The Codimension of the Zeros of a Stable Process in Random Scenery

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Summary. We show that for any $\alpha \in (1, 2]$, the (stochastic) codimension of the zeros of an α-stable process in random scenery is identically $1 - (2\alpha)^{-1}$. As an immediate consequence, we deduce that the Hausdorff dimension of the zeros of the latter process is almost surely equal to $(2\alpha)^{-1}$. This solves Conjecture 5.2 of [6], thereby refining a computation of [10].

Keywords Random walk in random scenery; stochastic codimension; Hausdorff dimension

AMS 2000 subject classification 60K37

1 Introduction

A stable process in random scenery is the continuum limit of a class of random walks in random scenery that is described as follows. A *random scenery* on $\mathbb{Z}$ is a collection, $\{y(0), y(\pm 1), y(\pm 2), \ldots\}$, of i.i.d. mean-zero variance-one random variables. Given a collection $x = \{x_1, x_2, \ldots\}$ of i.i.d. random variables, we consider the usual random walk $n \mapsto s_n = x_1 + \cdots + x_n$ which leads to the following *random walk in random scenery*:

$$w_n = y(s_1) + \cdots + y(s_n), \qquad n = 1, 2, \cdots \tag{1}$$

In words, w is obtained by summing up the values of the scenery that are encountered by the ordinary random walk s.

Consider the *local times* $\{l_n^a;\, a \in \mathbb{Z}, n = 1, 2, \ldots\}$ of the ordinary random walk s:

$$l_n^a = \sum_{j=1}^{n} \mathbf{1}_{\{a\}}(s_j), \qquad a \in \mathbb{Z},\, n = 1, 2, \ldots$$

Then, one readily sees from (1) that

$$w_n = \sum_{a \in \mathbb{Z}} l_n^a y(a), \qquad n = 1, 2, \ldots$$

* Research partially supported by grants from the NSF and NATO

As soon as s is in the domain of attraction of a stable process of index $\alpha \in (1,2]$, one might expect its local times to approximate those of the limiting stable process. Thus, one may surmise an explicit weak limit for a renormalization of w. Reference [4] has shown that this is the case. Indeed, let $S = \{S(t);\, t \geq 0\}$ denote a stable Lévy process with Lévy exponent

$$\mathbb{E}[\exp\{i\xi S(1)\}] = \exp\left(-|\xi|^{\alpha}\frac{1+i\nu\mathrm{sgn}(\xi)\tan(\alpha\pi/2)}{\chi}\right), \qquad \xi \in \mathbb{R}, \tag{2}$$

where $\nu \in [-1,1]$ and $\chi > 0$. If $\alpha \in (1,2]$, then it is well known ([1]) that S has continuous local times; i.e., there exists a continuous process $(x,t) \mapsto L_t(x)$ such that for all Borel measurable functions $f : \mathbb{R} \to \mathbb{R}$, and all $t \geq 0$,

$$\int_0^t f(S(u))\,du = \int_{-\infty}^{\infty} f(a) L_t(a)\,\mathrm{d}a. \tag{3}$$

Then, according to [4], as long as s is in the domain of attraction of S, the random walk in random scenery w can be normalized to converge weakly to the *stable process in random scenery* W defined by

$$W(t) = \int_{-\infty}^{\infty} L_t(x)\, B(\mathrm{d}x). \tag{4}$$

Here $B = \{B(t);\, -\infty < t < +\infty\}$ is a two-sided Brownian motion that is totally independent of the process S, and the stochastic integral above is defined in the sense of N. Wiener or, more generally, K. Itô.

References [5, 6] have established a weak notion of duality between iterated Brownian motion (i.e., $B \circ B'$, where B' is an independent Brownian motion) and Brownian motion in random scenery (i.e., the process W when $\alpha = 2$). Since the level sets of iterated Brownian motion have Hausdorff dimension $\frac{3}{4}$ ([2]), this duality suggests that when $\alpha = 2$ the level sets of W ought to have Hausdorff dimension $\frac{1}{4}$; cf. [6, Conjecture 5.2]. Reference [10] has shown that a randomized version of this assertion is true: For the $\alpha = 2$ case, and for any $t > 0$,

$$\mathbb{P}\left\{ \dim(W^{-1}\{W(t)\}) = \tfrac{1}{4} \right\} = 1,$$

where $W^{-1}A = \{s \geq 0 : W(s) \in A\}$ for any Borel set $A \subset \mathbb{R}$. In particular, Lebesgue-almost all level sets of W have Hausdorff dimension $\frac{1}{4}$ when $\alpha = 2$.

In this note, we propose to show that the preceeding conjecture is true for all level sets, and has an extension for all $\alpha \in (1,2]$. Indeed, we offer the following stronger theorem whose terminology will be explained shortly.

Theorem 1. *For any $x \in \mathbb{R}$,*

$$\operatorname{codim}\left(W^{-1}\{x\}\right) = 1 - \frac{1}{2\alpha}. \tag{5}$$

Consequently, if dim *represents Hausdorff dimension, then*

$$\dim\left(W^{-1}\{x\}\right) = \frac{1}{2\alpha}, \qquad \textit{almost surely.} \tag{6}$$

To conclude the introduction, we will define stochastic codimension, following the treatment of [8].

Given a random subset, K, of $\mathbb{R}_+$, we can define the *lower* (*upper*) *stochastic codimension* of K as the largest (smallest) number β such that for all compact sets $F \subset \mathbb{R}_+$ whose Hausdorff dimension is strictly less (greater) than β, K cannot (can) intersect F. We write $\underline{\operatorname{codim}}\,(K)$ and $\overline{\operatorname{codim}}\,(K)$ for the lower and the upper stochastic codimensions of K, respectively. When they agree, we write $\operatorname{codim}(K)$ for their common value, and call it the (stochastic) *codimension* of K. Note that the upper and the lower stochastic codimensions of K are not random, although K is a random set.

2 Supporting Lemmas

We recall from [8, Theorem 2.2] and its proof that when a random set $X \subseteq \mathbb{R}$ has a stochastic codimension,

$$\operatorname{codim} X + \dim X = 1, \qquad \mathbb{P}\text{-a.s.}$$

This shows that (5) implies (6). Thus, we will only verify (5). Throughout, P (E) denote the conditional probability measure (expectation) $\mathbb{P}$ ($\mathbb{E}$), given the entire process S.

With the above notation in mind, it should be recognized that, under the measure P, the process W is a centered Gaussian process with covariance

$$E\{W(s)W(t)\} = \langle L_s, L_t\rangle, \qquad s, t \geq 0, \tag{7}$$

where $\langle \bullet, \bullet\rangle$ denotes the usual $\mathcal{L}^2(\mathbb{R})$-inner product. Needless to say, the above equality holds with $\mathbb{P}$-probability one. In particular, $\mathbb{P}$-a.s., the P-variance of $W(t)$ is $\|L_t\|_2^2$, where $\|\bullet\|_r$ denotes the usual $\mathcal{L}^r(\mathbb{R})$-norm for any $1 \leq r \leq \infty$.

By the Cauchy–Schwarz inequality, $\langle f, g\rangle^2 \leq \|f\|_2^2 \cdot \|g\|_2^2$. We need the following elementary estimate for the slack in this inequality. It will translate to a P-correlation estimate for the process W.

Lemma 1. *For all $f, g \in \mathcal{L}^1(\mathbb{R}) \cap \mathcal{L}^\infty(\mathbb{R})$,*

$$\|f\|_2^2\|g\|_2^2 - \langle f, g\rangle^2 \geq \|g\|_2^2\|f-g\|_2^2 - \|g\|_\infty^2\|f-g\|_1^2.$$

Proof. One can check the following variant of the parallelogram law on $\mathcal{L}^2(\mathbb{R})$:

$$\|f\|_2^2\|g\|_2^2 - \langle f, g\rangle^2 = \|f-g\|_2^2\|g\|_2^2 - \langle f-g, g\rangle^2,$$

from which the lemma follows immediately.

Now, consider the random field

$$\varrho_{s,t} = \frac{\langle L_s, L_t\rangle}{\|L_s\|_2^2}, \qquad s,t \geq 0. \tag{8}$$

Under the measure P, $\{\varrho_{s,t};\ s,t\geq 0\}$ can be thought of as a collection of constants. Then, one has the following conditional regression bound:

Lemma 2 (Conditional Regression). *Fix* $1 \leq s < t \leq 2$*. Then, under the measure* P*,* $W(s)$ *is independent of* $W(t) - \varrho_{s,t}W(s)$*. Moreover,* $\mathbb{P}$*-a.s.,*

$$E\left\{|W(t) - \varrho_{s,t}W(s)|^2\right\} \geq \left(\|L_t - L_s\|_2^2 - \frac{\|L_2\|_\infty^2}{\|L_1\|_2^2}|t-s|^2\right)_+. \tag{9}$$

Proof. The independence assertion is an elementary result in linear regression. Indeed, it follows from the conditional Gaussian distribution of the process W, together with the following consequence of (7):

$$E\left\{W(s)\left[W(t) - \varrho_{s,t}W(s)\right]\right\} = 0, \qquad \mathbb{P}\text{-a.s.}$$

Similarly, (conditional) regression shows that $\mathbb{P}$-a.s.,

$$E\left\{\left[W(t) - \varrho_{s,t}W(s)\right]^2\right\} = \frac{\|L_t\|_2^2\|L_s\|_2^2 - \langle L_s, L_t\rangle^2}{\|L_s\|_2^2}, \tag{10}$$

$\mathbb{P}$-a.s. Thanks to Lemma 1, the numerator is bounded below by

$$\|L_s\|_2^2\|L_t - L_s\|_2^2 - \|L_s\|_\infty^2\|L_t - L_s\|_1^2.$$

By the occupation density formula (3), with $\mathbb{P}$-probability one, $\|L_t - L_s\|_1 = (t-s)$. Since $r \mapsto L_r(x)$ is non-increasing for any $x \in \mathbb{R}$, the lemma follows from (10).

Now, we work toward showing that the right hand side of (9) is essentially equal to the much simpler expression $\|L_t - L_s\|_2^2$. This will be done in a few steps.

Lemma 3. *If* $0 \leq s < t$ *are fixed, then the* $\mathbb{P}$*-distribution of* $\|L_t - L_s\|_2^2$ *is the same as that of* $(t-s)^{2-(1/\alpha)}\|L_1\|_2^2$.

Proof. Since the stable process S is Lévy, by applying the Markov property at time t, we see that the process $L_t(\cdot) - L_s(\cdot)$ has the same finite dimensional distributions as $L_{t-s}(\cdot)$. The remainder of this lemma follows from scaling; see [7, 5.4], for instance.

Next, we introduce a somewhat rough estimate of the modulus of continuity of the infinite-dimensional process $t \mapsto L_t$.

Lemma 4. *For each $\eta > 0$, there exists a $\mathbb{P}$-a.s. finite random variable V_4 such that for all $0 \le s < t \le 2$,*

$$\|L_t - L_s\|_2^2 \le V_4 |t-s|^{2-(1/\alpha)-\eta}.$$

Proof. Thanks to Lemma 3, for any $\nu > 1$, and for all $0 \le s < t$,

$$\mathbb{E}\{\|L_t - L_s\|_2^{2\nu}\} = (t-s)^{2\nu-(\nu/\alpha)} \mathbb{E}\{\|L_1\|_2^{2\nu}\}.$$

On the other hand, by the occupation density formula (3),

$$\|L_1\|_2^2 \le \|L_1\|_\infty \int_{-\infty}^{\infty} L_1(x)\, \mathrm{d}x = \|L_1\|_\infty.$$

According to [9, Theorem 1.4], there exists a finite $c > 0$ such that

$$\mathbb{P}\{\|L_1\|_\infty > \lambda\} \le \exp(-c\lambda^\alpha), \qquad \forall \lambda > 1.$$

The result follows from the preceeding two displays, used in conjunction with Kolmogorov's continuity criterion applied to the $\mathcal{L}^2(\mathbb{R})$-valued process $t \mapsto L_t$.

Up to an infinitesimal in the exponent, the above is sharp, as the following asserts.

Lemma 5. *For each $\eta > 0$, there exists a $\mathbb{P}$-a.s. finite random variable V_5 such that for all $1 \le s < t \le 2$,*

$$\|L_t - L_s\|_2^2 \ge V_5 |t-s|^{2-(1/\alpha)+\eta}.$$

Proof. According to [7, proof of Lemma 5.4], there exists a finite constant $c > 0$ such that for all $\lambda \in (0,1)$,

$$\mathbb{P}\{\|L_1\|_2^2 \le \lambda\} \le \exp(-c\lambda^{-\alpha}). \tag{11}$$

Combined with Lemma (3), this yields

$$\mathbb{P}\left\{\|L_{s+h} - L_s\|_2^2 \le h^{2-(1/\alpha)+\eta}\right\} \le \exp(-ch^{-\eta}), \qquad s \in [1,2],\, h \in (0,1).$$

Let

$$F_n = \{k2^{-n};\, 0 \le k \le 2^{n+1}\}, \qquad n = 0, 1, \ldots$$

Choose and fix some number $p > \eta^{-1}$ to see that

$$\mathbb{P}\left\{\min_{s \in F_n} \|L_{s+n^{-p}} - L_s\|_2^2 \le n^{-p\gamma}\right\} \le (2^{n+1}+1)\exp(-cn^{\eta p}),$$

where $\gamma = 2 - (1/\alpha) + \eta$. Since $p > \eta^{-1}$, the above probability sums in n. By the Borel–Cantelli lemma, $\mathbb{P}$-almost surely,

$$\min_{s \in F_n} \|L_{s+n^{-p}} - L_s\|_2^2 \geq n^{-p\gamma}, \quad \text{eventually.} \tag{12}$$

On the other hand, any for any $s \in [1,2]$, there exists $s' \in F_n$ such that $|s - s'| \leq 2^{-n}$. In particular,

$$\inf_{s \in [1,2]} \|L_{s+n^{-p}} - L_s\|_2^2 \geq \min_{s \in F_n} \|L_{s+n^{-p}} - L_s\|_2^2 - 4 \sup_{\substack{0 \leq u,v \leq 2 \\ |u-v| \leq 2^{-n}}} \|L_u - L_v\|_2^2.$$

We have used the inequality $|x+y|^2 \leq 2(x^2 + y^2)$ to obtain the above. Thus, by Lemma 4, and by (12), $\mathbb{P}$-almost surely,

$$\inf_{s \in [1,2]} \|L_{s+n^{-p}} - L_s\|_2^2 \geq (1 + o(1)) n^{-p\gamma}, \quad \text{eventually.}$$

Since $t \mapsto L_t(x)$ is increasing, the preceeding display implies the lemma.

3 Proof of Theorem 1

Not surprisingly, we prove Theorem 1 in two steps: First, we obtain a lower bound for codim $(W^{-1}\{x\})$. Then, we establish a corresponding upper bound.

In order to simplify the notation, we only work with the case $x = 0$; the general case follows by the very same methods.

3.1 The Lower Bound

The lower bound is quite simple, and follows readily from Lemma 4 and the following general result.

Lemma 6. *If $\{Z(t);\, t \in [1,2]\}$ is almost surely Hölder continuous of some nonrandom order $\gamma > 0$, and if $Z(t)$ has a bounded density function uniformly for every $t \in [1,2]$, then*

$$\underline{\text{codim}}\,(Z^{-1}\{0\}) \geq \gamma.$$

Proof. If $F \subset \mathbb{R}_+$ is a compact set whose Hausdorff dimension is $< \gamma$, then we are to show that almost surely, $Z^{-1}\{0\} \cap F = \varnothing$.

By the definition of Hausdorff dimension, and since $\dim(F) < \gamma$, for any $\delta > 0$ we can find closed intervals $F_1, F_2, \ldots$ such that *(i)* $F \subseteq \cup_{i=1}^{\infty} F_i$; and *(ii)* $\sum_{i=1}^{\infty} (\text{diam} F_i)^{\gamma} \leq \delta$. Let s_i denote the left endpoint of F_i, and observe that whenever $Z^{-1}\{0\} \cap F_j \neq \varnothing$, then with $\mathbb{P}$-probability one,

$$|Z(s_j)| \leq \sup_{s,t \in F_j} |Z(s) - Z(t)| \leq K_{\gamma} (\text{diam} F_j)^{\gamma},$$

where K_γ is an almost surely finite random variable that signifies the Hölder constant of Z. In particular, for any $M > 0$,

$$\begin{aligned}\mathbb{P}\big\{Z^{-1}\{0\} \cap F \neq \varnothing\big\} &\leq \sum_{j=1}^{\infty} \mathbb{P}\big\{|Z(s_j)| \leq M(\mathrm{diam} F_j)^\gamma\big\} + \mathbb{P}\{K_\gamma > M\} \\ &\leq 2DM \sum_{j=1}^{\infty} (\mathrm{diam} F_j)^\gamma + \mathbb{P}\{K_\gamma > M\},\end{aligned}$$

where D is the uniform bound on the density function of $Z(t)$, as t varies in $[1, 2]$. Consequently,

$$\mathbb{P}\big\{Z^{-1}\{0\} \cap F \neq \varnothing\big\} \leq 2DM\delta + \mathbb{P}\{K_\gamma > M\}.$$

Since δ is arbitrary,

$$\mathbb{P}\big\{Z^{-1}\{0\} \cap F \neq \varnothing\big\} \leq \mathbb{P}\{K_\gamma > M\},$$

which goes to zero as $M \to \infty$.

We can now turn to our

Proof (Theorem 1: Lower Bound). Since W is Gaussian under the measure P, for any $\nu > 0$, there exists a nonrandom and finite constant $C_\nu > 0$ such that for all $0 \leq s \leq t \leq 2$,

$$\begin{aligned}E\big\{|W(s) - W(t)|^\nu\big\} &= C_\nu \Big(E\big\{|W(s) - W(t)|^2\big\}\Big)^{\nu/2} \\ &= C_\nu \|L_t - L_s\|_2^\nu.\end{aligned}$$

Taking $\mathbb{P}$-expectations and appealing to Lemma 3 leads to

$$\mathbb{E}\big\{|W(s) - W(t)|^\nu\big\} = C'_\nu (t - s)^{\nu - (\nu/2\alpha)},$$

where $C'_\nu = C_\nu \mathbb{E}\{\|L_1\|_2^\nu\}$ is finite, thanks to [9, Theorem 1.4]. By Kolmogorov's continuity theorem, with probability one, $t \mapsto W(t)$ is Hölder continuous of any order $\gamma < 1 - (2\alpha)^{-1}$. We propose to show that the density function of $W(t)$ is bounded uniformly for all $t \in [1, 2]$. Lemma 6 would then show that $\mathrm{codim}\,(W^{-1}\{0\} \cap [1, 2]) \geq \gamma$ for any $\gamma < 1 - (2\alpha)^{-1}$; i.e., $\mathrm{codim}\,(W^{-1}\{0\} \cap [1, 2]) \geq 1 - (2\alpha)^{-1}$. The argument to show this readily implies that $\mathrm{codim}\,(W^{-1}\{0\}) \geq 1 - (2\alpha)^{-1}$, which is the desired lower bound.

To prove the uniform boundedness assertion on the density function, f_t, of $W(t)$, we condition first on the entire process S to obtain

$$f_t(x) = \frac{1}{\sqrt{2\pi}}\, \mathbb{E}\left[\frac{1}{\|L_t\|_2} \exp\left(-\frac{x^2}{2\|L_t\|_2^2}\right)\right], \qquad t \in [1, 2],\ x \in \mathbb{R}.$$

In particular,

$$\sup_{t \in [1,2]} \sup_{x \in \mathbb{R}} f_t(x) \leq \mathbb{E}\{\|L_1\|_2^{-1}\},$$

which is finite, thanks to (11).

3.2 The Upper Bound

We intend to show that for any $x \in \mathbb{R}$, and for any compact set $F \subset \mathbb{R}_+$ whose Hausdorff dimension is $> 1 - (2\alpha)^{-1}$, $\mathbb{P}\{W^{-1}\{x\} \cap F \neq \varnothing\} > 0$. It suffices to show that for all such F's,

$$P\{W^{-1}\{x\} \cap F \neq \varnothing\} > 0, \quad \mathbb{P}\text{-a.s.}$$

As in our lower bound argument, we do this merely for $x = 0$ and $F \subseteq [1, 2]$, since the general case is not much different. Henceforth, we shall fix one such compact set F without further mention.

Let $\mathcal{P}(F)$ denote the collection of probability measures on F, and for all $\mu \in \mathcal{P}(F)$ and all $\varepsilon > 0$, define

$$J_\varepsilon(\mu) = \frac{1}{2\varepsilon} \int \mathbf{1}_{\{|W(s)| \leq \varepsilon\}}\, \mu(\mathrm{d}s). \tag{13}$$

We proceed to estimate the first two moments of $J_\varepsilon(\mu)$.

Lemma 7. *There exists a $\mathbb{P}$-a.s. finite and positive random variable V_7 such that $\mathbb{P}$-almost surely,*

$$\liminf_{\varepsilon \to 0} E\{J_\varepsilon(\mu)\} \geq V_7,$$

for any $\mu \in \mathcal{P}(F)$.

Proof. Notice the explicit calculation:

$$E\{J_\varepsilon(\mu)\} = \frac{1}{2\sqrt{2\pi}\varepsilon} \int_F \int_{-\varepsilon}^{+\varepsilon} \|L_s\|_2^{-1} \exp\left(-\frac{x^2}{2\|L_s\|_2^2}\right) \mathrm{d}x\, \mu(\mathrm{d}s),$$

valid for all $\varepsilon > 0$ and all $\mu \in \mathcal{P}(F)$. Since $F \subseteq [1, 2]$, the monotonicity of local times shows that

$$E\{J_\varepsilon(\mu)\} \geq \frac{1}{\sqrt{2\pi}} \|L_2\|_2^{-1} \exp\left(-\frac{\varepsilon^2}{2\|L_1\|_2^2}\right).$$

The lemma follows with $V_7 = (2\pi)^{-\frac{1}{2}} \|L_2\|_2^{-1}$, which is $\mathbb{P}$-almost surely (strictly) positive, thanks to (11).

Lemma 8. *For any $\eta > 0$, there exists a $\mathbb{P}$-a.s. positive and finite random variable V_8 such that for all $\mu \in \mathcal{P}(F)$,*

$$\sup_{\varepsilon \in (0,1)} E\left\{|J_\varepsilon(\mu)|^2\right\} \leq V_8 \iint |s - t|^{-1+(1/2\alpha)-\eta}\, \mu(\mathrm{d}s)\, \mu(\mathrm{d}t), \qquad \mathbb{P}\text{-a.s.}$$

Proof. We recall $\varrho_{s,t}$ from (8), and observe that for any $1 \leq s < t \leq 2$, and for all $\varepsilon > 0$,

$$P\{|W(s)| \le \varepsilon\ ,\ |W(t)| \le \varepsilon\} = P\{|W(s)| \le \varepsilon\ ,\ |W(t) - \varrho_{s,t}W(s) + \varrho_{s,t}W(s)| \le \varepsilon\}$$
$$\le P\{|W(s)| \le \varepsilon\} \times \sup_{x\in\mathbb{R}} P\{|W(t) - \varrho_{s,t}W(s) + x| \le \varepsilon\},$$

since $W(s)$ and $W(t) - \varrho_{s,t}W(s)$ are P-independent; cf. Lemma 2. On the other hand, centered Gaussian laws are unimodal. Hence, the above supremum is achieved at $x = 0$. That is,

$$P\{|W(s)| \le \varepsilon\ ,\ |W(t)| \le \varepsilon\} \le P\{|W(s)| \le \varepsilon\} \times P\{|W(t) - \varrho_{s,t}W(s)| \le \varepsilon\}.$$

Computing explicitly, we obtain

$$\sup_{s\in[1,2]} P\{|W(s)| \le \varepsilon\} \le \varepsilon \|L_1\|_2^{-1}. \tag{14}$$

Likewise,

$$P\{|W(t) - \varrho_{s,t}W(s)| \le \varepsilon\} \le \frac{\varepsilon}{\sqrt{E\{|W(t) - \varrho_{s,t}W(s)|^2\}}}, \quad \mathbb{P}\text{-a.s.}$$

We can combine (14) with conditional regression (Lemma 2) and Lemma 5, after a few lines of elementary calculations.

Proof (Theorem 1: Upper Bound). Given a compact set $F \subset [1,2]$ with $\dim(F) > 1 - (2\alpha)^{-1}$, we are to show that $P\{W^{-1}\{0\} \cap F \neq \varnothing\} > 0$, $\mathbb{P}$-almost surely. But, for any $\mu \in \mathcal{P}(F)$, the following holds $\mathbb{P}$-a.s.:

$$P\{W^{-1}\{0\} \cap F \neq \varnothing\} \ge \liminf_{\varepsilon\to 0} P\{J_\varepsilon(\mu) > 0\}$$
$$\ge \liminf_{\varepsilon\to 0} \frac{\left|E\{J_\varepsilon(\mu)\}\right|^2}{E\{|J_\varepsilon(\mu)|^2\}},$$

thanks to the classical Paley–Zygmund inequality ([3, p. 8]). Lemmas 7 and 8, together imply that for any $\eta > 0$, $\mathbb{P}$-almost surely,

$$P\{W^{-1}\{0\} \cap F \neq \varnothing\} \ge \frac{V_7^2}{V_8 \cdot \inf_{\mu\in\mathcal{P}(F)} \iint |s-t|^{-1+(1/2\alpha)-\eta}\, \mu(\mathrm{d}s)\, \mu(\mathrm{d}t)}.$$

Note that the random variable V_8 depends on the value of $\eta > 0$. Now, choose η such that $\dim(F) > 1 - (2\alpha)^{-1} + \eta$, and apply Frostman's theorem ([3, p. 130]) to deduce that

$$\inf_{\mu\in\mathcal{P}(F)} \iint |s-t|^{-1+(1/2\alpha)-\eta}\, \mu(\mathrm{d}s)\, \mu(\mathrm{d}t) < +\infty.$$

This concludes our proof.

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