L_\lambda^x\right|}{\varphi\big(\rho(n)\big)} \ge 2\sqrt{L_\lambda^*(1-M^{-1})}.$$

However, as pointed out earlier,

$$N_{L_\lambda^x}^x\big(A(\rho(n))\big) = N\big(x,\rho(n)\big) + N_{L_\lambda^x}^x\big(A(M\rho(n))\big),$$

and $|u_\varepsilon(x,\lambda) - N_{L_\lambda^x}^x\big(A(\varepsilon)\big)| \le 1$. Therefore,

$$\begin{aligned}
&\liminf_{n\to\infty} \sup_{x\in\mathbb{R}^1} \frac{\left|2\rho(n)u_{\rho(n)}\big(x,\lambda\big) - L_\lambda^x\right|}{\varphi\big(\rho(n)\big)} \\
&\qquad \ge \liminf_{n\to\infty} \sup_{x\in\mathbb{R}^1} \frac{\left|2\rho(n)N\big(x,\rho(n)\big) - \big(1-M^{-1}\big)L_\lambda^x\right|}{\varphi\big(\rho(n)\big)} \\
&\qquad\qquad - \limsup_{n\to\infty} \sup_{x\in\mathbb{R}^1} \frac{\left|2\rho(n)N_{L_\lambda^x}^x\big(A(M\rho(n))\big) - M^{-1}L_\lambda^x\right|}{\varphi\big(\rho(n)\big)} \\
&\qquad \ge 2\sqrt{L_\lambda^*}\cdot\Big(\sqrt{1-M^{-1}} - \sqrt{M^{-1}}\Big).
\end{aligned}$$

The last inequality holding because of the (already proven) upper bound, (5.7). Since $M > 1$ is arbitrary, letting $M \uparrow \infty$ along a countable sequence, we see that

$$\liminf_{n\to\infty} \sup_{x\in\mathbb{R}^1} \frac{\left|2\rho(n)u_{\rho(n)}\big(x,\lambda\big) - L_\lambda^x\right|}{\varphi\big(\rho(n)\big)} \ge 2\sqrt{L_\lambda^*}.$$

By a monotonicity argument and taking inverse Laplace transforms as in the previous section, the lower bound follows. $\square$

6. Proof of Theorem 1.5.

As in Section four's proof of Theorem 1.3, we can reduce the problem to the case where $x = 0$ and $t = 1$. Moreover, much as in the derivation of (4.1), one can argue that it is sufficient to show that for all $w > 0$,

$$\limsup_{\varepsilon\downarrow 0} \frac{\left|(\pi\varepsilon/2)^{1/2}e_\varepsilon(0,1) - w\right|}{\kappa(\varepsilon)} = (2\pi)^{1/4}w^{1/2}, \qquad \text{a.s.} \tag{6.1}$$

where $\kappa(\varepsilon) \overset{df}{=} \varepsilon^{1/4}\sqrt{1 \vee \log\log(1/\varepsilon)}$. Therefore, from now on we shall hold fixed a $w > 0$.

Recall the process of excursions from zero: $\{e_s^0; s \geq 0\}$ and the associated counting process, $\{N_t^0(A); t \geq 0\}$ for measurable $A \subseteq U_0 \cup \{\delta\}$. Define,

$$A(\varepsilon) \overset{df}{=} \Big\{ f \in U_0 \cup \{\delta\} : \ R_0^f \geq \varepsilon \Big\}.$$

We recall that R_0^f was the lifetime of the excursion, f, from zero. Since $\mathbb{Q}_w$–almost surely,

$$|N_w(A(\varepsilon)) - e_\varepsilon(0,1)| \leq 1,$$

(where $\mathbb{Q}_w$ is defined in Section 4), by (6.1) it suffices to show that $\mathbb{P}$–almost surely,

$$\limsup_{\varepsilon \downarrow 0} \frac{|(\pi\varepsilon/2)^{1/2} N_w^0(A(\varepsilon)) - w|}{\kappa(\varepsilon)} = (2\pi)^{1/4} w^{1/2}.$$

However, by the exit system formula, $N_w^0(A(\varepsilon))$ is a Poisson random variable with mean $n^0(A(\varepsilon)) = (2/\pi\varepsilon)^{1/2} w$. The calculation of the latter excursion law can be found, for example, in Proposition XII.2.8 of Revuz and Yor (1991). Therefore, arguing as in the proof of Lemma 4.2, for all $\theta > 0$,

$$\begin{aligned} &\mathbb{P}\Big(|(\pi\varepsilon/2)^{1/2} N_w^0(A(\varepsilon)) - w| \geq (2\pi)^{1/4} (w\theta)^{1/2} \kappa(\varepsilon) \Big) \\ &\qquad = \frac{\sqrt{2/\pi}}{(2\theta \log\log(1/\varepsilon))^{1/2} (\log(1/\varepsilon))^\theta} \ (1 + o(1)), \end{aligned} \tag{6.2}$$

as $\varepsilon \downarrow 0$. Define for all $\varepsilon > 0$ and $\theta > 0$,

$$S_\theta(\varepsilon) \overset{df}{=} \Big\{ \omega : \ |(\pi\varepsilon)^{1/2} N_w^0(A(\varepsilon))(\omega) - w| \geq (2\pi)^{1/4} (\theta w)^{1/2} \kappa(\varepsilon) \Big\}.$$

First we will prove the upper bound for the limsup. Fix $\rho \in (0,1)$ and $\theta > 1$ and define $\rho(n) \overset{df}{=} \rho^n$. Then by (6.2),

$$\sum_{n=1}^\infty \mathbb{P}(S_\theta(n))) \leq C_{17}(\theta) \sum_{n=1}^\infty (\log n)^{-1/2} n^{-\theta} < \infty,$$

for some finite constant, $C_{17}(\theta)$. Therefore, by the Borel–Cantelli lemma (and letting $\theta \uparrow 1$) it follows that,

$$\limsup_{n \to \infty} \frac{|(\pi\rho(n)/2)^{1/2} N_w^0(\rho(n))) - w|}{\kappa(\rho(n))} \leq (2\pi)^{1/4} w^{1/2},$$

almost surely. However, as $\varepsilon \downarrow 0$, there are integers, N_ε, for which $\rho(N_\varepsilon + 1) \le \varepsilon \le \rho(N_\varepsilon)$. Since, $A(\rho(N_\varepsilon)) \subseteq A(\varepsilon) \subseteq A(\rho(N_\varepsilon + 1))$, it follows that,

$$N_w^0(A(\rho(N_\varepsilon))) \le N_w^0(A(\varepsilon)) \le N_w^0(A(\rho(N_\varepsilon + 1))).$$

Hence using some arithmetic,

$$\limsup_{\varepsilon \downarrow 0} \frac{|(\pi\varepsilon/2)^{1/2} N_w^0(A(\varepsilon)) - 2|}{\kappa(\varepsilon)} \le (2\pi)^{1/4}(w/\rho)^{1/2},$$

almost surely. Letting $\rho \uparrow 1$ along a countable sequence, we arrive at the desired upper bound.

For the proof of the lower bound define, $\rho(n) \stackrel{df}{=} n^{-n}$, and let

$$\begin{aligned} B(n+1) &\stackrel{df}{=} A(\rho(n+1)) \setminus A(\rho(n)); \\ N(n) &\stackrel{df}{=} N_w^0(B(n)). \end{aligned}$$

Let us note that $N(n) = N_w^0(A(\rho(n+1))) - N_w^0(A(\rho(n)))$. Since $\{B(n); n \ge 0\}$ is a disjoint sequence of sets in $U_0 \cup \{\delta\}$, by the exit system formula, $\{N(n); n \ge 0\}$ are independent Poisson random variables with mean,

$$\begin{aligned} n^0(A(\rho(n+1)) \setminus A(\rho(n))) &= n^0(A(\rho(n+1))) - n^0(A(\rho(n))) \\ &= (\pi\rho(n+1)/2)^{-1/2} w \ (1 + O(n^{-1/2})). \end{aligned}$$

Let

$$\widetilde{S}_\theta(n) \stackrel{df}{=} \Big\{\omega : \ |(\pi\rho(n+1)/2)^{1/2} N(n)(\omega) - w| \ge (2\pi)^{1/4}(w\theta)^{1/2}\kappa(\rho(n+1))\Big\},$$

where $\theta \in (0,1)$. Then arguing as in (6.2),

$$\mathbb{P}(\widetilde{S}_\theta(n)) = (\pi\theta \log n)^{-1/2} n^{-\theta} \ (1 + o(1)).$$

Therefore, since $\theta \in (0,1)$, $\sum_n \mathbb{P}(\widetilde{S}_\theta(n)) = \infty$. By the independence half of the Borel–Cantelli lemma, $\mathbb{P}(\widetilde{S}_\theta(n)$, i.o.$) = 1$. In other words, letting $\theta \uparrow 1$ along a countable sequence,

$$\limsup_{n \to \infty} \frac{|(\pi\varepsilon/2)^{1/2} N(n) - w|}{\kappa(\rho(n+1))} \ge (2\pi)^{1/4} w^{1/2},$$

almost surely. Therefore,

$$\begin{aligned}\limsup_{\varepsilon\downarrow 0}\frac{|(\pi\varepsilon/2)^{1/2}N_w^0(A(\varepsilon))-w|}{\kappa(\varepsilon)} &\geq \limsup_{n\to\infty}\frac{|(\pi\varepsilon/2)^{1/2}N_w^0(A(\rho(n+1)))-w|}{\kappa(\rho(n+1))}\\ &\geq (2\pi)^{1/4}w^{1/2}-\limsup_{n\to\infty}\frac{|(\pi\varepsilon/2)^{1/2}N_w^0(A(\rho(n)))-w|}{\kappa(\rho(n+1))}\\ &= (2\pi)^{1/4}w^{1/2},\end{aligned}$$

by the upper bound half (which we have already proven) together with the fact that $\kappa(\rho(n)) = \kappa(\rho(n+1))(en)^{-1/4}\ (1+o(1)) = o(\kappa(\rho(n+1))$, as $n\to\infty$. This completes the proof. $\square$

7. Proof of Theorem 1.6.

Theorem 1.6 is proved much like Theorem 1.4. Therefore, for the sake of brevity, we shall point out the differences in proofs. As in Section 5, we start with the proof of the upper bound. The first result is the analogue of Lemma 5.3.

Lemma 7.1. *Let $\theta>0$ be fixed. Then as $\varepsilon\downarrow 0$,*

$$\mathbb{P}\Big(|(\pi\varepsilon/2)^{1/2}e_\varepsilon(x,\lambda)-L_\lambda^x|\geq 2\kappa(\varepsilon)\sqrt{\theta L_\lambda^x}, T_x<\lambda\Big) = \big(\theta\pi\log(1/\varepsilon)\big)^{-1/2}\varepsilon^\theta e^{-\sqrt{2}|x|}\ (1+o(1)),$$

where the $o(1)$ term is independent of $x\in\mathbb{R}^1$.

Proof. We proceed almost exactly as in the proof of Lemma 5.3 with some minor adjustments. Let

$$D(\varepsilon)\stackrel{df}{=}\bigcup_{x\in\mathbb{R}^1}\Big\{f\in U_x : R_x^f\geq\varepsilon\Big\}. \tag{7.1}$$

Following the argument in Lemma 5.3,

$$\begin{aligned}&\mathbb{P}\Big(|(\pi\varepsilon/2)^{1/2}N^x_{L^x_\lambda}\big(D(\varepsilon)\big)-L_\lambda^x|\geq(2\pi)^{1/4}\kappa(\varepsilon)\sqrt{\theta L_\lambda^x}, T_x<\lambda\Big)\\ &\quad=\sqrt{2}e^{-\sqrt{2}|x|}\int_0^{\theta\log(1/\varepsilon)}\mathbb{P}\Big(|(\pi\varepsilon/2)^{1/2}N_w^0\big(D(\varepsilon)\big)-w|\geq(2\pi)^{1/4}\kappa(\varepsilon)\sqrt{\theta w}\Big)e^{-\sqrt{2}w}dw\\ &\qquad+R(x,\varepsilon),\end{aligned}$$

where $R(x,\varepsilon)\geq 0$ for all $x\in\mathbb{R}^1$ and $\varepsilon>0$; moreover,

$$\sup_{x\in\mathbb{R}^1} R(x,\varepsilon)\leq \varepsilon^{\sqrt{2}\theta}. \tag{7.2}$$

From Section 6, $N_w^0\big(D(\varepsilon)\big)$ is a Poisson random variable with mean $(2/\pi\varepsilon)^{1/2}w$. Therefore, as in Lemma 5.3,

$$\mathbb{P}\bigg(\big|(\pi\varepsilon/2)^{1/2}N_w^0\big(D(\varepsilon)\big)-w\big|\geq(2\pi)^{1/4}\kappa(\varepsilon)\sqrt{\theta w}\bigg)=\big(\theta\pi\log(1/\varepsilon)\big)^{-1/2}\varepsilon^{\theta}\ (1+o(1)),$$

uniformly over all $w\in\big(0,\theta\log(1/\varepsilon)\big)$. The lemma follows from (7.2). $\square$

Fix $R,\eta,\theta>0$ and define as in Section 5,

$$K(\varepsilon,\eta,R)\stackrel{df}{=}\Big\{s\in\mathbb{R}^1 : s\in\varepsilon^{\eta}\big(\mathbb{Z}^1\cap[-R,R]\big)\Big\}$$
$$V(x;\theta,\varepsilon)\stackrel{df}{=}\Big\{\omega : \big|(\pi\varepsilon/2)^{1/2}e_\varepsilon(x,\lambda)(\omega)-L_\lambda^x(\omega)\big|\geq(2\pi)^{1/4}\kappa(\varepsilon)\sqrt{\theta L_\lambda^x(\omega)}\Big\}.$$

Then exactly as in Section 5, (using Lemma 7.1 instead of Lemma 5.3)

$$\mathbb{P}\bigg(\bigcup_{x\in K(\varepsilon,\eta,R)}V(x;\theta,R)\bigcap\big\{T_x<\lambda\big\}\bigg)\leq C_{18}\big(\theta,\eta,R\big)\big(\log(1/\varepsilon)\big)^{-1/2}\varepsilon^{\theta-\eta}.$$

Fix $\rho\in(0,1)$, $\eta>1$ and $\theta>\eta$. Replace ε by $\rho(n)\stackrel{df}{=}\rho^n$ to see that

$$\sum_n\mathbb{P}\bigg(\bigcup_{x\in K(\rho(n),\eta,R)}V(x;\theta,\rho(n))\cap\{T_x<\lambda\}\bigg)<\infty.$$

Therefore, as in the argument leading to (5.5):

$$\limsup_{n\to\infty}\sup_{x\in K(\rho(n),\theta,R)}\frac{\big|\big(\pi\rho(n)/2\big)^{1/2}e_{\rho(n)}(x,\lambda)-L_\lambda^x\big|}{\kappa\big(\rho(n)\big)}\leq(2\pi)^{1/4}\sqrt{\theta L_\lambda^*}. \tag{7.3}$$

Temporarily fix some $\zeta\in(0,1/2)$. By the modulus of continuity of $t\mapsto Z(t)$, (see (2.1b)), a sample path argument reveals that uniformly over all $x\in\mathbb{R}^1$, and $t\in[0,1]$:

$$e_{\varepsilon_0}(x-\delta,t)\wedge e_{\varepsilon_0}(x+\delta,t)\leq e_\varepsilon(x,t)\leq e_{\varepsilon_1}(x-\delta,t)\vee e_{\varepsilon_1}(x+\delta,t),$$

where $\varepsilon_0\stackrel{df}{=}\varepsilon+\delta^\zeta$, $\varepsilon_1\stackrel{df}{=}\varepsilon-\delta^\zeta$, and $\varepsilon>\delta>0$ are small. Arguing as in the proof of (5.7), we are led to the following:

$$\limsup_{\varepsilon\downarrow 0}\sup_{x\in\mathbb{R}^1}\frac{|(\pi\varepsilon/2)^{1/2}e_\varepsilon(x,\lambda)-L_\lambda^x|}{\kappa(\varepsilon)}\leq(2\pi)^{1/4}\sqrt{L_\lambda^*}, \tag{7.4}$$

which is the desired upper bound. To get the lower bound, the ideas are again similar to those appearing in Section 5. Fix $M > 1$, $x \in \mathbb{R}^1$, $\theta \in (0, 1/2)$ and $\varepsilon \in \big(0, M^{-1}\big)$. Define,

$$D(x, \varepsilon) \stackrel{df}{=} \Big\{ f \in U_x : R_x^f \geq \varepsilon \text{ and } \sup_{0 \leq r \leq R_x^f} |f(r)| \leq x + (M\varepsilon)^{1/2} \Big\}.$$

Also define,

$$W(x, \varepsilon) \stackrel{df}{=} \Big\{ f \in U_x : \sup_{0 \leq r \leq R_x^f} |f(r)| \geq x + (M\varepsilon)^{1/2} \Big\}.$$

Let $\widetilde{N}$ be the counting process associated to $D(x, \varepsilon)$, i.e., $\widetilde{N}(x, \varepsilon) \stackrel{df}{=} N^x_{L^x_\lambda}\big(D(x, \varepsilon)\big)$. Define, $n(\varepsilon) \stackrel{df}{=} (\pi\varepsilon/2)^{1/2} n^0\big(D(0, \varepsilon)\big)$ and $\bar{n}(\varepsilon) \stackrel{df}{=} (\pi\varepsilon/2)^{1/2} \big[\; n^0\big(D(0, \varepsilon)\big) - n^0\big(D(0, \varepsilon)\big) \;\big]$. Based on the above definitions, we define:

$$\widetilde{D}(x; \theta, \varepsilon) \stackrel{df}{=} \Big\{ \omega : |(\pi\varepsilon/2)^{1/2} \widetilde{N}(x, \varepsilon)(\omega) - n(\varepsilon) L^x_\lambda(\omega)| \leq (2\pi)^{1/4} \kappa(\varepsilon) \sqrt{\theta n(\varepsilon) L^x_\lambda(\omega)} \Big\}.$$

The event $\widetilde{D}(x; \theta, \varepsilon)$ is this section's analogue of $\widetilde{G}(x; \theta, \varepsilon)$ of Section 5. The following lemma estimates $n(\varepsilon)$:

Lemma 7.2. *For all $\varepsilon \in (0, M^{-1})$:*

$$\begin{aligned} \bar{n}(\varepsilon) &\in \big[0, (2/M\pi)^{1/2}\big] \\ n(\varepsilon) &\in \big[1 - (2/M\pi)^{1/2}, 1\big]. \end{aligned}$$

Proof. Recall the definition of $D(\varepsilon)$ from (7.1). Then for all $w \in \mathbb{R}^1$ and $\varepsilon > 0$,

$$N^0_w\big(D(\varepsilon)\big) \geq N^0_w\big(D(0, \varepsilon)\big) \geq N^0_w\big(D(\varepsilon)\big) - N^0_w\big(W(0, \varepsilon)\big).$$

Letting $w = 1$ and taking expectations:

$$(\pi\varepsilon/2)^{1/2} n^0\big(D(\varepsilon)\big) \geq n(\varepsilon) \geq (\pi\varepsilon/2)^{1/2} \big[\; n^0\big(D(\varepsilon)\big) - n^0\big(W(0, \varepsilon)\big) \;\big].$$

The lemma follows since from Section 6: $n^0\big(D(\varepsilon)\big) = (2/\pi\varepsilon)^{1/2}$ and from Section 5 and the exit sytem formula: $n^0\big(W(0, \varepsilon)\big) = (M\varepsilon)^{-1/2}$. $\square$

Let $x_j \stackrel{df}{=} j(M\varepsilon)^{1/2}$ for $j = 0, \ldots, \nu \stackrel{df}{=} \big[(M\varepsilon)^{-1/2}\big]$. Writing $\widetilde{D}(x)$ for $\widetilde{D}(x; \theta, \varepsilon)$, as in Section 5 we obtain:

$$\begin{aligned} \mathbb{P}\big(\widetilde{D}(0) \cap \widetilde{D}(x_1) \cap \ldots \cap \widetilde{D}(x_\nu) \cap \{T x_\nu \leq \lambda\}\big) &\leq C_{19} \varepsilon^{-1/2} \exp\big(- C_{20} \nu \big(\log(1/\varepsilon)\big)^{-1/2} \varepsilon^\theta \big) \\ &\leq C_{21} \exp\big(- C_{22} \varepsilon^{\theta - 1/2} \big), \end{aligned}$$

for some positive finite constants, $C_j \overset{df}{=} C_j(\theta, M)$, $j = 19, \cdots, 22$. Fix $\rho \in (0,1)$ and define $\rho(m) \overset{df}{=} \rho^m$ for $m \geq 1$. By Lemma 7.2 and the Borel–Cantelli arguments of Section 5, since $\theta \in (0, 1/2)$ is arbitrary,

$$\liminf_{m\to\infty} \sup_{x\in\mathbb{R}^1} \frac{|(\pi\rho(m)/2)^{1/2}\widetilde{N}\big(x,\rho(m)\big) - n\big(\rho(m)\big)L_\lambda^x|}{\kappa\big(\rho(m)\big)} \geq (\pi/2)^{1/4}\sqrt{\beta_M L_\lambda^*}, \tag{7.5}$$

where $\beta_M \overset{df}{=} \big(1 - (2/M\pi)^{1/2}\big)$. On the other hand, let us define,

$$\widetilde{\widetilde{N}}(x,\varepsilon) \overset{df}{=} N_{L_\lambda^x}^x\big(D(\varepsilon) \setminus D(x,\varepsilon)\big) = N_{L_\lambda^x}^x\big(D(\varepsilon)\big) - \widetilde{N}(x,\varepsilon).$$

Then as in (7.3), we see that for all $\theta_0 > 1$,

$$\begin{aligned} \limsup_{m\to\infty} \sup_{x\in K(\rho(m),\theta_0,\infty)} & \frac{|\big(\pi\rho(m)/2\big)^{1/2}\widetilde{\widetilde{N}}\big(x,\rho(m)\big) - \bar{n}\big(\rho(m)\big)L_\lambda^x|}{\kappa\big(\rho(m)\big)} \\ & \leq (2\pi)^{1/4}\sqrt{(2/M\pi)\theta_0 L_\lambda^*} \\ & = 2^{3/4}\pi^{-1/4}M^{-1/2}\sqrt{\theta_0 L_\lambda^*}. \end{aligned} \tag{7.6}$$

Since $\widetilde{\widetilde{N}}(x,\varepsilon) + \widetilde{N}(x,\varepsilon)$ differs from $e_\varepsilon(x,\lambda)$ by at most one, (7.5) and (7.6) together yield the following:

$$\liminf_{m\to\infty} \sup_{x\in\mathbb{R}^1} \frac{|\big(\pi\rho(m)/2\big)^{1/2} e_{\rho(m)}(x,\lambda) - L_\lambda^x|}{\kappa\big(\rho(m)\big)} \geq (\pi/2)^{1/4}\alpha_M\sqrt{L_\lambda^*},$$

where $\alpha_M = \alpha_M(\theta_0) \overset{df}{=} \beta_M^{1/2} - 2(\theta_0/M\pi)^{1/2}$. But $M > 1$ is arbitrary. Therefore, we can let $M \uparrow \infty$ along a countable sequence. Since $\lim_{M\to\infty}\beta_M = 1$, $\lim_{M\to\infty}\alpha_M = 1$ as well. Hence the desired lower bound follows from the monotonicity argument of the previous sections. □

References.

M.T. Barlow, E.A. Perkins and S.J. Taylor (1986a) The behavior and construction of local times for Lévy processes. *Sem. Stoch. Proc.* 1984, Birkhäuser, Boston, pp. 151–164

M.T. Barlow, E.A. Perkins and S.J. Taylor (1986b) Two uniform intrinsic construction for the local time of a class of Lévy processes. *Ill. J. Math.* **30**, pp. 19–65

A.N. Borodin (1986) On the character of convergence to Brownian local time, I. *Z. Wahr. verw. Geb.* **72**, pp. 231–250

R.V. Chacon, Y. LeJan, E. Perkins and S.J. Taylor (1981) Generalized arc length for Brownian motion and Lévy processes. *Z. Wahr. verw. Geb.* **57**, pp. 197–211

K.L. Chung and R. Durrett (1976) Downcrossings and local time. *Z. Wahr. verw. Geb.* **35**, pp. 147–149

M. Csörgő and P. Révész (1981)*Strong Approximations in Probability and Statistics*, Academic Press, New York

M. Csörgő and P. Révész (1986) Mesure de voisinage and occupation density. *Prob. Th. Rel. Fields*, **73**, pp. 211–226

E.B. Dynkin (1984) Gaussian and non–Gaussian random fields associated with Markov processes, *J. Funct. Anal.* **55**, pp. 344–376

B.E. Fristedt and S.J. Taylor (1983) Construction of local time for a Markov process. *Z. Wahr. verw. Geb.* **62**, pp. 73–112

L. Horváth (1990) Strong laws and limit theorems for local time of Markov processes. *Prob. Th. Rel. Fields* **85**, pp. 413–424

I.A. Ibragimov and Yu V. Linnik (1971) *Independent and Stationary Sequences of Random Variables.* Wolters–Noordhoff Publishing, Groningen, ed. 1

K. Itô and H.P. McKean (1965) *Diffusion Processes and Their Sample Paths.* Springer, Berlin–Heidelberg

F.B. Knight (1981) *Essentials of Brownian Motion and Diffusion*, Amer. Math. Soc., Providence, Rhode Island

S.B. Kochen and C.J. Stone (1964) A note on the Borel–Cantelli problem. *Ill. J. Math.* **8**, pp. 248–251

H.P. McKean (1962) A Hölder condition for Brownian local time. *J. Math. Kyoto Univ.* **1-2**, pp. 195–201

B. Maisonneuve (1981) On Lévy's downcrossing theorem and various extnesions. *Sém. de prob.* LNM 850, Springer, Berlin–Heidelberg, pp. 191–205

M.B. Marcus and J. Rosen (1993) Exact rates of convergence to the local times of symmetric Lévy processes. Preprint.

E. Perkins (1981) A global intrinsic characterization of Brownian local time. *Ann. Prob.* Vol. 9, No. 5, pp. 800–817

D.B. Ray (1963) Sojourn times of a diffusion process. *Ill. J. Math.* **7**, pp.615–630

D. Revuz and M. Yor (1991) *Continuous Martingales and Brownian Motion.* Springer–Verlag, Berlin–Heidelberg

H.F. Trotter (1958) A property of Brownian motion paths. *Ill. J. Math.* **2**, pp. 425–433.

D. Williams (1974) Path decomposition and continuity of local time for one dimensional diffusions, I. *Proc. London Math. Soc. (3)*, **28**, pp. 738–768

D. Williams (1977) On Lévy's downcrossing theorem. *Z. Wahr. verw. Geb.* **40**, pp. 157–158

D. Williams (1979) *Diffusions, Markov Processes and Martingales, Vol. 1: Foundations.* Wiley and Sons, New York