Vector Calculus Review

It may have been a while since you have played around with Vector Calculus and transport equations , this lecture will hopefully serve to jog your memory a bit!

Vectors

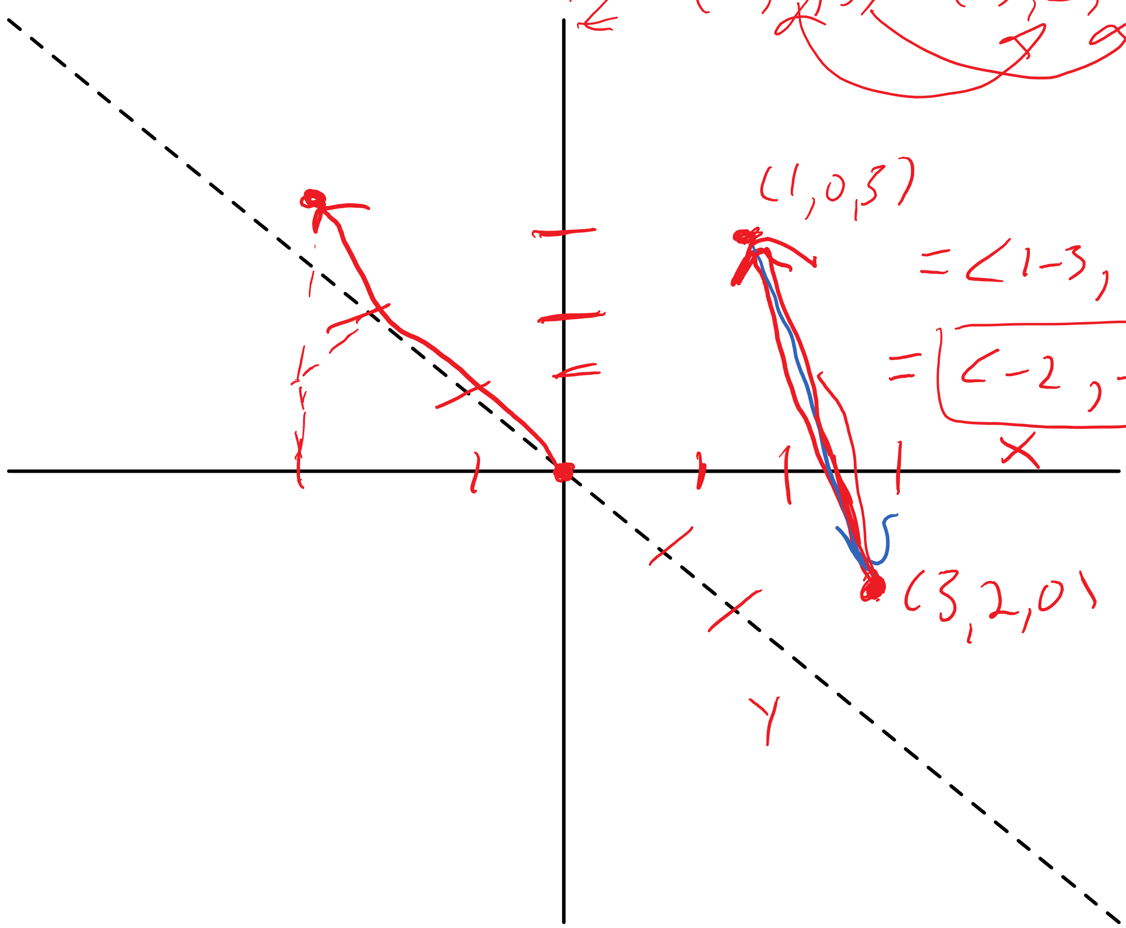
Recall that a vector is a kind of number that has both magnitude *and* direction.   This kind of number is particularly useful in science. For example a train can travel at 90 MPH but if we don't know which direction , say east or west, it is moving we won't know where it ends up.

The speed of the train would be the magnitude of the velocity vector and east or west it's direction

Vectors are defined in terms of components, one in each direction of space. A vector between two points can be found by taking the "tip" minus the "tail"

Let's find and draw the vector between the points (1,0,3) and (3,2,0)

$(1,0,3)-(3,2,0)$



$= \langle 1-3, 0-2, 3-0 \rangle$

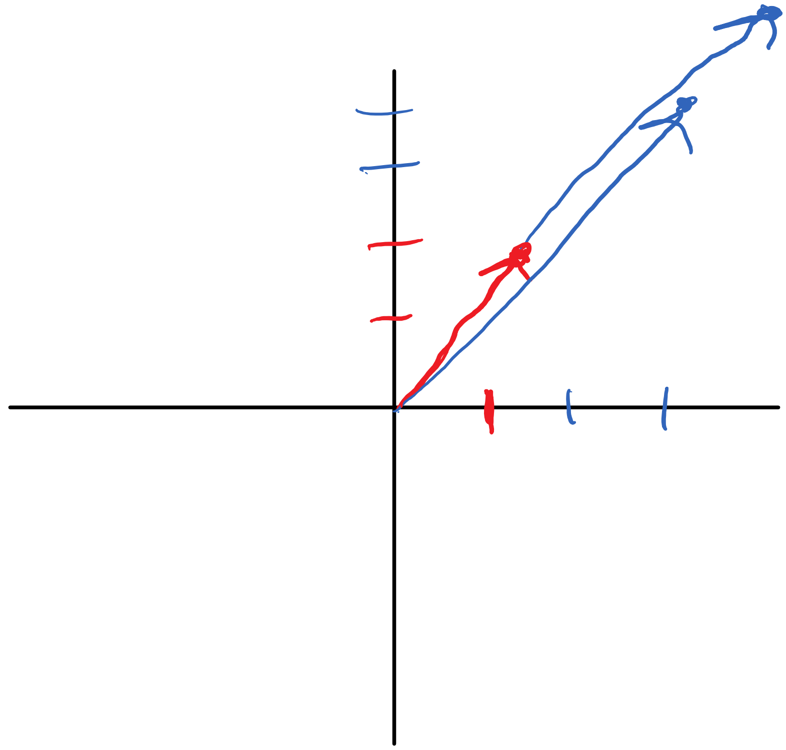
$= \boxed{\langle -2, -2, 3 \rangle}$

Vector Calculus Review

Vector Addition and Multiplication.

We add vectors component wise:

Example add $\vec{u} = <1,2>$ and $\vec{v} = <3,4>$



$$\vec{u} + \vec{v} = <1,2> + <3,4>$$

$$<1+3, 2+4>$$

$$<4, 6>$$

One way to multiply vectors is using the dot product. It produces a scalar from two vectors.

Dot product:
$\vec{u} \cdot \vec{v} = u_1v_1 + u_2v_2 + u_3v_3 + \cdots u_nv_n$ where n is the number of dimesions. Let's do a 3d example

Let $\vec{u} = <1,2,3>$ and $\vec{v} = <4,5,6>$ find the dot product.

$$\vec{u} \cdot \vec{v} = <1,2,3> \cdot <4,5,6>$$

$$1(4) + 2(5) + 3(6)$$

$$4 + 10 + 18 = \boxed{32}$$

Vector Calculus Review

Summary of some useful vector facts.

1. The vector dot product in 2D:

$$\vec{a}\cdot\vec{b} = \langle a_1, a_2\rangle \cdot \langle b_1, b_2\rangle = a_1b_1 + a_2b_2 = ||\vec{a}||\,||\vec{b}||\cos(\theta)$$

2. In n-dimensions the dot product is

$$\vec{a}\cdot\vec{b} = \sum_{i=1}^{n} a_ib_i = \boxed{||\vec{a}||\,||\vec{b}||\cos(\theta)}$$

3. Vector magnitude $||\vec{a}|| = \sqrt{\sum_{i=1}^{n} a_i^2} = \sqrt{\vec{a}\cdot\vec{a}}$

$u = \langle 1, 2, 3\rangle$

$||u|| = \sqrt{1^2 + 2^2 + 3^2}$

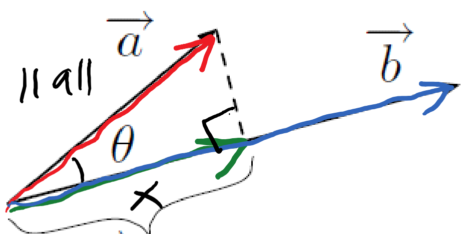
$\sqrt{1+4+9}$

$\sqrt{14}$

4. A scalar projection of one vector $\vec{a}$ onto another $\vec{b}$ is termed

$$\boxed{comp(\vec{a})_{\vec{b}} = \frac{\vec{b}\cdot\vec{a}}{||\vec{b}||}}$$

which measures the length of vector $\vec{a}$ that is pointing in the direction of vector $\vec{b}$.



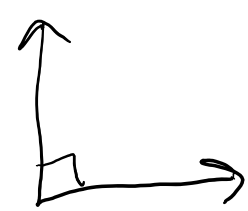
$$\frac{\vec{b}\cdot\vec{a}}{||\vec{b}||}$$

$\cos\theta = \dfrac{x}{||a||} \Rightarrow x = ||a||\cos\theta$

$\dfrac{\vec{a}\cdot\vec{b}}{||b||} = \dfrac{||a||\,||b||\cos\theta}{||b||}$

$\dfrac{\vec{a}\cdot\vec{b}}{||b||} = \overbrace{||a||\cos\theta}^{x}$

5. Two vectors are said to be orthogonal if $\vec{a}\cdot\vec{b} = 0$, implying $\theta = \pi/2$.



$\vec{a}\cdot\vec{b} = ||a||\,||b||\cos\theta = 0$

$\theta = \pi/2$

$\cos\theta = 0$

Vector Calculus Review

We can also have functions which are vectors. We call them vector functions. They look like

$\vec{F}(x,y,z) = < f_1(x,y,z), f_2(x,y,z), f_3(x,y,z) >$ $= f_1 \hat{i} + f_2 \hat{j} + f_3 \hat{k}$

Example:

Let $\vec{F}(x,y,z) = < x+y, y, z-x >$ draw $\vec{F}$ when (x,y,z)=(1,1,1) and (1,2,3)

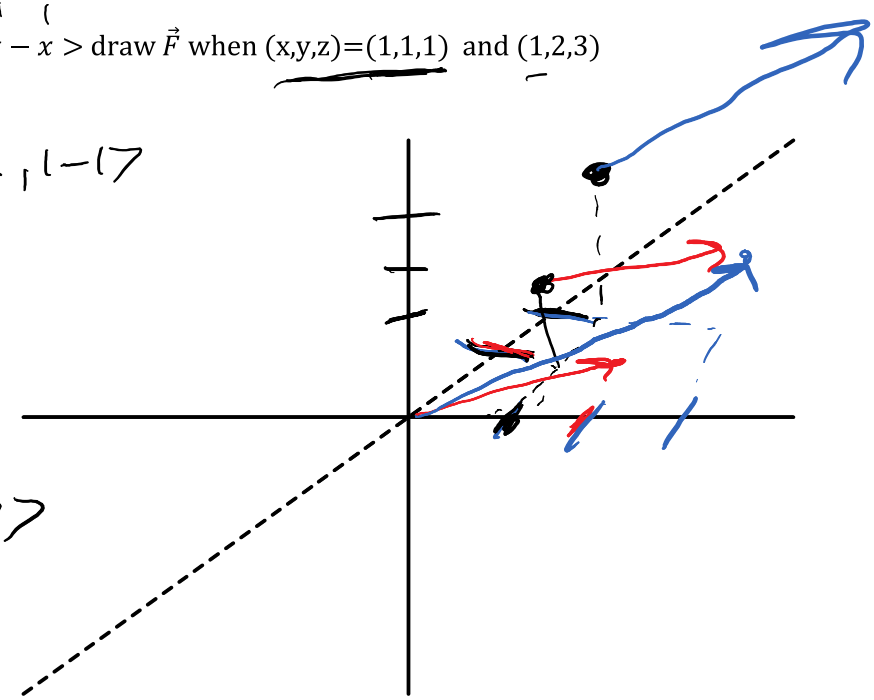
$\vec{F}(1,1,1) = <1+1, 1, 1-1>$

$= <2, 1, 0>$

$\vec{F}(1,2,3)$

$<1+2, 2, 3-1>$

$<3, 2, 2>$



A vector function can also depend only on time. Let $\vec{F}(t) = < t, t^2 >$ Sketch the graph.

$t = 0$

$<0,0>$

$t = 1$

$<1, 1^2>$

$<1,1>$

$t = 2$

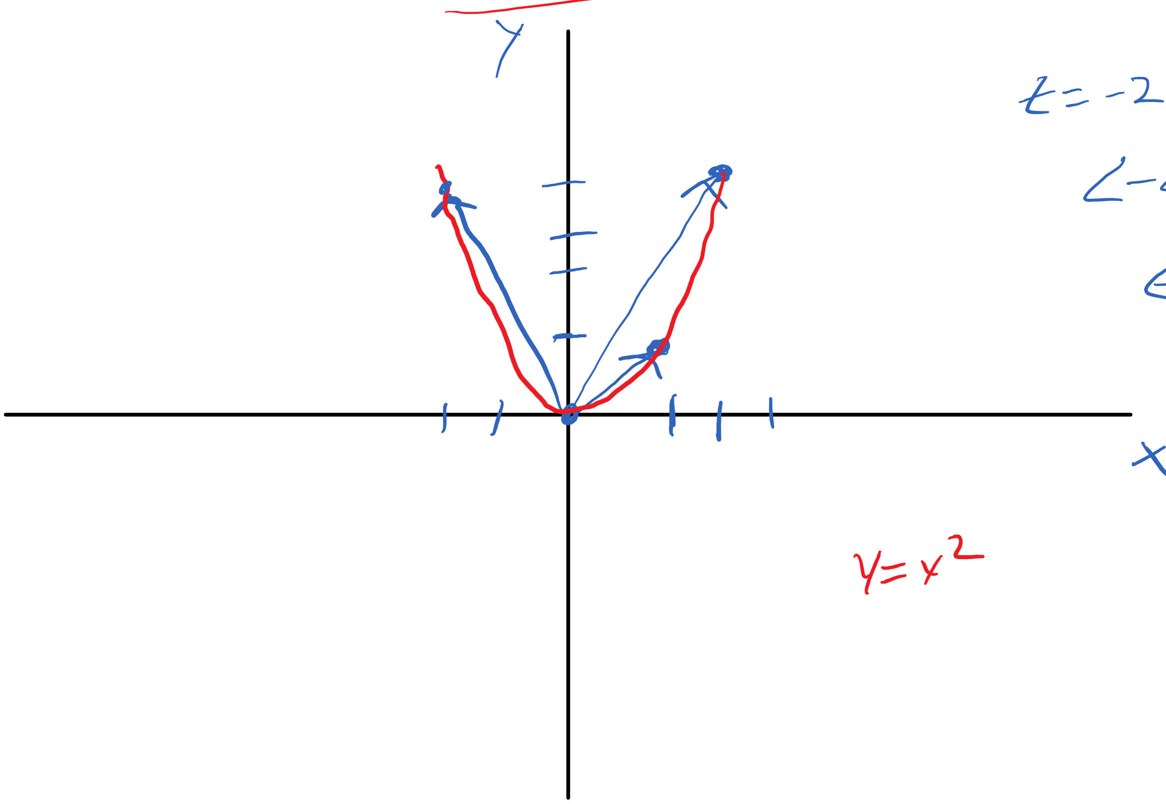
$<2, 2^2>$

$<2, 4>$

$t = -2$

$<-2, (-2)^2>$

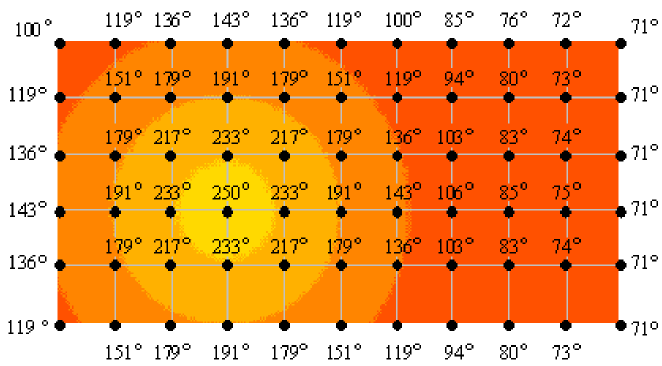
$<-2, 4>$

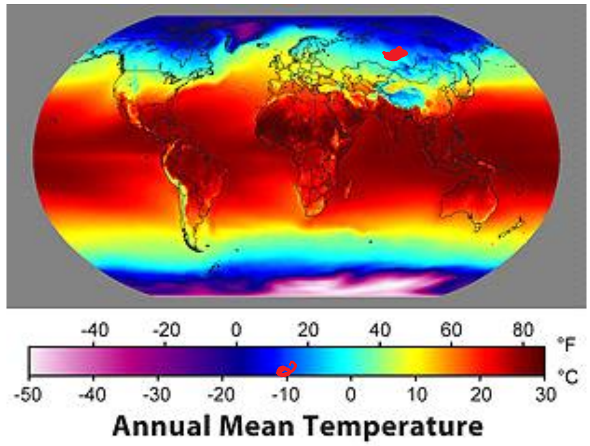


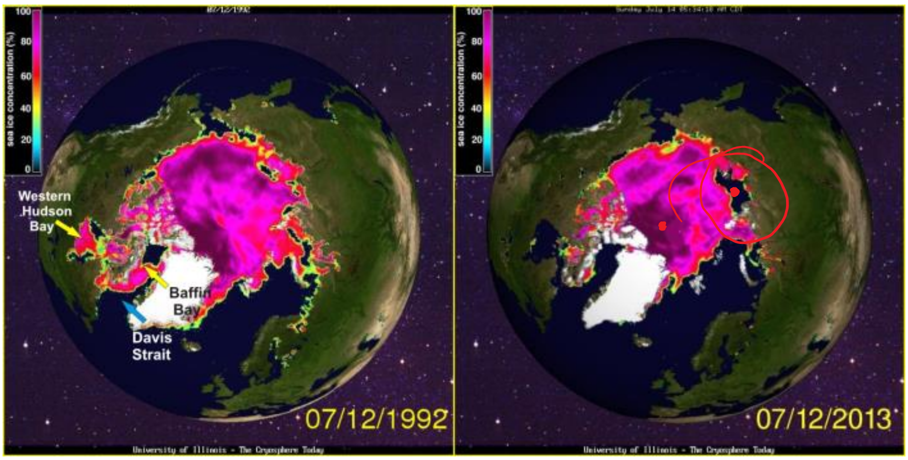
Vector Calculus Review

Fields. What is a field? We will deal with two types main types of fields in this course, scalar fields and vector fields.

Scalar field: A scalar field comes from a function which outputs a scalar value at a given point in space and time. The field itself can be thought of those scalars sitting at their respective points is space. For example the temperature in a room can be considered a scalar field, at each position in space in the room we associate a scalar number that represents the temperature of the room.



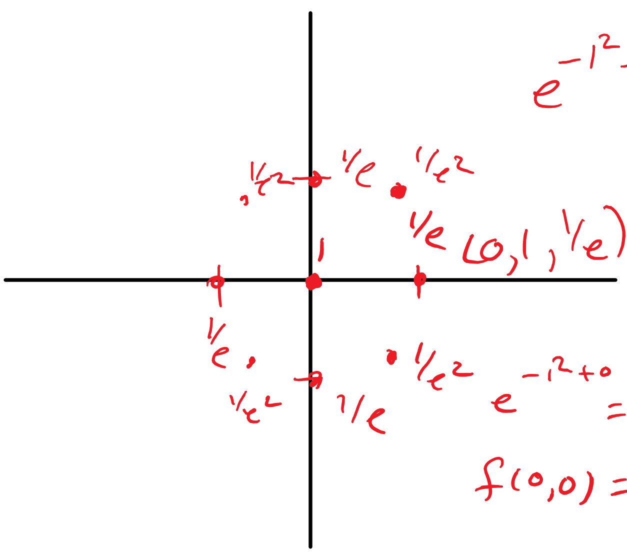




Example: Draw the scalar field defined by the function $f(x,y) = e^{-x^2-y^2}$

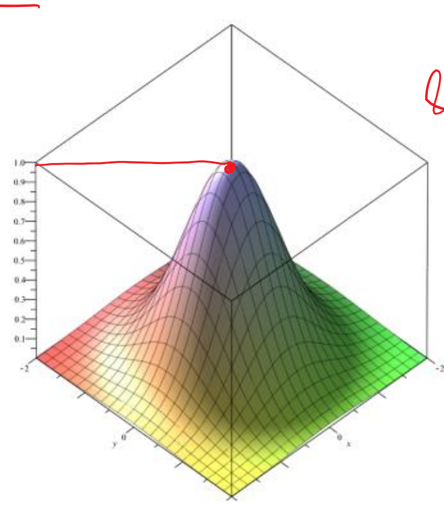
$e^{-1^2-1^2} = e^{-2}$

$1/e^2$



$e^{-1^2+0} = e^{-1} = 1/e$

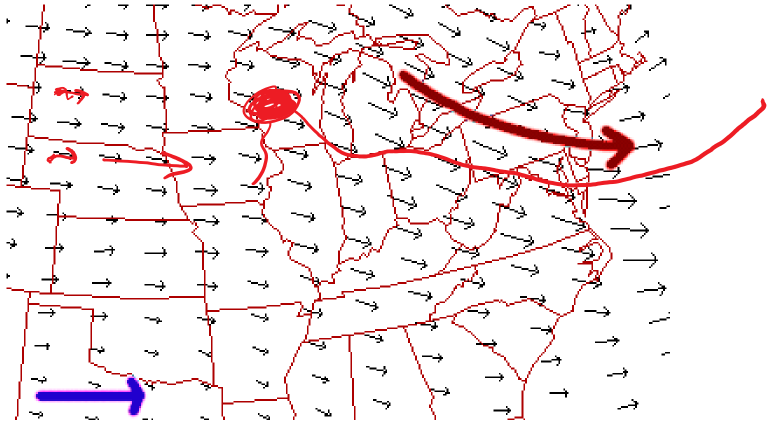
$f(0,0) = e^{-0^2-0^2}$
$= e^0 = 1$

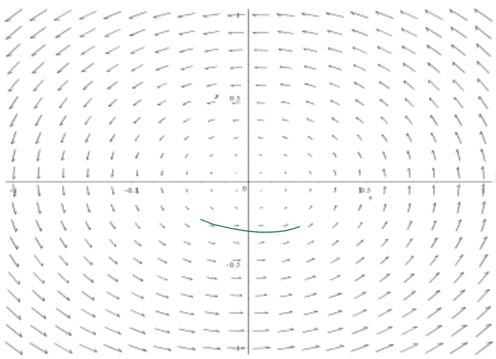


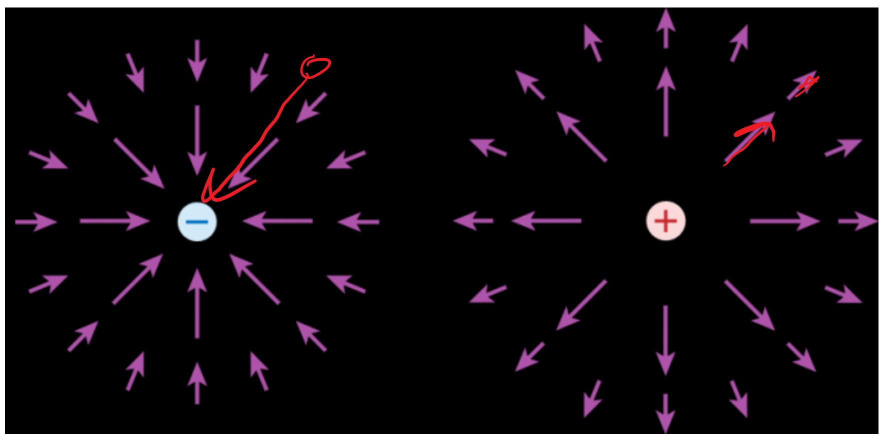
$f(x,y) = e^{-x^2-y^2}$

Vector Calculus Review

Vector Field: A vector field comes from a vector function which assigns a vector to points in space. An example of a vector field would be wind velocities in the atmosphere, water velocities in a river or electric forces around a charge.



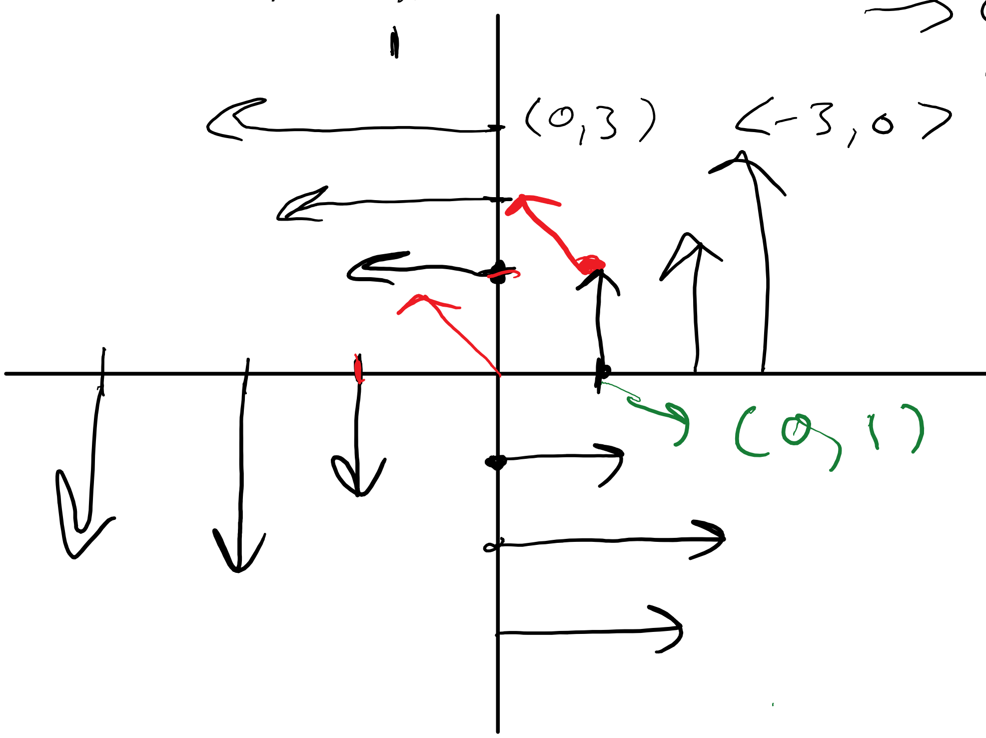




Example: Sketch the vector field defined by: $\vec{F} = < -y, x >$

(1,1)

<-1,1>



x y
(0,1)
<-1,0>
x y
(1,0)
<0,1>
(0,-1)
<-y,x>
<-(-1),0>
<1,0>

Vector Calculus Review

Derivatives of functions of several variables (partial derivatives) or the rate of change with respect to a particular variable.

Let $f(x,y,z) = x^2 + y^2 + z^2 + \sin(xyz)$ find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^3 f}{\partial x \partial y \partial z}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + y^2 + z^2 + \sin(xyz)\right)$$

$$= 2x + 0 + 0 + \cos(xyz)(yz)$$

$$\frac{\partial f}{\partial y} = 2y + \cos(xyz)\,xz \qquad \frac{\partial f}{\partial z} = 2z + \cos(xyz)(xy)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(2y + \cos(xyz)(xz)\right) = -\sin(xyz)(yz)(xz) + \cos(xyz)(z)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\cancel{2x}^{\,0} + \cos(xyz)(yz)\right) = -\sin(xyz)(xz)(yz) + \cos(xyz)(z)$$

Vector Calculus Review

Vector Derivative of a scalar function of several variables (Gradient) $\nabla = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$

∇ is what we call an operator, when written to the left of a scalar function it tells us to create a vector made up of the partial derivatives of the function which it is operating on. The prescription for operating is given by $\nabla f = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} >$ or in $\vec{\imath}, \vec{\jmath}, \vec{k}$ notation $\nabla f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial y}\vec{k}$

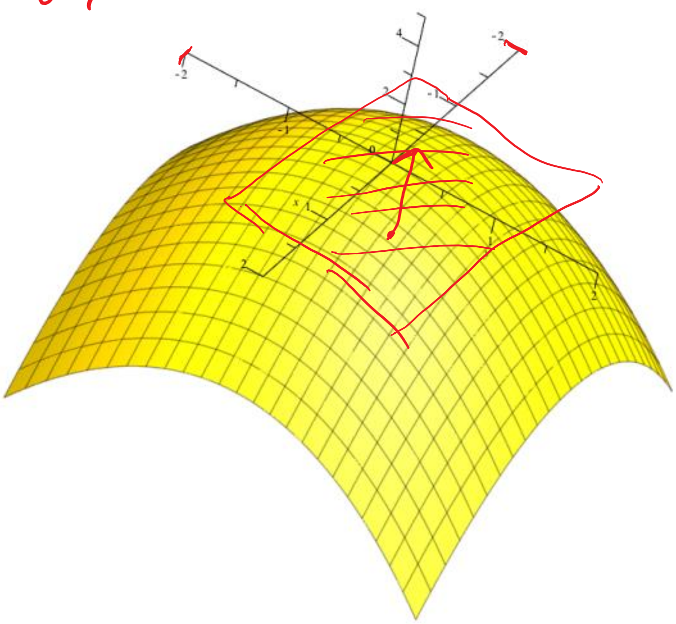
Example: Let $f(x,y) = -x^2 - y^2$. Calculate the gradient of f. Evaluate the gradient at the point (1,1) and sketch both f and its gradient vector.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle -2x, -2y \rangle$$

$$\frac{\partial f}{\partial x} = -2x$$

$$\nabla f \Big|_{(1,1)} = \langle -2, -2 \rangle$$

$$\frac{\partial f}{\partial y} = -2y$$



$\langle -2, -2, 1 \rangle$

$(1,1)$

$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$

$\langle a, b, c \rangle$

What does this gradient represent? What else can we use it for?

2 Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$\boxed{z - z_0 = \overset{a}{f_x(x_0, y_0)}(x - x_0) + \overset{b}{f_y(x_0, y_0)}(y - y_0)}$$

Vector Calculus Review

Scalar derivatives of a vector function of several variables, Divergence $\nabla\cdot$. The Divergence is another operator we have that operates on vector functions. $\nabla\cdot=<\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}>\cdot$ and is applied to a vector function of the form $\vec{F}(x,y,z)=<f_1(x,y,z),f_2(x,y,z),f_3(x,y,z)>$. That is:

$$\nabla\cdot\vec{F}=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}$$

Example Calculate the divergence of $\vec{F}(x,y,z)=<xy,xz,z>$

$$\nabla\cdot\vec{F}=<\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}>\cdot<xy,xz,z>$$

$$=\frac{\partial}{\partial x}xy+\frac{\partial}{\partial y}xz+\frac{\partial}{\partial z}z=y+0+1=y+1$$

The scalar second derivative of a scalar function, the Laplicain $\nabla\cdot\nabla=\nabla^2=\Delta$.

Let $f(x,y,z)=x^2+y^2+z^2+\cancel{\sin(xyz)}$ calculate $\nabla^2 f$

$$\nabla f=<\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}>=<2x,2y,2z>$$

$$\nabla\cdot\nabla f=2+2+2=6$$

$$\nabla^2 f=\Delta f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}$$

Vector Calculus Review

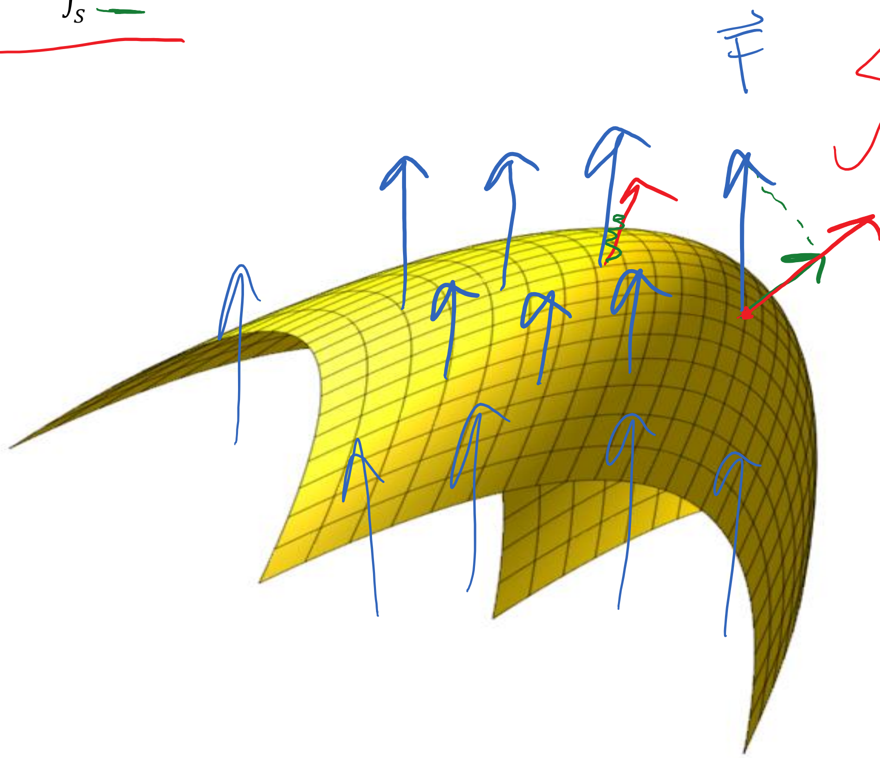
One very important idea we will need in this course is that of flux which cannot , sadly, be capacitated. :(

What is flux?

Mathematically, flux is a measure of how much a vector field passes through a surface in the normal(orthogonal) direction. That is:

$Flux = \int_S \vec{F} \cdot \vec{n}\, ds$ *Where $\vec{n}$ is the **outer** unit normal vector*

$\boxed{comp(\vec{a})_{\vec{b}} = \frac{\vec{b} \cdot \vec{a}}{||\vec{b}||}}$



$\langle 0, 1 \rangle$

$\sqrt{0^2+1^2} = 1$

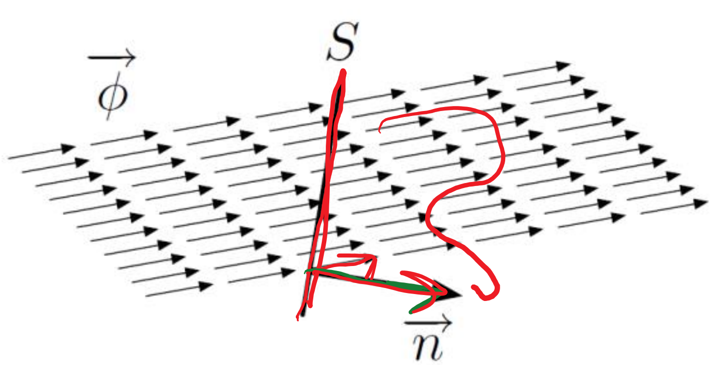
$comp(\vec{F})_{\vec{n}} = \frac{\vec{n} \cdot \vec{F}}{||n||}$

$= \vec{n} \cdot \vec{F}$

$= \vec{F} \cdot \vec{n}$

For its use in physical problems the flux measures the flow of some quantity, such as mass or heat, moving through an area per unit time. For example , we can define the mass flux $\vec{\phi} = \rho \vec{v}$. Where ρ is a density and $\vec{v}$ a velocity vector field. Incidentally, when flux is positive stuff is leaving the surface and when it is negative stuff is coming in.

In 2D we would measure the flux through a line.



$flux = (\vec{n} \cdot \vec{\phi}) \times length(S)$

$flux = \int_S \vec{F} \cdot \vec{n}\, ds$

Vector Calculus Review

Your first quiz problem:

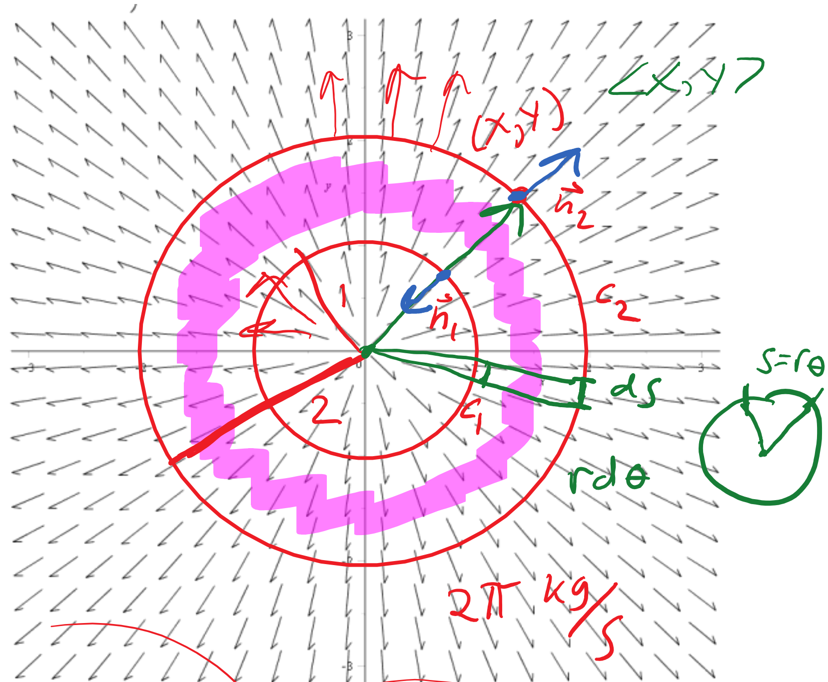
$\frac{\text{mass}}{\text{time}}$

Calculate the net mass flux through the annular region centered around the origin with inner radius 1 and outer radius 2 with the velocity vector field $\vec{v} = < \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} >$ for a fluid of constant density ρ.

$\vec{\Phi} = \rho \vec{v} = \frac{kg}{m^{\cancel{2}}} \frac{\cancel{m}}{s} = kg/ms$

$\int_C \vec{\Phi} \cdot \vec{n}\, ds = \int \frac{kg}{m\,sec} ds = \frac{kg}{\cancel{m}\,sec} \cancel{m}$

$= kg/sec$



$\vec{n}_1 = -\vec{n}_2$

$\int_{C_1} \vec{\Phi} \cdot \vec{n}_1 ds + \int_{C_2} \vec{\Phi} \cdot \vec{n}_2 ds = \text{net flux}$

$\int_{C_1} \vec{\Phi} \cdot (-\vec{n}_2) + \int_{C_2} \vec{\Phi} \cdot \vec{n}_2 = -\int_{C_1} \vec{\Phi} \cdot \vec{n}_2 ds + \int_{C_2} \vec{\Phi} \cdot \vec{n}_2 ds$

$\frac{\langle x,y\rangle}{\|\langle x,y\rangle\|} = \frac{\langle x,y\rangle}{\sqrt{x^2+y^2}} = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle = \vec{n}_2$ $\quad \Phi = \rho \vec{v}$

$\vec{\Phi} = \rho \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle$

net flux $= -\int_{C_1} \rho \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle \cdot \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle ds + \int_{C_2} \text{"} \quad \text{"} ds$

$x = r\cos\theta$
$y = r\sin\theta$
$r = \sqrt{x^2+y^2}$

$-\int_{C_1} \langle \frac{\cancel{r}\cos\theta}{\cancel{r}}, \frac{\cancel{r}\sin\theta}{\cancel{r}} \rangle \cdot \langle \frac{\cancel{r}\cos\theta}{\cancel{r}}, \frac{\cancel{r}\sin\theta}{\cancel{r}} \rangle + \int_{C_2} \text{"} \quad \text{"} ds$

$= -\int_{C_1} (\cos^2\theta + \sin^2\theta) ds + \int_{C_2} (\cos^2\theta + \sin^2\theta) ds$

$= -\int_{C_1} ds + \int_{C_2} ds = -\int_0^{2\pi} r\,d\theta + \int_0^{2\pi} r\,d\theta = -\int_0^{2\pi} d\theta + \int_0^{2\pi} 2\,d\theta = 2\pi$

C_1: $r \to 1$ $\quad$ C_2: $r \to 2$

$$= -\theta\Big|_0^{2\pi} + 2\theta\Big|_0^{2\pi} = -2\pi - (-0) + 2(2\pi) - 2(0)$$

$$-2\pi + 2(2\pi) = \boxed{2\pi}$$

Vector Calculus Review

The continuity equation

The continuity equation is extremely important in PDE's and is related to conserved quantities such as mass, heat and energy.

It basically says that the time rate change of a conserved quantity q in a volume V increases when more q flows in through the boundary of that volume and decreases when u flows out through the boundary.

Let's take mass as an example. Suppose we are interested in the time rate of change of the mass of oil with density ρ through a section of pipe with of volume V.

density

$$\frac{dm}{dt} = \frac{d}{dt}\underbrace{\int_V \rho dV}_{m} = \int_V \frac{d\rho}{dt} dV$$

The flux of mass through the surface of the pipe would be given by $\int_S \rho\vec{v}\cdot\vec{n}dS$

Let's examine the units here:

$$\frac{dm}{dt} = \frac{mass}{second}$$

$$\int_S \rho\vec{v}\cdot\vec{n}dS = \overset{\rho}{\frac{mass}{\cancel{m^3}}} * \overset{v}{\frac{\cancel{m}}{second}} * \overset{dS}{\cancel{m^2}} = \frac{mass}{second}$$

dec $\frac{dm}{dt} < 0$

We might then suppose:

$$\frac{dm}{dt} = \frac{d}{dt}\int_V \rho dV = \int_V \frac{d\rho}{dt} dV = -\int_S \rho\vec{v}\cdot\vec{n}dS$$

Inc $\frac{dm}{dt} > 0$

Why the minus sign? Well recall that when flux is negative that means stuff is flowing in through the surface. When say, mass , flows in through a surface we would expect it to increase that means its derivative would be positive, thus we need the minus sign to obtain a positive time derivative. If mass is flowing out the flux is positive and in the same way, the minus sign e nsures the derivative would be negative corresponding to decreasing mass.

For most useful applications this integral relation is cumbersome and in most situations we use a differential form of the continuity equation which we can obtain from the divergence theorem. We will derive it on the next page but it reads

$$\boxed{\frac{dm}{dt} = -\int \underbrace{\vec{\phi}}_{\rho\vec{v}}\cdot\vec{n}dS}$$

$$\boxed{\frac{\partial u}{\partial t} = -\nabla\cdot\vec{\phi}}$$

$u = \rho$

Where $\vec{\phi}$ is the flux field, in the above example $\vec{\phi} = \rho\vec{v}$, the mass flux field.

Vector Calculus Review

The Divergence Theorem

Suppose we wish to measure the flux through a closed surface, like a sphere. We can do this most easily with the divergence theorem. The divergence theorem relates the integral around the closed surface to a volume integral, which is often easier to compute.

The Divergence Theorem: If S is a closed surface then the flux through S of a vector field $\vec{F}$ can be computed using.

$$Flux = \int_S \vec{F}\cdot\vec{n}ds = \int_V \nabla\cdot\vec{F}dV \qquad \int_a^b f'(x)\,dx = \boxed{f(b) - f(a)}$$

$$\int_V \to \iiint$$

Example: Find the flux of the vector field $\vec{F} =< x, y, z >$ through the surface of the sphere of radius 2.

$$\int_S <x,y,z> \cdot \frac{<x,y,z>}{\sqrt{x^2+y^2+z^2}}$$

$$<\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}> \cdot <x,y,z>$$

$$\int_S \vec{F}\cdot\vec{n}ds = \int_V \nabla\cdot<x,y,z>dV = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}y + \frac{\partial}{\partial z}z$$

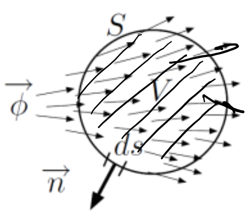
$$Vol = 4/3\,\pi r^3$$

$$r=2$$

$$= \int_V 1+1+1\,dv = \int_V 3dv = 3\int_V dv = \cancel{3}\frac{4}{\cancel{3}}\pi 2^3$$

$$= 4\pi\, 8 = \boxed{32\pi}$$

In 2d the divergence theorem relates the integral of an area to that around a closed curve.



$$\int_C \vec{F}\cdot\vec{n}\,ds = \int_S \nabla\cdot\vec{F}dA$$

Vector Calculus Review

Derivation of the differential form of the continuity equation.

We have that for a conserved quantity q

$$\frac{\partial q}{\partial t} = -\int_S \vec{\phi}\cdot\vec{n}\,dS$$

$$\int_V \frac{\partial u}{\partial t}\,dV$$

$$\cancel{\frac{kg}{m^3}\,m^3}$$

Now suppose that u is the density of the quantity q. For example, If q is mass then $u = \rho$. Then we can say $\boxed{\int_V u\,dV} = \boxed{q}$

With this in mind we can say:

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial t}\overbrace{\int_V u\,dV}^{q} = \int_V \frac{\partial u}{\partial t}\,dV$$

Which means

$$\longrightarrow \int_V \frac{\partial u}{\partial t}\,dV = -\int_S \vec{\phi}\cdot\vec{n}\,dS$$

Now we apply the divergence theorem

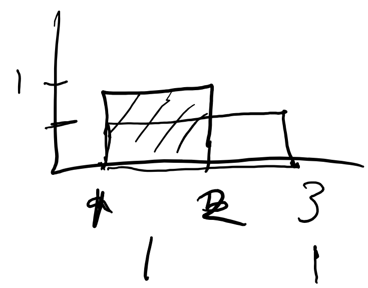
$$\int_S \vec{F}\cdot\vec{n}\,dS = \int_V \vec{\nabla}\cdot\vec{F}\,dV$$

div th.

$$\boxed{\frac{dq}{dt}} = \int_V \frac{\partial u}{\partial t}\,dV = -\int_S \vec{\Phi}\cdot\vec{n}\,dS = -\int_V \nabla\cdot\vec{\Phi}\,dV$$

$\vec{F}$

$$\int_V \frac{\partial u}{\partial t}\,dV = -\int_V \nabla\cdot\vec{\Phi}\,dV$$



$$\int_V \frac{\partial u}{\partial t}\,dV + \int_V \nabla\cdot\vec{\Phi}\,dV = 0$$

$$\int_V \frac{\partial u}{\partial t} + \nabla\cdot\vec{\Phi}\,dV = 0 \Longrightarrow \frac{\partial u}{\partial t} + \nabla\cdot\vec{\Phi} = 0$$

$$\boxed{\frac{\partial u}{\partial t} = -\nabla\cdot\vec{\Phi}}$$

$q = \text{mass}$

$u = \rho$

Vector Calculus Review

The continuity equation when there is internal forcing.

Suppose that for our conserved quantity q we have a source of that quantity inside the volume V that we are interested in. For example, if q is heat we may have a heating element producing more heat.

In this case we simply add in the rate at which the quantity is being created(or destroyed) per unit time per unit volume(kind of a mouth full)

That is:

$$\frac{\partial u}{\partial t} = -\nabla \cdot \vec{\phi} + \boxed{\sigma}$$

Where

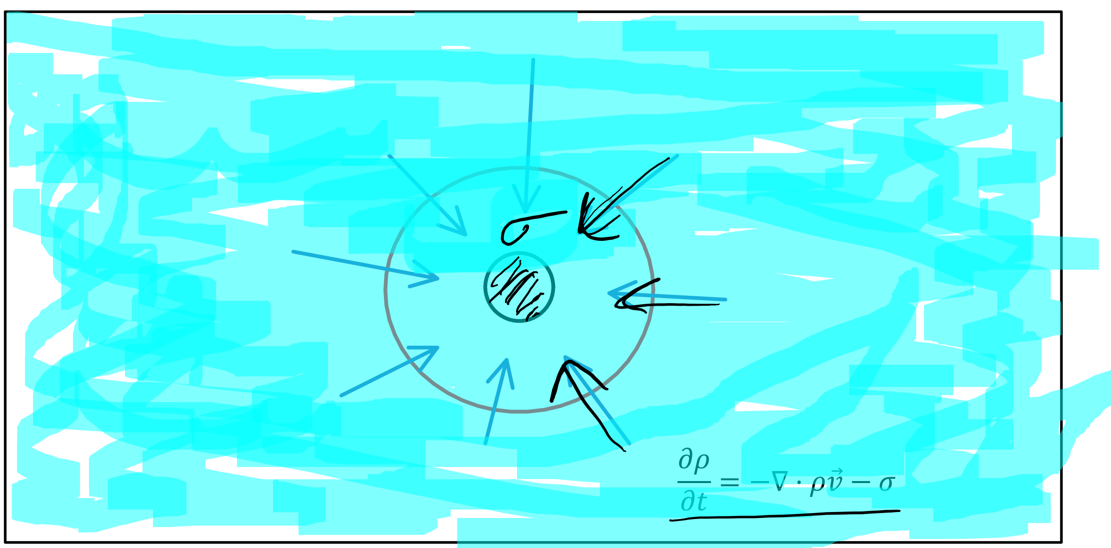
u = the amount of q per unit volume

$\vec{\phi}$ = the flux field of the quantity q

σ=rate of generation of quantity q per unit time per unit volume

When $\sigma > 0$ q is being created which we call a source, when $\sigma < 0$ q is being destroyed which we call a sink.

A 2D example: Suppose we have a very shallow pool with a drain in the middle, Let's think about the rate of change of water mass through a circle around the drain

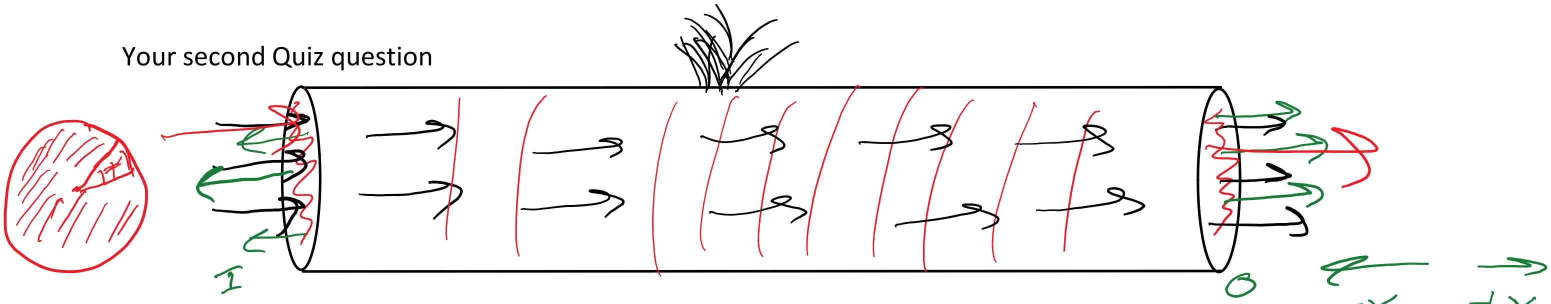


$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \underset{\substack{\downarrow \\ \rho\vec{v}}}{\phi} - \sigma$$

In this case the continuity equation gives us our first example of a PDE!

Vector Calculus Review

Your second Quiz question



Crude oil with a density of $850kg/m^3$ is flowing through a pipeline with radius 0.5 meters. The velocity of the oil in the pipe is measured at 2m/s at the input and remains constant throughout its trip.

a) Calculate the net flux of oil into and out of the pipe assuming no leaks:

$$\boxed{\int_I \vec{\phi}\cdot\vec{n}_I\,dS} + \int_O \vec{\phi}\cdot\vec{n}_O\,dS$$

$$\vec{\phi} = \rho\vec{v} = 850\langle 2,0,0\rangle$$

$$\vec{n}_I = \langle -1,0,0\rangle \quad \vec{n} = \langle 1,0,0\rangle$$

$$\int_I 850\langle 2,0,0\rangle\cdot\langle -1,0,0\rangle\,dS + \int_O 850\langle 2,0,0\rangle\cdot\langle 1,0,0\rangle\,dS$$

$$\int_I 850(-2)\,dS + \int_O 850(2)\,dS = -850(2)\underbrace{\int_I dS}_{\pi/4} + 850(2)\underbrace{\int_O dS}_{\pi/4}$$

$r = 1/2$

$$-850(2)(\pi/4) + 850(2)(\pi/4) = 0$$

$\pi(1/2)^2$

$\pi\frac{1}{4}$

$\pi/4$

b) Suppose the flux at the end of the pipe is measured to be $1300kg/s$. Does this imply we have a leak? If so, determine the rate at which oil is leaking from the pipe.

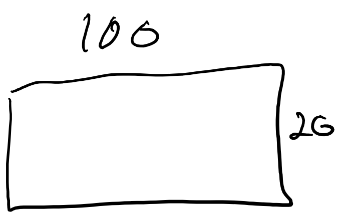
$$\int_I \vec{\Phi}\cdot\vec{n}_I\,dS = -850(2)\pi/4 = -850\,\pi/2 = -425\,\pi$$

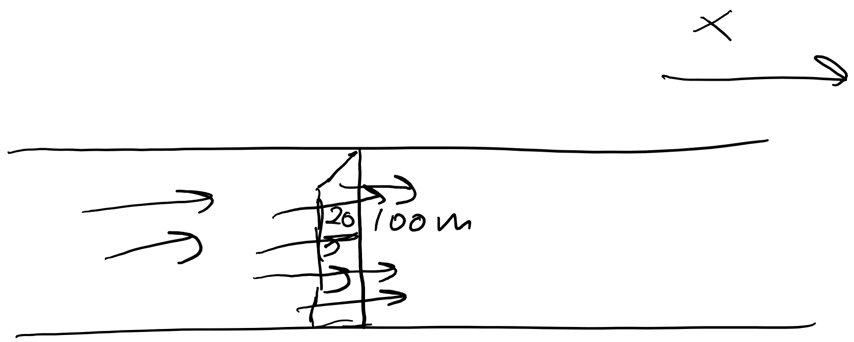
$$-1335.17 + 1300 = -35.17$$

$$\frac{dm}{dt} = -35.17\ kg/s$$

Problem 4

In candy land they measure the deliciousness of a treat in $yums$. It turns out that molasses has a yum density of 100 $yums/m^3$. In this strange land the mighty molasses river flows through the gum drop forest with a constant velocity of $v = 0.5m/s$. If at its widest point the river is 100 m across and 20 m deep, find the yum flux of the molasses through the rectangular area connecting the banks at this widest point.





$$\int_S \vec{\Phi}\cdot\vec{n}\,dS \qquad \Phi = \rho\vec{v} = 100(.5\,m/s)$$

$$\int_S \langle 100(.5), 0, 0\rangle \cdot \langle 1, 0, 0\rangle = \int_S \underline{100(.5)}\,dS = 100(.5)\int_S dS$$

$$= 100(.5)A = 100(.5)(100)(20)$$