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Local times of additive Lévy processes

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Abstract

Let $X = \{X(t);\ t \in \mathbb{R}_+^N\}$ be an additive Lévy process in $\mathbb{R}^d$ with

$$X(t) = X_1(t_1) + \cdots + X_N(t_N) \qquad \forall t \in \mathbb{R}_+^N,$$

where $X_1, \ldots, X_N$ are independent, classical Lévy processes on $\mathbb{R}^d$ with Lévy exponents $\Psi_1, \ldots, \Psi_N$, respectively. Under mild regularity conditions on the Ψ_i's, we derive moment estimates that imply joint continuity of the local times in question. These results are then refined to precise estimates for the local and uniform moduli of continuity of local times when all of the X_i's are strictly stable processes with the same index $\alpha \in (0, 2]$.

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1. Introduction

An N-parameter, d-dimensional random field $X = \{X(t);\ t \in \mathbb{R}_+^N\}$ is an *additive Lévy process*, if X has the following pathwise decomposition:

$$X(t) = X_1(t_1) + \cdots + X_N(t_N) \qquad \forall t \in \mathbb{R}_+^N,$$

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where $X_1,\dots,X_N$ are independent, classical Lévy processes on $\mathbb{R}^d$. Using tensor notation, we will often write $X = X_1 \oplus \cdots \oplus X_N$ for brevity. Throughout this paper, we will always be assuming that $X(0) = 0$.

Since they locally resemble Lévy sheets, and since they are more amenable to analysis, additive Lévy processes first arose to simplify the study of Lévy sheets (see Dalang and Walsh, 1993a, b; Ehm, 1981; Kahane, 1968; Kendall, 1980). They also arise in the theory of intersection and self-intersection of Lévy processes (see LeGall et al., 1989; Fitzsimmons and Salisbury, 1989; Khoshnevisan and Xiao, 2002). Moreover, recent progress has shown that additive Lévy processes have a rich and interesting structure on their own; especially noteworthy in this regard is their various connections to potential kernels and operators not found in classical probabilistic potential theory. We mention Hirsch and Song (1995), Khoshnevisan (1999), Khoshnevisan and Shi (1999), and Khoshnevisan and Xiao (2002) and refer the reader to the detailed discussion and the bibliography of the last reference for further works in this area.

In this, and a companion paper, we study the local times of additive Lévy processes. Formally, local times are defined by

$$L(a,I) = \int_I \delta_a(X(s))\,\mathrm{d}s,$$

where δ_a denotes Dirac's delta function at a. Here, we seek to find conditions that ensure continuity of $L(a,\bullet)$, as a measure-valued process, assuming that such local times exist. In a companion paper, we describe a necessary and sufficient condition for the existence of the mentioned local times.

While the existing literature on local times is too vast to mention here, in the context of Lévy processes and, more generally, Markov processes, we mention Bertoin (1996), Blumenthal and Getoor (1964), and Getoor and Kesten (1972). In the context of random fields, a good deal of mathematical, as well as historical, information can be found in Geman and Horowitz (1980) (see also Ehm, 1981; Vares, 1983; Geman et al., 1984; Lacey, 1990; Xiao, 1997).

The rest of the paper is organized as follows. Section 2 contains the definitions and some basic facts about ordinary, as well as additive, Lévy processes. Some sufficient conditions for the existence of local times of additive Lévy processes are also derived. In a companion paper, we will show that one of them is also necessary; see Theorem 2.1 for the precise statement and a proof of the easy half which is sufficiency. The hard half, i.e., necessity, will be presented in Khoshnevisan et al. (2002).

In Section 3, we prove the joint continuity of the local times of additive Lévy processes under a mild regularity condition. Our argument is based on deriving sharp moment estimates.

Section 4 establishes upper bounds for the moduli of continuity of the local times of additive stable processes (Theorem 4.3).

In Section 5, we compute lower envelopes for the oscillations of the sample functions of additive stable processes. Amongst other implications, these results show that the almost sure estimates of Section 4 are sharp, up to multiplicative constants. An inspection of our arguments reveals that the special structure of additive Lévy processes plays a very important rôle in our derivations.

This paper raises many questions about additive Lévy processes and Lévy sheets. We state some of them in Section 6.

2. Preliminaries

In this section, we present some notation and collect facts about Lévy processes, additive Lévy processes, as well as local times.

2.1. General notation

The underlying parameter space is $\mathbb{R}^N$, or $\mathbb{R}_+^N = [0,\infty)^N$, throughout. A typical parameter, $t \in \mathbb{R}^N$, is written as $t = (t_1, \ldots, t_N)$, coordinatewise. We frequently write t as $\langle c \rangle$, if $t_1 = t_2 = \cdots = t_N = c \in \mathbb{R}$.

There is a natural partial order, "$\preccurlyeq$", on $\mathbb{R}^N$. Namely, $s \preccurlyeq t$ if and only if $s_j \leqslant t_j$ for all $j = 1, \ldots, N$. When $s \preccurlyeq t$, we define the *interval*

$$[s,t] = \prod_{\ell=1}^{N} [s_\ell, t_\ell].$$

This partial order is, in fact, one of 2^N useful partial orders on $\mathbb{R}^N$ that we describe next.

Let $\Pi = \{1, \ldots, N\}$ and for all $A \subseteq \Pi$, we define $\preccurlyeq_{(A)}$ via

$$s \underset{(A)}{\preccurlyeq} t \Leftrightarrow \begin{cases} s_i \leqslant t_i & \text{for all } i \in A, \\ s_i \geqslant t_i & \text{for all } i \in A^{\complement}. \end{cases}$$

Thus, $\preccurlyeq$ is nothing other than $\preccurlyeq_{(\Pi)}$. We shall often also use $\succcurlyeq$ and $\succcurlyeq_{(A)}$; they mean the obvious thing: $s \succcurlyeq t$ if and only if $t \preccurlyeq s$, and so on. In particular, we note that $\succcurlyeq$ is the same relation as $\preccurlyeq_{(\varnothing)}$.

Throughout, we will let $\mathscr{A}$ denote the class of all N-dimensional intervals $I \subset \mathbb{R}^N$ that are parallel to the axes. That is $I \in \mathscr{A}$ is of the form $I = [s,t]$, where $s \preccurlyeq t$ are both in $\mathbb{R}^N$. If all the sides of I are of the same lengths, then I is called a cube. We always write λ_m for Lebesgue's measure on $\mathbb{R}^m$, no matter the value of the integer m.

The state space, $\mathbb{R}^d$, is endowed with the ℓ^2 Euclidean norm $\|\cdot\|$ and the corresponding dot product $\langle x, y\rangle = \sum_{j=1}^d x_j y_j$ ($x, y \in \mathbb{R}^d$). Furthermore, for any $x \in \mathbb{R}^d$, $|x| = \max_{1 \leqslant \ell \leqslant d} |x_\ell|$ denotes the ℓ^∞ norm of x.

We will use $K, K_1, K_2, \ldots$, to denote unspecified positive finite constants that may not necessarily be the same in each occurrence.

2.2. Lévy processes

Recall that a stochastic process $Z = \{Z(t);\ t \geqslant 0\}$, with values in $\mathbb{R}^d$, is called a *Lévy process*, if it has stationary and independent increments, such that $t \mapsto Z(t)$ is continuous in probability. It is well known that for $t \geqslant s \geqslant 0$, the characteristic function of $Z(t) - Z(s)$ is given by

$$\mathbb{E}[e^{i\langle \xi, Z(t)-Z(s)\rangle}] = e^{-(t-s)\psi(\xi)},$$

where by the Lévy–Khintchine formula,

$$\psi(\xi) = i\langle a, \xi\rangle + \frac{1}{2}\langle \xi, \Sigma\xi'\rangle + \int_{\mathbb{R}^d} \left[1 - e^{i\langle x,\xi\rangle} + \frac{i\langle x, \xi\rangle}{1+\|x\|^2}\right] \mathsf{L}(dx) \quad \forall \xi \in \mathbb{R}^d,$$

and $\mathsf{a} \in \mathbb{R}^d$ is fixed, Σ is a non-negative definite, symmetric, $(d \times d)$ matrix, and L is a Borel measure on $\mathbb{R}^d \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}^d} \frac{\|x\|^2}{1+\|x\|^2} \mathsf{L}(dx) < \infty.$$

The function ψ is the *Lévy exponent* of Z, and L is the corresponding *Lévy measure*. In this regard, we also note that

$$Re\,\psi(\xi) \geqslant 0 \quad \text{and} \quad Re\,\psi(-\xi) = Re\,\psi(\xi) \quad \forall \xi \in \mathbb{R}^d.$$

A Lévy process, Z, is *symmetric* if $-Z$ and Z have the same finite dimensional distributions. It is clear that Z is symmetric if and only if $\psi(\xi) \geqslant 0$ for all $\xi \in \mathbb{R}^d$.

Strictly stable processes on $\mathbb{R}^d$ with index $\alpha \in (0, 2]$ are Lévy processes on $\mathbb{R}^d$, whose Lévy exponent has the form

$$\psi(\xi) = \sigma\|\xi\|^\alpha \int_{\mathbb{S}_d} w_\alpha(\xi, y)\, \mathsf{M}(dy).$$

Here, $\sigma > 0$ is some fixed constant,

$$w_\alpha(\xi, y) = \left[1 - i\,\mathrm{sgn}(\langle \xi, y\rangle) \tan\left(\frac{\pi\alpha}{2}\right)\right] \left|\left\langle \frac{\xi}{\|\xi\|}, y\right\rangle\right|^\alpha \quad \text{if } \alpha \neq 1,$$

$$w_1(\xi, y) = \left|\left\langle \frac{\xi}{\|\xi\|}, y\right\rangle\right| + \frac{2i}{\pi}\langle \xi, y\rangle \log|\langle \xi, y\rangle|,$$

and M is a probability measure on the centered unit sphere $\mathbb{S}_d \subset \mathbb{R}^d$. When $\alpha = 1$, M must have the origin as its center of mass, i.e.,

$$\int_{\mathbb{S}_d} y\, \mathsf{M}(dy) = 0.$$

See, for example, Samorodnitsky and Taqqu (1994, p. 73). In particular, we note that the completely asymmetric Cauchy process is not strictly stable.

Throughout, we will tacitly assume that all stable distributions are *non-degenerate*; that is, the measure M is not supported by any diametral plane of $\mathbb{S}_d$. Then, it is possible to see that there exists a positive and finite constant K, such that

$$Re\,\psi(\xi) \geqslant K\|\xi\|^\alpha \quad \forall \xi \in \mathbb{R}^d. \tag{2.1}$$

Strictly stable processes of index α are $(1/\alpha)$-self-similar. A particularly interesting class arises when we let M be the uniform distribution on $\mathbb{S}_d$. In this case, $\psi(\xi) = \chi\|\xi\|^\alpha$ for some constant $\chi > 0$, and Z is the *isotropic stable process* with index α. Isotropic processes are sometimes also known as radial processes in the literature.

As discovered in Taylor (1967), it is natural to distinguish between two types of strictly stable processes: those of *Type A* and those of *Type B*. A strictly stable process, Z, is of *Type A*, if

$$p(t,y) > 0 \quad \forall t > 0, \ \ y \in \mathbb{R}^d,$$

where $p(t,y)$ is the density function of $Z(t)$; all other stable processes are called of *Type B*. Taylor (1967) has shown that if $\alpha \in (0,1)$, and if the measure M is concentrated on a hemisphere, then, Z is of *Type B*, while all other strictly stable processes of index $\alpha \neq 1$ are of *Type A*.

Blumenthal and Getoor (1961) have introduced the *lower index*, β^{low}, of a Lévy process Z as

$$\beta^{\text{low}} = \sup\left\{\gamma \geqslant 0\colon \lim_{\|\xi\|\to\infty} \|\xi\|^{-\gamma} Re\,\psi(\xi) = \infty\right\}. \tag{2.2}$$

It is always the case that $0 \leqslant \beta^{\text{low}} \leqslant 2$. Moreover, when the process Z is strictly stable with index α, $\beta^{\text{low}} = \alpha$. For more information on various indices for Lévy processes, their relationships and their usefulness in characterizing sample path properties of Lévy processes, we refer to Pruitt and Taylor (1996) and its bibliography.

2.3. Additive Lévy Processes

Let $X_1,\ldots,X_N$ denote N independent Lévy processes on $\mathbb{R}^d$, whose Lévy exponents are denoted by $\Psi_1,\ldots,\Psi_N$, respectively. For each $t \in \mathbb{R}_+^N$, the characteristic function of $X(t) = \sum_{j=1}^N X_j(t_j)$ is given by

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{\mathrm{i}\langle \xi, X(t)\rangle}] &= \mathrm{e}^{-\sum_{j=1}^N t_j \Psi_j(\xi)}\\ &= \mathrm{e}^{-\langle t, \Psi(\xi)\rangle}, \quad \xi \in \mathbb{R}^d,\end{aligned} \tag{2.3}$$

where $\Psi(\xi) = (\Psi_1(\xi),\ldots,\Psi_N(\xi))$.

We say that the additive Lévy process $X = X_1 \oplus \cdots \oplus X_N$ is *absolutely continuous* if for each $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$, the function $\xi \mapsto \exp\{-\langle t, \Psi(\xi)\rangle\} \in L^1(\mathbb{R}^d)$. In this case, for every $t \in \mathbb{R}_+^N \setminus \partial\mathbb{R}_+^N$, $X(t)$ has a density function $p(t;\bullet)$ that is described, by the Fourier inversion formula, as

$$p(t;x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \mathrm{e}^{-\mathrm{i}\langle \xi, x\rangle - \langle t, \Psi(\xi)\rangle}\, \mathrm{d}\xi \quad \forall x \in \mathbb{R}^d.$$

The *gauge function*, Φ, for the multiparameter process, X, is defined by

$$\Phi(s) = (2\pi)^{-d} \int_{\mathbb{R}^d} \exp\left\{-\sum_{j=1}^N |s_j| \Psi_j([\operatorname{sgn}(s_j)]\xi)\right\} \mathrm{d}\xi \quad \forall s \in \mathbb{R}^N.$$

It is important to observe that $\Phi(s) = \tilde{p}(s,0) \geqslant 0$, where $\tilde{p}(s,\bullet)$ is the density function of the random variable

$$\tilde{X}(s) = \sum_{j=1}^N \operatorname{sgn}(s_j) X_j(|s_j|). \tag{2.4}$$

When X is absolutely continuous, this density function exists, since the fact that $\xi \mapsto e^{-\langle t, \Psi(\xi)\rangle}$ is in $L^1(\mathbb{R}^d)$ readily implies the integrability of the characteristic function of the random variable in Eq. (2.4). In fact, it is easy to verify that X is absolutely continuous if and only if the characteristic function of $\tilde{X}(s)$ is Lebesgue integrable for all

$$s \in \mathbb{R}^N \setminus \left\{ t \in \mathbb{R}^N \colon \min_{1 \leqslant j \leqslant N} |t_j| = 0 \right\}.$$

In this case, we say that the additive Lévy process $\tilde{X}$ is *absolutely continuous*. We remark, further that the gauge function, Φ, is the continuous-time analogue of the gauge function for the additive random walks of Khoshnevisan and Xiao (2000). Moreover, when $X_1, \ldots, X_N$ are all symmetric Lévy processes, our definition of gauge function agrees with that of Khoshnevisan and Xiao (2002).

2.4. Local times

We end this section by briefly recalling aspects of the theory of local times. More information on local times of random, as well as non-random, functions can be found in Geman and Horowitz (1980), Geman et al. (1984), and Xiao (1997).

Let $X(t)$ be a Borel vector field on $\mathbb{R}^N$ with values in $\mathbb{R}^d$. For any Borel set $B \subseteq \mathbb{R}^N$, the occupation measure of X on B is defined as the following measure on $\mathbb{R}^d$:

$$\mu_B(\bullet) = \lambda_N\{t \in B \colon X(t) \in \bullet\}.$$

If μ_B is absolutely continuous with respect to λ_d, we say that $X(t)$ has *local times* on B and define its local times, $L(\bullet, B)$, as the Radon–Nikodým derivative of μ_B with respect to λ_d, i.e.,

$$L(x, B) = \frac{d\mu_B}{d\lambda_d}(x) \quad \forall x \in \mathbb{R}^d.$$

In the above, x is the so-called *space variable*, and B is the *time* variable. Sometimes, we write $L(x, t)$ in place of $L(x, [0, t])$.

By standard martingale and monotone class arguments, one can deduce that the local times have a measurable modification that satisfies the following *occupation density formula*: for every Borel set $B \subseteq \mathbb{R}^N$, and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_B f(X(t))\, dt = \int_{\mathbb{R}^d} f(x) L(x, B)\, dx. \tag{2.5}$$

Suppose we choose and fix a rectangle $T = \prod_{i=1}^N [a_i, a_i + h_i]$ in $\mathscr{A}$. Then, whenever we can choose a continuous modification of $\mathbb{R}^d \times \prod_{i=1}^N [0, h_i] \ni (x, t_1, \ldots, t_N) \mapsto L(x, \prod_{i=1}^N [a_i, a_i + t_i])$, X is said to have *jointly continuous local times* on T. When these local times are jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$X_T^{-1}(x) = \{t \in T \colon X(t) = x\}. \tag{2.6}$$

In fact, the null set in question can be chosen to be independent of x; see Adler (1981) for further details. In other words, local times often act as a Frostman measure on the level sets of X. As such, they are useful in studying the various fractal properties of the vector field X. In this regard, see Berman (1972), Ehm (1981), Monrad and Pitt (1987), Rosen (1984), LeGall et al. (1989), and Xiao (1997).

With the aid of some Fourier analysis, one can easily find the sufficiency portion of the following theorem. Proving necessity is more difficult, and is the subject of Khoshnevisan et al. (2002). When $N = 1$, the following is due to Hawkes (1986).

Theorem 2.1 (Khoshnevisan et al. (2002)). *Let $X = X_1 \oplus \cdots \oplus X_N$, where $X_1, \ldots, X_N$ are independent Lévy processes in $\mathbb{R}^d$ whose Lévy exponents are $\Psi_1, \ldots, \Psi_N$, respectively. If*

$$\int_{\mathbb{R}^d} \prod_{j=1}^N Re\left(\frac{1}{1+\Psi_j(u)}\right) \mathrm{d}u < +\infty, \tag{2.7}$$

then X admits square integrable local times on every interval $I \in \mathscr{A}$.

If, in addition, there exists a positive constant C_1 such that

$$Re\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) \geqslant C_1 \prod_{j=1}^N Re\left(\frac{1}{1+\Psi_j(\xi)}\right), \tag{2.8}$$

then Condition (2.7) is also necessary for the existence of local times.

In particular, whenever $X_1 \oplus \cdots \oplus X_N$ has square integrable local times, so does the additive Lévy process $X_1 \oplus \cdots \oplus X_N \oplus Y_1 \oplus \cdots \oplus Y_M$, where $Y_1, \ldots, Y_M$ are independent Lévy processes in $\mathbb{R}^d$, that are also totally independent of $X_1, \ldots, X_N$.

Remark 2.2. Examples of additive Lévy processes which satisfy Condition (2.8) are given in Khoshnevisan et al. (2002). Here we just mention that under the condition

$$\text{At least } N-1 \text{ of the Lévy processes } X_1, \ldots, X_N \text{ are symmetric}, \tag{2.9}$$

we always have

$$Re\left(\prod_{j=1}^N \frac{1}{1+\Psi_j(\xi)}\right) = \prod_{j=1}^N Re\left(\frac{1}{1+\Psi_j(\xi)}\right), \tag{2.10}$$

which can be verified by using induction. Hence, in this case, Condition (2.8) is satisfied with $C_1 = 1$.

It is easy to prove the sufficiency half of Theorem 2.1, which we do next for the sake of completeness.

Proof of sufficiency. Throughout, we assume $\prod_{j=1}^N Re\{1+\Psi_j\}^{-1} \in L^1(\mathbb{R}^d)$ and adapt the argument of Hawkes (1986, Theorem 1.1) to the present, multiparameter setting.

Define a Borel measure v on $\mathbb{R}^d$ by

$$v(\bullet) = \int_{\mathbb{R}_+^N} \mathrm{e}^{-\sum_{j=1}^N s_j} \mathbb{1}_\bullet(X(s))\,\mathrm{d}s.$$

Then, v is a random probability measure on the closure of $X(\mathbb{R}^N_+)$. It is easy to see that $v(A)=0$ if and only if $\mu_I(A)=0$ for every interval $I\in\mathscr{A}$. We prove that both v and μ_I are absolutely continuous with respective to λ_d and their densities are square integrable.

Denote the Fourier transform of v by $\hat{v}$, so that

$$\hat{v}(u)=\int_{\mathbb{R}^N_+} \mathrm{e}^{-\sum_{j=1}^N s_j}\ \mathrm{e}^{\mathrm{i}\langle X(s),u\rangle}\,\mathrm{d}s \quad \forall u\in\mathbb{R}^d.$$

By Fubini's theorem, and by the independence of the X_j's,

$$\begin{aligned}
&\mathbb{E}\left\{\int_{\mathbb{R}^d}|\hat{v}(u)|^2\,\mathrm{d}u\right\}\\
&\quad=\int_{\mathbb{R}^d}\int\int_{\mathbb{R}^N_+\times\mathbb{R}^N_+}\mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle u,X(s)-X(t)\rangle}\}\mathrm{e}^{-\sum_{j=1}^N s_j-\sum_{j=1}^N t_j}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\\
&\quad=\int_{\mathbb{R}^d}\int\int_{\mathbb{R}^N_+\times\mathbb{R}^N_+}\mathrm{e}^{-\sum_{\ell=1}^N|s_\ell-t_\ell|\Psi_\ell([\mathrm{sgn}(s_\ell-t_\ell)]u)-\sum_{j=1}^N s_j-\sum_{j=1}^N t_j}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u\\
&\quad=\int_{\mathbb{R}^d}\prod_{j=1}^N\left[\int_0^\infty\int_0^\infty \mathrm{e}^{-s-t-|s-t|\Psi_j([\mathrm{sgn}(s-t)]u)}\,\mathrm{d}s\,\mathrm{d}t\right]\,\mathrm{d}u.
\end{aligned}$$

We now do the natural thing and break up the double integral into two regions: one where $s\geqslant t$, and one where $s\leqslant t$, and use the fact that for all $z\in\mathbb{C}$, $\{1+z\}^{-1}+\{1+\bar{z}\}^{-1}=2\,Re\{1+z\}^{-1}$. Namely,

$$\begin{aligned}
&\mathbb{E}\left\{\int_{\mathbb{R}^d}|\hat{v}(u)|^2\,\mathrm{d}u\right\}\\
&\quad=\int_{\mathbb{R}^d}\prod_{j=1}^N\left[\int_0^\infty\mathrm{d}t\int_t^\infty(\cdots)\,\mathrm{d}s\right]\,\mathrm{d}u+\int_{\mathbb{R}^d}\prod_{j=1}^N\left[\int_0^\infty\mathrm{d}s\int_s^\infty(\cdots)\,\mathrm{d}t\right]\,\mathrm{d}u\\
&\quad=\int_{\mathbb{R}^d}\prod_{j=1}^N\frac{1}{2}\left[\frac{1}{1+\overline{\Psi_j(u)}}+\frac{1}{1+\Psi_j(u)}\right]\,\mathrm{d}u\\
&\quad=\int_{\mathbb{R}^d}\prod_{j=1}^N Re\left(\frac{1}{1+\Psi_j(u)}\right)\,\mathrm{d}u<\infty.
\end{aligned}$$

By the Riesz–Fisher theorem and/or Plancherel's theorem, v is, almost surely, absolutely continuous with respect to Lebesgue's measure λ_d, and its density is, almost surely, in $L^2(\mathbb{R}^d)$.

To prove that for every $I\in\mathscr{A}$, X almost surely has square integrable local times $L(\bullet,I)$ on I, we first note that there exists a positive and finite constant K—it depends on I—such that

$$\mathbb{E}(|\widehat{\mu_I}(u)|^2)\leqslant K\,\mathbb{E}(|\hat{v}(u)|^2)\quad\forall u\in\mathbb{R}^d.$$

Hence, we almost surely have $\widehat{\mu}_I \in L^2(\mathbb{R}^d)$, which implies that X has a square integrable local time on I. □

We present two useful corollaries of this theorem that are stated in terms of easy-to-check conditions; one in terms of lower indices and the other in terms of the gauge function, when it exists.

Corollary 2.3. *Suppose $X = X_1 \oplus \cdots \oplus X_N$, where $X_1,\dots,X_N$ are independent Lévy processes in $\mathbb{R}^d$ with lower indices $\beta_1^{\text{low}},\dots,\beta_N^{\text{low}}$, respectively. Then, X has square integrable local times on every finite interval $I \in \mathscr{A}$, as long as $\sum_{\ell=1}^N \beta_\ell^{\text{low}} > d$.*

Proof. By the definition of the lower indices (Eq. (2.2)), for all $\varepsilon > 0$, there exists $n > 0$, such that whenever $\|u\| > n$, $Re\,\Psi_\ell(u) \geqslant \|u\|^{\beta_\ell^{\text{low}}-\varepsilon}$, for all $\ell = 1,\dots,N$. Consequently, for all $u \in \mathbb{R}^d$,

$$\prod_{\ell=1}^N Re\left(\frac{1}{1+\Psi_\ell(u)}\right) \leqslant \prod_{\ell=1}^N \frac{1}{1+Re\,\Psi_\ell(u)} \leqslant \|u\|^{N\varepsilon - \sum_{\ell=1}^N \beta_\ell^{\text{low}}} \wedge 1.$$

This integrates [du], if we choose $0 < \varepsilon < (1/N)[\sum_{\ell=1}^N \beta_\ell^{\text{low}} - d]$. Theorem 2.1 does the rest. □

Corollary 2.4. *Let $X = X_1 \oplus \cdots \oplus X_N$ be an additive Lévy process in $\mathbb{R}^d$ satisfying Condition* (2.8). *We assume that X is absolutely continuous with gauge function Φ. Then, X has square integrable local times on every interval $I \in \mathscr{A}$ if and only if $\Phi \in L^1_{\text{loc}}(\mathbb{R}^N)$.*

Proof. A calculation similar to the one made in the sufficiency proof of Theorem 2.1 reveals that as long as Φ exists,

$$\int_{\mathbb{R}^N} \mathrm{e}^{-\sum_{j=1}^N |s_j|} \Phi(s)\,\mathrm{d}s = (2\pi)^{-d} \int_{\mathbb{R}^d} \prod_{j=1}^N Re\left(\frac{1}{1+\Psi_j(u)}\right) \mathrm{d}u.$$

Hence, thanks to Theorem 2.1, the existence of square integrable local times implies that the left-hand side of the above display is finite. This, in turn, implies that $\Phi \in L^1_{\text{loc}}$. Conversely, given $\Phi \in L^1_{\text{loc}}$, it suffices to show that for all $I \in \mathscr{A}$, $\widehat{\mu}_I \in L^2(\mathbb{R}^d)$ a.s., keeping in mind that μ_I is the occupation measure over I (cf. Section 2.4). For then, by Plancherel's theorem, μ_I is absolutely continuous with respect to λ_d, and its Radon–Nykodým derivative is in $L^2(\mathbb{R}^d)$. However,

$$\begin{aligned}
\mathbb{E}\left\{\int_{\mathbb{R}^d} |\widehat{\mu}_I(u)|^2\,\mathrm{d}u\right\} &= \int_{\mathbb{R}^d} \iint_{I\times I} \mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle u, X(s)-X(t)\rangle}\}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u \\
&= \int_{\mathbb{R}^d} \iint_{I\times I} \mathrm{e}^{-\sum_{j=1}^N |s_j-t_j|\Psi_j([\operatorname{sgn}(s_j-t_j)]\,u)}\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}u
\end{aligned}$$

$$= \iint_{I\times I} \Phi(s-t)\,\mathrm{d}s\,\mathrm{d}t$$

$$\leqslant \lambda_N(I) \int_{I\ominus I} \Phi(s)\,\mathrm{d}s,$$

where $I \ominus I = \{a-b:\ a,b \in I\}$. This completes our proof. □

3. Joint continuity

Now, we turn to the problem of studying the existence of jointly continuous local times of additive Lévy processes. Throughout, we assume the following regularity condition. In the light of Corollary 2.3, this condition, a priori, implies the existence of square integrable local times:

(C) If $\beta_\ell^{\mathrm{low}}$ denotes the lower index of Ψ_ℓ for $\ell = 1,\dots,N$, then $\sum_{\ell=1}^N \beta_\ell^{\mathrm{low}} > d$.

Remark 3.1. An equivalent formulation of Condition (C) is that there exists constants $K>0$, $r_0>0$ and $\beta_1,\dots,\beta_N \geqslant 0$, such that $\sum_{\ell=1}^N \beta_\ell > d$, and $Re\,\Psi_\ell(\xi) \geqslant K\|\xi\|^{\beta_\ell}$ for all $\|\xi\| > r_0$. In particular, thanks to Eq. (2.1), all additive stable processes with index α and $N\alpha > d$ satisfy Condition (C).

Our main general result is the following:

Theorem 3.2. *Let $X = X_1 \oplus \cdots \oplus X_N$, where $X_1,\dots,X_N$ are independent Lévy processes in $\mathbb{R}^d$. If Condition (C) holds, then for any $I=[a,a+h] \in \mathscr{A}$, almost surely X has jointly continuous local times, $L=\{L(x,[a,a+t]);\ (x,t) \in \mathbb{R}^d \times [0,h]\}$.*

Proof. Throughout, we will assume and use the notation of Remark 3.1 regarding Condition (C). It follows from Geman and Horowitz (1980, Eqs. (25.2) and (25.7)) that for any $x,y \in \mathbb{R}^d$, $I=\prod_{\ell=1}^N [a_\ell, a_\ell+h_\ell] \in \mathscr{A}$ and any integer $k \geqslant 1$, we have

$$\mathbb{E}\{[L(x,I)]^k\} = (2\pi)^{-kd} \int_{I^k} \int_{\mathbb{R}^{kd}} \mathrm{e}^{-\mathrm{i}\sum_{j=1}^k \langle u^j, x\rangle} \mathbb{E}\{\mathrm{e}^{\mathrm{i}\sum_{j=1}^k \langle u_j, X(t^j)\rangle}\}\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{t} \tag{3.1}$$

and for any even integer $k \geqslant 2$,

$$\mathbb{E}\{[L(x,I)-L(y,I)]^k\}$$

$$=(2\pi)^{-kd} \int_{I^k} \int_{\mathbb{R}^{kd}} \prod_{j=1}^k [\mathrm{e}^{-\mathrm{i}\langle u^j,x\rangle} - \mathrm{e}^{-\mathrm{i}\langle u^j,y\rangle}] \mathbb{E}\{\mathrm{e}^{\mathrm{i}\sum_{j=1}^k \langle u^j, X(t^j)\rangle}\}\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{t}, \tag{3.2}$$

where $\bar{u}=(u^1,\dots,u^k) \in \mathbb{R}^{kd}$ and $\bar{t}=(t^1,\dots,t^k) \in I^k$ (see also Geman et al., 1984). [N.B. Written coordinatewise, $t^j=(t_1^j,\dots,t_N^j)$.]

In order to prove the joint continuity of L, we first establish appropriate upper bounds for (3.1) and (3.2), and then apply the continuity lemma of Garsia (1972).

By the elementary inequality

$$|\mathrm{e}^{\mathrm{i}u}-1| \leqslant 2^{1-\gamma}|u|^\gamma \quad \text{for any } u \in \mathbb{R},\ 0<\gamma<1,$$

we see that for any even integer $k \geqslant 2$ and any $0 < \gamma < 1$, (3.2) is bounded above by

$$(2\pi)^{-kd}2^{(1-\gamma)k}\|x-y\|^{k\gamma}\int_{I^k}\int_{\mathbb{R}^{kd}}\prod_{j=1}^{k}\|u^j\|^{\gamma}|\mathbb{E}\{\mathrm{e}^{-\mathrm{i}\sum_{j=1}^{k}\langle u^j,X(t^j)\rangle}\}|\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{t}. \tag{3.3}$$

For convenience, we introduce the following quantity:

$$\mathscr{J}(I,k,\gamma)=\int_{I^k}\int_{\mathbb{R}^{kd}}\prod_{j=1}^{k}\|u^j\|^{\gamma}|\mathbb{E}\{\mathrm{e}^{\mathrm{i}\sum_{j=1}^{k}\langle u^j,X(t^j)\rangle}\}|\,\mathrm{d}\bar{u}\,\mathrm{d}\bar{t}, \tag{3.4}$$

where $I\in\mathscr{A}$, $\gamma\in[0,1)$, and $k\geqslant 1$ is an integer. We note that

$$\mathbb{E}\{[L(x,I)]^k\}\leqslant(2\pi)^{-kd}\mathscr{J}(I,k,0). \tag{3.5}$$

For each fixed $\bar{t}\in I^k$,

$$\int_{\mathbb{R}^{kd}}\prod_{j=1}^{k}\|u^j\|^{\gamma}|\mathbb{E}\{\mathrm{e}^{\mathrm{i}\sum_{j=1}^{k}\langle u^j,X(t^j)\rangle}\}|\,\mathrm{d}\bar{u}$$

$$=\int_{\mathbb{R}^{kd}}\prod_{j=1}^{k}\|u^j\|^{\gamma}\prod_{\ell=1}^{N}|\mathbb{E}\{\mathrm{e}^{\mathrm{i}\sum_{j=1}^{k}\langle u^j,X_\ell(t_\ell^j)\rangle}\}|\,\mathrm{d}\bar{u}. \tag{3.6}$$

By Eq. (3.6), we may, and will, assume with no loss of generality, that $\beta_\ell>0$ for all $\ell=1,\ldots,N$. Since $\sum_{\ell=1}^{N}\beta_\ell>d$, by induction, we see that there exist N positive numbers $p_1,\ldots,p_N$ such that

$$\sum_{\ell=1}^{N}\frac{1}{p_\ell}=1\quad\text{and}\quad p_\ell\beta_\ell>d\ \ \forall\ell=1,\ldots,N. \tag{3.7}$$

It follows from the independence of $X_1,\ldots,X_N$ and the generalized Hölder inequality (Hardy, 1934, p. 140) that $\mathscr{J}(I,k,\gamma)$ is bounded above by

$$\int_{I^k}\prod_{\ell=1}^{N}\left[\int_{\mathbb{R}^{kd}}\prod_{j=1}^{k}\|u^j\|^{\gamma}|\mathbb{E}\{\mathrm{e}^{\mathrm{i}\sum_{j=1}^{k}\langle u^j,X_\ell(t_\ell^j)\rangle}\}|^{p_\ell}\,\mathrm{d}\bar{u}\right]^{1/p_\ell}\mathrm{d}\bar{t}. \tag{3.8}$$

Fix ℓ for the moment, and let r_ℓ^j $(1\leqslant j\leqslant k)$ denote the jth order statistic of the k-tuple $(t_\ell^1,\ldots,t_\ell^k)$. That is, $(r_\ell^1,\ldots,r_\ell^k)$ is a permutation of $(t_\ell^1,\ldots,t_\ell^k)$ that satisfies $r_\ell^1\leqslant\cdots\leqslant r_\ell^k$. To keep the notation from getting overbearing, we continue to write $(u^1,\ldots,u^k)$ for the corresponding permutation of $\bar{u}$.

Since the Lévy process X_ℓ has stationary and independent increments,

$$\int_{\mathbb{R}^{kd}}\prod_{j=1}^{k}\|u^j\|^{\gamma}|\mathbb{E}\{\mathrm{e}^{\mathrm{i}\sum_{j=1}^{k}\langle u^j,X_\ell(t_\ell^j)\rangle}\}|^{p_\ell}\,\mathrm{d}\bar{u}$$

$$\leqslant\int_{\mathbb{R}^{kd}}\prod_{j=1}^{k}\|u^j\|^{\gamma}\left|\mathbb{E}\left\{\exp\left(\mathrm{i}\sum_{j=1}^{k}\left\langle\sum_{m=j}^{k}u^m,X_\ell(r_\ell^j)-X_\ell(r_\ell^{j-1})\right\rangle\right)\right\}\right|^{p_\ell}\mathrm{d}\bar{u}$$

$$= \int_{\mathbb{R}^{kd}} \prod_{j=1}^{k} \|u^j\|^{\gamma} \prod_{j=1}^{k} \left| \mathbb{E}\left\{ \exp\left(\mathrm{i} \left\langle \sum_{m=j}^{k} u^m, X_\ell(r_\ell^j) - X_\ell(r_\ell^{j-1}) \right\rangle \right) \right\} \right|^{p_\ell} \mathrm{d}\bar{u}$$

$$= \int_{\mathbb{R}^{kd}} \prod_{j=1}^{k} [\|u^j\|^{\gamma} \exp\left(-p_\ell (r_\ell^j - r_\ell^{j-1}) \, Re\, \Psi_\ell \left(\sum_{m=j}^{k} u^m \right) \right) \Bigg] \, \mathrm{d}\bar{u}.$$

In the above we have written $r_\ell^0 = a_\ell$. By letting

$$v^j = \sum_{m=j}^{k} u^m \quad \forall 1 \leqslant j \leqslant k,$$

we see that the above integral is equal to

$$\int_{\mathbb{R}^{kd}} \prod_{j=1}^{k} \|v_\ell^j - v_\ell^{j+1}\|^{\gamma} \exp(-p_\ell (r_\ell^j - r_\ell^{j-1}) \, Re\, \Psi_\ell(v^j)) \, \mathrm{d}\bar{v}, \tag{3.9}$$

where $v^{k+1} = 0$. On the other hand, for any $0 < \gamma < 1$, $|a+b|^{\gamma} \leqslant |a|^{\gamma} + |b|^{\gamma}$, we have

$$\prod_{j=1}^{k} \|v^j - v^{j+1}\|^{\gamma} \leqslant \sum{}' \prod_{j=1}^{k} \|v^j\|^{q_j \gamma},$$

where $\sum'$ denotes summation over all $(q_1, \ldots, q_k) \in \{0,1,2\}^k$ such that $\sum_{j=1}^k q_j = k$. Hence, the integral in (3.9) is bounded above by

$$\int_{\mathbb{R}^{kd}} \sum{}' \prod_{j=1}^{k} \|v^j\|^{q_j\gamma} \exp(-p_\ell (r_\ell^j - r_\ell^{j-1}) \, Re\, \Psi_\ell(v^j)) \, \mathrm{d}\bar{v}$$

$$= \sum{}' \prod_{j=1}^{k} \left[\int_{\mathbb{R}^d} \|v^j\|^{q_j\gamma} \exp\left(-p_\ell (r_\ell^j - r_\ell^{j-1}) \, Re\, \Psi_\ell(v^j) \right) \mathrm{d}v^j \right]$$

$$\leqslant \sum{}' K^k \prod_{j=1}^{k} (r_\ell^j - r_\ell^{j-1})^{-(q_j\gamma + d)/\beta_\ell}, \tag{3.10}$$

where we have used Condition (C) in deriving the last inequality, and $K > 0$ is a finite constant depending on Ψ_ℓ, p_ℓ and d only (cf. also Remark 3.1). Combining Eqs. (3.4)–(3.10), and noting the $k!$ permutations of $\{1, \ldots, k\}$, we obtain

$$\mathscr{J}(I, k, \gamma) \leqslant \sum{}' K^k \prod_{\ell=1}^{N} k! \int_{a_\ell = r_\ell^0 \leqslant r_\ell^1 \cdots \leqslant r_\ell^k \leqslant a_\ell + h_\ell} \times \prod_{j=1}^{k} (r_\ell^j - r_\ell^{j-1})^{-(q_j\gamma + d)/\beta_\ell p_\ell} \, \mathrm{d}t_\ell^1 \cdots \mathrm{d}t_\ell^k. \tag{3.11}$$

We take $\gamma \in [0, 1)$ such that $(2\gamma + d)/(\beta_\ell p_\ell) < 1$, which is legitimate, thanks to (3.7). We also need the following elementary calculation: for all $k \geqslant 1$, $h > 0$

and $b_j < 1$,

$$\int_{0\leqslant s^1\leqslant\cdots\leqslant s^k\leqslant h}\prod_{j=1}^k (s^j - s^{j-1})^{-b_j}\,\mathrm{d}s^1\cdots\mathrm{d}s^k = h^{k-\sum_{j=1}^k b_j}\,\frac{\prod_{j=1}^k \Gamma(1-b_j)}{\Gamma(1+k-\sum_{j=1}^k b_j)},$$

where $s^0 = 0$. [It follows from induction on k.] Thus, we obtain

$$\begin{aligned}\mathscr{J}(I,k,\gamma) &\leqslant {\sum}' K^k \prod_{\ell=1}^N \Bigg[k!\,h_\ell^{k-\sum_{j=1}^k (q_j\gamma+d)/(\beta_\ell p_\ell)} \\ &\qquad\times \frac{\prod_{j=1}^k \Gamma(1-(q_j\gamma+d)/(\beta_\ell p_\ell))}{\Gamma(1+k-\sum_{j=1}^k (q_j\gamma+d)/(\beta_\ell p_\ell))}\Bigg] \\ &\leqslant K^k (k!)^N \prod_{\ell=1}^N \frac{h_\ell^{k(1-(\gamma+d)/(\beta_\ell p_\ell))}}{\Gamma(1+k(1-(\gamma+d)/(\beta_\ell p_\ell)))}.\end{aligned} \tag{3.12}$$

It follows from Eqs. (3.1)–(3.3), (3.5) and (3.12) that

$$\mathbb{E}\{[L(x,I)]^k\} \leqslant K^k (k!)^N \prod_{\ell=1}^N \frac{h_\ell^{k(1-d/(\beta_\ell p_\ell))}}{\Gamma(1+k(1-d/(\beta_\ell p_\ell)))}, \tag{3.13}$$

while

$$\begin{aligned}\mathbb{E}\{[L(x,I)-L(y,I)]^k\} &\leqslant K^k (k!)^N \|x-y\|^{k\gamma} \prod_{\ell=1}^N \\ &\qquad \times \frac{h_\ell^{k(1-(\gamma+d)/(\beta_\ell p_\ell))}}{\Gamma(1+k(1-(\gamma+d)/(\beta_\ell p_\ell)))}.\end{aligned} \tag{3.14}$$

By Eqs. (3.13) and (3.14), and by the triangle inequality, for all even integers $k \geqslant 2$,

$$\mathbb{E}\{[L(x,[a,a+s]) - L(y,[a,a+t])]^k\} \leqslant K_1^k (k!)^N |(x,s)-(y,t)|^{k\gamma}. \tag{3.15}$$

The asserted joint continuity of $(x,t) \mapsto L(x,[a,a+t])$ follows immediately from Eq. (3.15), and the continuity lemma of Garsia (1972). This completes our proof of Theorem 3.2. $\square$

The following is an immediate corollary of Theorem 3.2.

Corollary 3.3. *Suppose $X = X_1 \oplus \cdots \oplus X_N$, where $X_1,\ldots,X_N$ are independent, strictly stable processes in $\mathbb{R}^d$ with indices $\alpha_1,\ldots,\alpha_N \in (0,2]$, respectively. If $\sum_{\ell=1}^N \alpha_\ell > d$, then for each $I = [a,a+h]$, local times, $L = \{L(x,[a,a+t]),(x,t)\in\mathbb{R}^d\times[0,h]\}$, exist that are jointly continuous in (x,t).*

Remark 3.4. Ehm (1981, Theorem 1.1) states that the stable sheets have jointly continuous local times. Moreover, under the same conditions, upper bounds are given for the moduli of continuity of these local times. The arguments of Ehm (1981) rely on decomposing an N-parameter stable sheet of index α as a sum of an N-parameter additive stable process, and a negligible remainder (see Ehm, 1981, Eq. (1.9)). Viewed as such, Ehm (1981, Theorem 1.1) is, in fact, a theorem on N-parameter additive stable processes of index α, and is refined in Theorem 3.2 above.

Remark 3.5. Thanks to Theorem 2.1, the condition $\sum_{\ell=1}^N \alpha_\ell > d$ in Corollary 3.3 is best possible in the sense that whenever $\sum_{\ell=1}^N \alpha_\ell \leqslant \mathrm{d}$, square integrable local times do not exist. Moreover, under Condition (2.8), the condition $\sum_{\ell=1}^N \alpha_\ell \leqslant d$ implies that for every point $x \in \mathbb{R}^d$, $X^{-1}(x) = \emptyset$ a.s., that is, X does not hit points in $\mathbb{R}^d$; this follows from Theorem 1.5 in Khoshnevisan et al. (2002). In the special case that $X_1, \ldots, X_N$ are symmetric stable Lévy processes in $\mathbb{R}^d$, this was proved directly in Khoshnevisan and Xiao (2002, Theorems 1.1 and 2.9).

If $X_1, \ldots, X_N$ are strictly stable processes with the same index α, then our proof of Theorem 3.2 yields the following estimates for the even moments of the local time that will be used in Section 4: If $N\alpha > d$, then for any $\gamma \in (0, 1 \wedge \frac{1}{2}(N\alpha - d))$, there are finite constants K_2, K_3 such that for any $I = [a, a+h] \in \mathscr{A}$, all x, $y \in \mathbb{R}^d$, and even number k

$$\mathbb{E}\left\{\left[\frac{L(X(t)+x, I)}{[\lambda_N(I)]^{1-d/(N\alpha)}}\right]^k\right\} \leqslant K_2^k \left[\frac{k!}{\Gamma[1+k(1-d/(N\alpha))]}\right]^N \tag{3.16}$$

and

$$\mathbb{E}\left\{\left[\frac{(L(X(t)+x, I) - L(X(t)+y, I)}{[\lambda_N(I)]^{1-(d+\gamma)/(N\alpha)}}\right]^k\right\}$$

$$\leqslant K_3^k \|x-y\|^{k\gamma} \left[\frac{k!}{\Gamma[1+k(1-(d+\gamma)/(N\alpha))]}\right]^N. \tag{3.17}$$

Here, t may be either of the time points $t = 0$ or a. In case $t = 0$, Eqs. (3.16) and (3.17) follow from Eqs. (3.13) and (3.14) with $p_\ell = N$ $(\ell = 1, 2, \ldots, N)$, respectively. On the other hand, it is clear that the inequality in (3.12) remains valid if we replace the random variables $X(t^j)$ in definition (3.4) of $\mathscr{J}(I, k, \gamma)$ by $X(t^j) - X(a)$. Hence, Eqs. (3.16) and (3.17) also hold for $t = a$.

4. Hölder laws: upper bounds

In this section, we are interested in deriving Hölder-type estimates for the moduli of continuity for the local times of additive stable processes. In the context of classical one-parameter processes, such works can be found, for example, in (Donsker and Varadhan (1977) and Kesten (1965), while in Ehm (1981) upper bounds for the moduli of continuity of the local times of (multi-parameter) stable sheets can be found. Further limit laws for two-parameter, real Brownian sheet are found in Lacey (1990).

In this section, we continue along the lines of the aforementioned works by establishing upper bounds for the local, as well as uniform, moduli of continuity of the local times of additive stable processes. We will see later in Section 5 that these upper bounds are, in fact, sharp up to multiplicative constants. Throughout Sections 4 and 5, we assume that the strictly stable processes $X_1, \ldots, X_N$ are of *Type A*. For brevity, we will say that X is of *Type A*.

Lemma 4.1. *Let X be a Type A additive stable process in $\mathbb{R}^d$ with index $\alpha \in (0,2]$. Then, there exists a positive and finite constant K such that for all $I = [0,a] \in \mathscr{A}$ and $\xi > 0$,*

$$\mathbb{P}\left\{\sup_{t\in I}\|X(t)\| \geqslant \xi\right\} \leqslant K|a|\xi^{-\alpha}. \tag{4.1}$$

Proof. Inequality (4.1) follows easily from $\|X(t)\| \leqslant \sum_{\ell=1}^{N} |X_\ell(t_\ell)|$ and a well known fact about ordinary stable processes of index α (see Bertoin, 1996, p. 221). □

The following lemma is a consequence of (3.16), (3.17) and Chebyshev's inequality. The proof is standard, and hence omitted.

Lemma 4.2. *Let X be a Type A additive Lévy process in $\mathbb{R}^d$ with index $\alpha \in (0,2]$. For any number $\gamma \in (0, 1 \wedge \frac{1}{2}(N\alpha - d))$, there are constants $b_1, b_2 > 0$, and $0 < K_4,\ K_5 < +\infty$, such that for all $I = [a, a+h] \in \mathscr{A}$, $x, y \in \mathbb{R}^d$, and all $u > 0$,*

$$\mathbb{P}\{L(X(t)+x, I) \geqslant [\lambda_N(I)]^{1-d/(N\alpha)} u^{d/\alpha}\} \leqslant K_4 \mathrm{e}^{-b_1 u} \tag{4.2}$$

and

$$\mathbb{P}\{|L(X(t)+x, I) - L(X(t)+y, I)| \geqslant [\lambda_N(I)]^{1-(d+\gamma)/(N\alpha)} \|x-y\|^{\gamma} u^{(d+\gamma)/\alpha}\}$$
$$\leqslant K_5 \mathrm{e}^{-b_2 u}, \tag{4.3}$$

where either $t = 0$ or a.

Theorem 4.3. *Let X be a Type A additive stable process in $\mathbb{R}^d$ with index $\alpha \in (0,2]$ and $N\alpha > d$. Let L be its jointly continuous local time, and for $I \in \mathscr{A}$, write $L^*(I) = \sup_{x\in\mathbb{R}^d} L(x, I)$. Then, there are finite constants $K_6,\ K_7$, such that for every $\tau \in (0,\infty)^N$ and every $T \in \mathscr{A}$,*

$$\limsup_{r\to 0} \frac{L^*([\tau - \langle r\rangle, \tau + \langle r\rangle])}{r^{N-d/\alpha}(\log\log r^{-1})^{d/\alpha}} \leqslant K_6, \quad \text{a.s.} \tag{4.4}$$

and

$$\limsup_{r\to 0} \sup_{\substack{I\in\mathscr{A}, I\subset T\\ \lambda_N(I)<r}} \frac{L^*(I)}{[\lambda_N(I)]^{1-d/(N\alpha)}|\log \lambda_N(I)|^{d/\alpha}} \leqslant K_7, \quad \text{a.s.} \tag{4.5}$$

Proof. Our proof is based on Lemmas 4.1 and 4.2, together with a chaining argument similar to that employed in Ehm (1981) and Xiao (1997). In the following, we only verify Eq. (4.4). Eq. (4.5) follows along similar lines; see Ehm (1981) and Xiao (1997) for further details.

For brevity, we write $g(r) = r^{N-d/\alpha}(\log\log r^{-1})^{d/\alpha}$ for small $r > 0$. Since for any $\tau \in \mathbb{R}_+^N$ and $r > 0$, the cube $[\tau - \langle r\rangle, \tau + \langle r\rangle]$ can be covered by at most 2^N subcubes of sides r in $\mathscr{A}$, we see that (4.4) will follow from a standard monotonicity argument, once we prove that for any $s \in \mathbb{R}_+^N$,

$$\limsup_{n\to\infty} \frac{L^*(C_n)}{g(2^{-n})} \leqslant K_6, \quad \text{a.s.,} \tag{4.6}$$

where $C_n = [s, s + \langle 2^{-n} \rangle]$, $n \geqslant 1$.

Having dispensed with the requisite preliminaries, we divide our proof of (4.6) into four parts.

(a) Lemma 4.1 implies for any $\beta > 0$,

$$\mathbb{P}\left\{ \sup_{t\in C_n} \|X(t) - X(s)\| \geqslant 2^{-n/\alpha} n^{\beta} \right\} \leqslant K n^{-\alpha\beta} \quad \forall n \geqslant 1.$$

We select $\beta > 1/\alpha$ and appeal to the Borel–Cantelli lemma, to deduce that with probability one,

$$\sup_{t\in C_n} \|X(t) - X(s)\| < 2^{-n/\alpha} n^{\beta} \quad \forall n \text{ large enough.} \tag{4.7}$$

(b) Let $\theta_n = (2^n \log n)^{-1/\alpha}$, $n \geqslant 1$, and define

$$G_n = \{x \in \mathbb{R}^d \colon \|x\| \leqslant 2^{-n/\alpha} n^{\beta} \text{ and } \exists p \in \mathbb{Z}^d \colon x = \theta_n p, \}.$$

It follows from (4.2) that

$$\begin{aligned} \mathbb{P}\left\{ \max_{x\in G_n} L(X(s) + x, C_n) \geqslant a_1^{d/\alpha} g(2^{-n}) \right\} &\leqslant K\,(2^{-n/\alpha} n^{\beta}/\theta_n)^d \mathrm{e}^{-b_1 a_1 \log\log 2^n} \\ &\leqslant K\,(\log n)^{d/\alpha}\; n^{-(b_1 a_1 - d\beta)}. \end{aligned}$$

Choose $a_1 > (1 + d\beta)/b_1$ and apply the Borel–Cantelli lemma, once more, to obtain the following: with probability one,

$$\max_{x\in G_n} L(X(s) + x, C_n) < a_1^{d/\alpha} g(2^{-n}) \quad \forall n \text{ large enough.} \tag{4.8}$$

(c) For any two integers $n, k \geqslant 1$, and any $x \in G_n$, let

$$F(n,k,x) = \left\{ y \in \mathbb{R}^d \colon\ y = x + \theta_n \sum_{j=1}^{k} \varepsilon_j 2^{-j},\ \ \varepsilon \in \{0,1\}^d,\ \ 1 \leqslant j \leqslant k \right\}.$$

We select number $\gamma > 0$ first, and then, we choose $\delta > 0$ such that it satisfies

$$\frac{\delta}{\alpha}(d + \gamma) < \gamma < 1 \wedge \frac{1}{2}(N\alpha - d).$$

Consider the event B_n that is defined as

$$\begin{aligned} &\bigcup_{x\in G_n} \bigcup_{k=1}^{\infty} \bigcup_{y_1,y_2} \{ |L(X(s) + y_1, C_n) - L(X(s) + y_2, C_n)| \\ &\qquad \geqslant \Lambda \|y_1 - y_2\|^{\gamma} (a_2 2^{\delta k} \log n)^{(d+\gamma)/\alpha} \}, \end{aligned}$$

where $\Lambda = [\lambda_N(C_n)]^{1-(d+\gamma)/(N\alpha)}$, and "$\bigcup_{y_1,y_2}$", signifies the union over all $y_1, y_2 \in F(n,k,x)$, such that $y_1 - y_2 = \theta_n \varepsilon 2^{-k}$ for some $\varepsilon \in \{0,1\}^d$.

From (4.3), we see that for any constant $a_2 > 0$,

$$\mathbb{P}\{\mathsf{B}_n\} \leqslant K(2^{-n/\alpha}n^{\beta}/\theta_n)^d \sum_{k=1}^{\infty} 2^{(d+1)k} \mathrm{e}^{-b_2 a_2 2^{\delta k}\log n}$$

$$= K(\log n)^{d/\alpha} n^{-(b_2 a_2/2 - d\beta)}.$$

We have used the elementary fact that for $x > 0$ large enough,

$$\sum_{k=1}^{\infty} 2^{(d+1)k} \exp(-x 2^{\delta k}) \leqslant \mathrm{e}^{-x/2}.$$

Hence, we can choose $a_2 > 0$ large so that $\sum_n \mathbb{P}\{\mathsf{B}_n\} < \infty$. The Borel–Cantelli lemma implies that almost surely, B_n occurs only finitely often.

(d) Fix an integer n, together with some $y \in \mathbb{R}^d$ that satisfies $\|y\| < 2^{-n/\alpha}n^{\beta}$. We can represent y in the form $y = \lim_{k\to\infty} y_k$, with $y_k = x + \theta_n \sum_{j=1}^{k} \varepsilon_j 2^{-j}$, $x \in G_n$ and $\varepsilon_j \in \{0,1\}^d$. As local times are continuous in the space variable, we see from this expansion, and the triangle inequality, that, on the event $\mathsf{B}_n^{\complement}$,

$$\begin{aligned}
&|L(X(s)+y, C_n) - L(X(s)+x, C_n)| \\
&\quad \leqslant \sum_{k=1}^{\infty} |L(X(s)+y_k, C_n) - L(X(s)+y_{k-1}, C_n)| \\
&\quad \leqslant \sum_{k=1}^{\infty} [\lambda_N(C_n)]^{1-(d+\gamma)/(N\alpha)} \|y_k - y_{k-1}\|^{\gamma} (a_2 2^{\delta k}\log n)^{(d+\gamma)/\alpha} \\
&\quad \leqslant 2^{-n(N-(d+\gamma)/\alpha)} (a_2 \log n)^{(d+\gamma)/\alpha} \sum_{k=1}^{\infty} [\sqrt{d}(2^n \log n)^{-1/\alpha}]^{\gamma} 2^{-k(\gamma - \delta(d+\gamma)/\alpha)} \\
&\quad \leqslant K g(2^{-n}) \quad (y_0 = x),
\end{aligned} \tag{4.9}$$

where the finite constant K is independent of s and n.

When n is large enough, we combine (4.8) and (4.9) to get

$$\sup_{\|x\| \leqslant 2^{-n/\alpha}n^{\beta}} L(X(s)+x, C_n) \leqslant K\, g(2^{-n}).$$

That is,

$$\sup_{\|x - X(s)\| \leqslant 2^{-n/\alpha}\, n^{\beta}} L(x, C_n) \leqslant K g(2^{-n}).$$

Therefore,

$$L^*(C_n) = \sup\{L(x, C_n);\ x \in \overline{X(C_n)}\} \leqslant Kg(2^{-n}).$$

This proves Eq. (4.6), and (4.4) follows readily thereafter. □

5. Hölder laws: lower bounds

Our purpose, in this section, is to derive lower bounds for the moduli of continuity of the local times of additive stable processes. We achieve this by first establishing the following lim inf result about the oscillations of additive stable processes.

Theorem 5.1. *Let X be a Type A additive stable process of index α. Let $\tau \in \mathbb{R}_+^N$ and $T \in \mathscr{A}$ be fixed. Then, there exist finite constants $K_8, K_9 > 1$, such that, a.s.,*

$$K_8^{-1} \leqslant \liminf_{r\to 0} \sup_{t\in[\tau,\tau+\langle r\rangle]} \frac{|X(t)-X(\tau)|}{(r/\log\log r^{-1})^{1/\alpha}} \leqslant K_8, \quad \text{a.s} \tag{5.1}$$

and

$$\begin{aligned} K_9^{-1} &\leqslant \liminf_{r\to 0} \inf_{\tau\in T} \sup_{t\in[\tau,\tau+\langle r\rangle]} \frac{|X(t)-X(\tau)|}{(r/\log r^{-1})^{1/\alpha}} \\ &\leqslant \limsup_{r\to 0} \inf_{\tau\in T} \sup_{t\in[\tau,\tau+\langle r\rangle]} \frac{|X(t)-X(\tau)|}{(r/\log r^{-1})^{1/\alpha}} \leqslant K_9, \quad \text{a.s.} \end{aligned} \tag{5.2}$$

Our proof of Eq. (5.1) is based on the arguments of Taylor (1967). However, the multi-parameter nature of the process X introduces new and interesting difficulties that need to be overcome. First, we need a small ball estimate.

Lemma 5.2. *Under the conditions of Theorem* 5.1, *there exists a constant $K_{10} > 1$ such that*

$$\exp(-K_{10} r^{-\alpha}) \leqslant \mathbb{P}\left\{\sup_{t\in[0,1]^N} |X(t)| \leqslant r\right\} \leqslant \exp(-K_{10}^{-1} r^{-\alpha}).$$

Proof. Clearly,

$$\begin{aligned} \bigcap_{j=1}^N \left\{\sup_{t_j\in[0,1]} |X_j(t_j)| \leqslant rN^{-1/2}\right\} &\subseteq \left\{\sup_{t\in[0,1]^N} \|X(t)\| \leqslant r\right\} \\ &\subseteq \bigcap_{j=1}^N \left\{\sup_{t_j\in[0,1]} |X_j(t_j)| \leqslant r\right\}. \end{aligned}$$

Therefore, Lemma 5.2 follows from the corresponding result of Taylor (1967, p. 1240) for ordinary stable processes of *Type A*. □

Proof of Theorem 5.1. The lower bound in (5.1) follows from (4.4) of Theorem 4.3 and the inequality

$$\begin{aligned}\lambda_N(I) &= \int_{\mathbb{R}^d} L(x,I)\,\mathrm{d}x \\ &\leqslant L^*(I)\sup_{s,t\in I}\|X(s)-X(t)\|^d \\ &\leqslant 2^d L^*(I)\sup_{s\in I}\|X(s)-X(\tau)\|^d \qquad (5.3)\end{aligned}$$

with $I=[\tau,\tau+\langle r\rangle]$. The lower bound in (5.2) follows from (5.3) and (4.5) in a similar way. Both lower bounds can also be obtained by using Lemma 5.2 and the first part of the Borel–Cantelli lemma, with possibly different constants.

To prove the upper bound in (5.1), we assume, without loss of generality, that $\tau=0$. For $r>0$, denote

$$\mathscr{M}(r)=\sup_{t\in[0,r]^N}\|X(t)\|, \quad \text{and} \quad \phi_1(r)=(r/\log\log r^{-1})^{1/\alpha}.$$

Given

$$\eta_k=\mathrm{e}^{-k^2}, \quad k=1,2,\ldots,$$

it suffices to show that there exists a finite constant $K_8>0$ such that

$$\mathbb{P}\{\mathscr{M}(\eta_k)\leqslant K_8\phi_1(\eta_k) \text{ infinitely often}\}=1. \qquad (5.4)$$

In order to create independence, we will first replace $\mathscr{M}(\eta_k)$ by a sum of two random variables. Recall that $\Pi=\{1,2,\ldots,N\}$, and for every $A\subseteq\Pi\setminus\varnothing$, define

$$\mathscr{M}_A(\eta_k)=\sup_{\langle\eta_{k+1}\rangle\underset{(A)}{\preccurlyeq} t\preccurlyeq\langle\eta_k\rangle}\left\|\sum_{j\in A}[X_j(t_j)-X_j(\eta_{k+1})]\right\|$$

and

$$\mathscr{M}_\varnothing(\eta_k)=\sup_{t\preccurlyeq\langle\eta_{k+1}\rangle}\|X(t)\|=\mathscr{M}(\eta_{k+1}).$$

We note that

$$\mathscr{M}_\Pi(\eta_k)=\sup_{\langle\eta_{k+1}\rangle\preccurlyeq s\preccurlyeq\langle\eta_k\rangle}\|X(s)-X(\langle\eta_{k+1}\rangle)\|,$$

and thanks to the triangle inequality,

$$\mathscr{M}(\eta_k)\leqslant\mathscr{M}(\eta_{k+1})+\max_{A\subseteq\Pi,\ A\neq\varnothing}\mathscr{M}_A(\eta_k). \qquad (5.5)$$

Let $\gamma>0$ be a constant whose value will be determined later, and consider the following events:

$$D_k=\{\mathscr{M}(\eta_k)>2\gamma\phi_1(\eta_k)\},$$

$$G_k=\{\max_{A\subseteq\Pi,\ A\neq\varnothing}\mathscr{M}_A(\eta_k)>\gamma\phi_1(\eta_k)\},$$

$$H_k=\{\mathscr{M}(\eta_{k+1})>\gamma\phi_1(\eta_k)\}.$$

Then, Eq. (5.5) implies that for all $k \geqslant 1$ $D_k \subseteq G_k \cup H_k$, and for all $m \geqslant 1$

$$\bigcap_{k=m}^{2m} D_k \subseteq \left(\bigcap_{k=m}^{2m} G_k\right) \cup \left(\bigcup_{k=m}^{2m} H_k\right).$$

The point is that $G_m, \ldots, G_{2m}$ are independent events.

Let us write $p_k = \mathbb{P}\{G_k^{\complement}\}$ for brevity. By our proof of Lemma 5.2, we have

$$\begin{aligned}
p_k &= \mathbb{P}\left\{\max_{A \subseteq \Pi,\ A \neq \varnothing} \mathscr{M}_A(\eta_k) \leqslant \gamma\phi_1(\eta_k)\right\} \\
&\geqslant \mathbb{P}\left\{\sup_{\eta_{k+1} \leqslant t_j \leqslant \eta_k} |X_j(t) - X_j(\eta_{k+1})| \leqslant \frac{\gamma}{\sqrt{N}}\,\phi_1(\eta_k),\ \forall 1 \leqslant j \leqslant N\right\} \\
&= \prod_{j=1}^{N} \mathbb{P}\left\{\sup_{0 \leqslant t \leqslant 1} |X_j(t)| \leqslant \frac{\gamma\phi_1(\eta_k)}{\sqrt{N}(\eta_k - \eta_{k+1})^{1/\alpha}}\right\} \\
&\geqslant \exp(-K\gamma^{-\alpha} \log\log \eta_k^{-1}),
\end{aligned}$$

where the second inequality follows from the self-similarity and the stationarity of the increments of the processes $X_1, \ldots, X_N$, and the last inequality follows from the result of Taylor (1967, p. 1240). We now take $\gamma > 0$ so large that

$$p_k \geqslant k^{-1/2}. \tag{5.6}$$

On the other hand, Lemma 4.1 implies that

$$\mathbb{P}\{H_k\} \leqslant K \frac{\eta_{k+1}}{\eta_k} \log\log \eta_k^{-1} \leqslant \mathrm{e}^{-k} \tag{5.7}$$

for all k large enough. It follows from (5.6), (5.7) and the aforementioned independence of $G_m, \ldots, G_{2m}$ that for all m large,

$$\begin{aligned}
\mathbb{P}\left\{\bigcap_{k=m}^{2m} D_k\right\} &\leqslant \prod_{k=m}^{2m} (1 - p_k) + \sum_{k=m}^{2m} \mathrm{e}^{-k} \\
&\leqslant \mathrm{e}^{-\sum_{k=m}^{2m} p_k} + \frac{1}{1 - \mathrm{e}^{-1}}\, \mathrm{e}^{-m} \\
&\leqslant \exp(-m^{1/4}).
\end{aligned}$$

By the Borel–Cantelli lemma, we see that Eq. (5.4) holds with $K_8 = 2\gamma$, which proves the upper bound in Eq. (5.1).

Now, we prove the upper bound in (5.2). Without loss of generality, we assume $T = [0,1]^N$. Let $\gamma > 0$ be a parameter that will be determined later, and denote

$$\phi_2(r) = r^{1/\alpha} |\log r|^{-1/\alpha}.$$

For each integer $n \geqslant 1$, we divide $[0,1]^N$ into n^N subcubes, $\{C_{\mathbf{i}}\}$, of sides n^{-1}, where $\mathbf{i} = (i_1, \ldots, i_N)$ and $1 \leqslant i_j \leqslant n$, and denote the lower left vertex of the cube $C_{\mathbf{i}}$ by $\tau_{\mathbf{i}}$. In order to create independence, we will only use the subcubes whose lower left vertex lies on the diagonal $\{s = (s_1, \ldots, s_N) \in \mathbb{R}_+^N\colon s_1 = \cdots = s_N\}$.

Letting $\Gamma = \{\mathbf{i}\colon i_1 = \cdots = i_N\}$, we obtain the following from Lemma 5.2:

$$\begin{aligned}
&\mathbb{P}\left\{\min_{\mathbf{i}\in\Gamma} \sup_{t\in[0,1/n]^N} \|X(t+\tau_{\mathbf{i}}) - X(\tau_{\mathbf{i}})\| \geqslant \gamma\phi_2(1/n)\right\}\\
&\quad = \left[\mathbb{P}\left\{\sup_{t\in[0,1/n]^N} \|X(t)\| \geqslant \gamma\phi_2(1/n)\right\}\right]^n\\
&\quad \leqslant [1-\exp(-K\gamma^{-\alpha}\log n)]^n\\
&\quad =[1-n^{-K\gamma^{-\alpha}}]^n\\
&\quad \leqslant \exp(-n^{1-K\gamma^{-\alpha}}).
\end{aligned}$$

We can take $\gamma > 0$ large enough so that $1 - K\gamma^{-\alpha} > 0$, and apply the Borel–Cantelli lemma, to deduce that with probability one,

$$\limsup_{n\to\infty} \inf_{\tau\in[0,1]^N} \sup_{t\in[\tau,\tau+\langle 1/n\rangle]} \frac{|X(t)-X(\tau)|}{\phi_2(1/n)} \leqslant \gamma. \tag{5.8}$$

The upper bound in Eq. (5.2) follows from Eq. (5.8) and monotonicity. □

Combining Theorems 4.3 and 5.1 yields the following Hölder estimates on the smoothness of the local times of additive stable processes.

Theorem 5.3. *Let X denote a Type A additive stable process in $\mathbb{R}^d$ with index $\alpha \in (0,2]$, where the number of parameters $N\alpha > d$. Then, jointly continuous local times exist, and for every $\tau \in (0,\infty)^N$ and every $T \in \mathscr{A}$, there exist positive constants K_{11} and K_{12} such that*

$$K_{11}^{-1} \leqslant \limsup_{r\to 0} \frac{L^*([\tau-\langle r\rangle, \tau+\langle r\rangle])}{r^{N-d/\alpha}(\log\log r^{-1})^{d/\alpha}} \leqslant K_{11}, \quad \text{a.s.}, \tag{5.9}$$

$$K_{12}^{-1} \leqslant \limsup_{r\to 0} \sup_{\substack{I\in\mathscr{A},\ I\subset T\\ \lambda_N(I)<r}} \frac{L^*(I)}{[\lambda_N(I)]^{1-d/N\alpha}|\log\lambda_N(I)|^{d/\alpha}} \leqslant K_{12}, \quad \text{a.s.} \tag{5.10}$$

Proof. The upper bounds in Eqs. (5.9) and (5.10) are contained in Theorem 4.3, whereas the lower bounds follow from Theorem 5.1 and Eq. (5.3). □

6. Open problems

The results, and methods, of the present paper raise several open questions about the local times of additive Lévy processes and Lévy sheets. We list some of them below as concluding remarks.

Problem 6.1. Can Condition (C) be replaced by a more "geometric" condition in Theorem 3.2? The $L^k(\mathbb{P})$-norm of the local time difference induces a psuedo-norm on

Euclidean space. Thus, one might expect that there is a metric entropy improvement on Condition (C). For classical one-parameter processes, some related results can be found in Getoor and Kesten (1972), Barlow (1985, 1988), and Barlow and Hawkes (1985).

Problem 6.2. Question 6.1 leads to the deeper, but also more difficult, problem of finding a natural, necessary and sufficient condition for the joint continuity of the local times of additive Lévy processes. This seems to be outside the reach of the techniques that are known to us. When $N = 1$, a necessary and sufficient condition for joint continuity is found in Barlow (1988). While the methods of the latter reference are unlikely to be of use in the present random fields setting, those of Marcus and Rosen (1992) are quite robust, and likely to lead to interesting conclusions along these directions.

Problem 6.3. Considering in conjunction with Xiao (1997, Proposition 4.1), our Theorem 4.3, shows that for every $x \in \mathbb{R}^d$, and for all $T \in \mathscr{A}$, the Λ-Hausdorff measure of the level set, $X_T^{-1}(x)$, is positive, where $\Lambda(r) = r^{N-d/\alpha}(\log\log r^{-1})^{d/\alpha}$, and the level sets $X_T^{-1}(x)$ are defined in Eq. (2.6). Furthermore, it is not hard to show that

$$\dim_{\mathrm{H}} X_T^{-1}(x) = N - \frac{d}{\alpha}, \quad \text{a.s.},$$

where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. Thus, we are led to conjecture that Λ is the correct Hausdorff measure function for the level sets of X.

Problem 6.4. In its analysis of the local times of stable sheets, Ehm (1981) considers time points that remain strictly away from the axes of $\mathbb{R}_+^N$. In fact, very little is known about the behavior of local times of stable sheets, as we consider time points closer and closer to the axes. Even in the simplest case of Brownian sheet, the subtle behavior of the Brownian sheet near $\partial\mathbb{R}_+^N$ (cf. Talagrand, 1994) suggests the delicate behavior of the local times on time sets that intersect, or are approaching, $\partial\mathbb{R}_+^N$. For some related works in the special case of Brownian sheet (see Lacey, 1990, 2.21, p. 69; Khoshnevisan et al., 2001).

Problem 6.5. In this paper, we have not considered regularity results for the local times of additive stable processes of *Type B*. Although classical stable processes of *Type B* do not have local times, N-parameter additive stable processes of *Type B* do, in many instances where $N > 1$; cf. Theorem 2.1 for precise conditions. In these cases, what can be said about the Hölder regularity of such local times? Related questions about the fractal measures of the sample paths of additive processes of *Type B* are also open. Some results, in this direction, on the image of two-parameter additive stable subordinators are given in Hu (1994).

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