Fluid Dynamics - Math 6750

Generalized Newtonian Fluids

1 Generalized Newtonian Fluid

For a Newtonian fluid, the constitutive equation for the stress is $\underline{\underline{T}} = -p\underline{\underline{I}} + 2\mu\underline{\underline{E}}$, where $\underline{\underline{E}} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the rate of strain tensor. In deriving this constitutive equation, we have used the fact that the flow is incompressible and the stress isotropic and isothermal. In general, every tensor $\underline{\underline{T}}$ can be written as a diagonal part and a trace free tensor. Therefore, we write $\underline{\underline{T}} = -p\underline{\underline{I}} + \underline{\underline{\sigma}}$, where $\mathrm{tr}(\underline{\underline{\sigma}}) = 0$ upon renaming the pressure if necessary. It follows that one particular way of looking at the constitutive equation is $\underline{\underline{\sigma}} = 2\mu\underline{\underline{E}}$, or $\underline{\underline{\sigma}} = f(\underline{\underline{E}})$ with f linear. The fact that f depends on $\underline{\underline{E}}$ only and not on $\underline{\underline{W}}$ is because $\underline{\underline{W}}$ corresponds to rigid body rotation.

For simplicity, we will consider only incompressible and unidirectional flow. For example, for a parallel plate set-up with $\mathbf{u} = u(y)\mathbf{e}_1$, the rate of strain tensor becomes

$$2E_{ij} = \partial_y u \quad \text{if} \quad i = 1, j = 2 \text{ or } i = 2, j = 1 \quad E_{ij} = 0 \quad \text{else.}$$

Therefore the only non-zero component of $\underline{\underline{\sigma}}$ is $\sigma = \sigma_{12}(y) = \sigma_{21}(y) = \mu\partial_y u(y)$. We note that if $u(y)$ is linear, then $\partial_y u = \dot{\gamma}$ is a constant called the shear rate and $\sigma = \mu\dot{\gamma}$. By looking at the units, it is obvious that $\dot{\gamma}$ is indeed a rate. The relationship between σ and $\dot{\gamma}$ is called a stress-shear rate relationship. For a Newtonian fluid, it is linear. Examples of liquids which do not follow the Newtonian constitutive equation include blood, synovial fluid, mucus, slurries, chocolates, nail polish, shaving foam, lotion, yogurt, fresh cream, fire fighting foam, jam, mayonnaise, ale, salad dressing, lubricating oil, mine tailing, magma, paint, sludge.

Definition 1. A generalized Newtonian fluid (GNF) is a fluid, such that the value of $\dot{\gamma}$ depends only on the current time point, i.e there is no memory or history in shear rate.

Before talking about GNF, we discuss the physical meaning of viscosity. Since σ is a stress, it has units of pressure or force per area. Since $\dot{\gamma}$ is a rate, it has units of inverse time. Therefore, μ has units of pressure times time or of mass per length per time. It is usually given in Pa·s. Below are a few examples of viscosity measured at room temperature.

Liquid	Air	Water	Ethyl Alcohol	Mercury	Ethylene Glycol	Olive Oil
Viscosity (Pa·s)	10^{-5}	10^{-3}	$1.2 \cdot 10^{-3}$	$1.5 \cdot 10^{-3}$	$2 \cdot 10^{-2}$	10^{-1}

Liquid	Glycerol	Honey	Corn Syrup			
Viscosity (Pa·s)	1.5	10	100			

Experimentally, the viscosity can be obtained using a parallel plate experiment in which the top plate is driven at velocity U and the bottom plate is held stationary. We denote by A the area of the plates and by d the separation distance. The force F required to keep the top plate moving can be obtained. Changing A, U, d, it can be experimentally established that $F \sim \frac{AU}{d}$. The constant of proportionality is the viscosity μ. Dividing by A to get a pressure, we have $\frac{F}{A} = \mu\frac{U}{d}$.

Figure 1 is an illustration of the different functional behavior of the stress as a function of the shear rate. For example, a Bingham fluid will only flow if the stress is high enough, it behaves like a solid at low stress. While shear thinning is a general term meaning that the rate of change of the stress decreases as the shear rate increases, it is usually characterized by a horizontal asymptote as $\dot{\gamma} \to \infty$. In some context (and in the absence of an asymptote), it might be referred to as pseudoplastic.

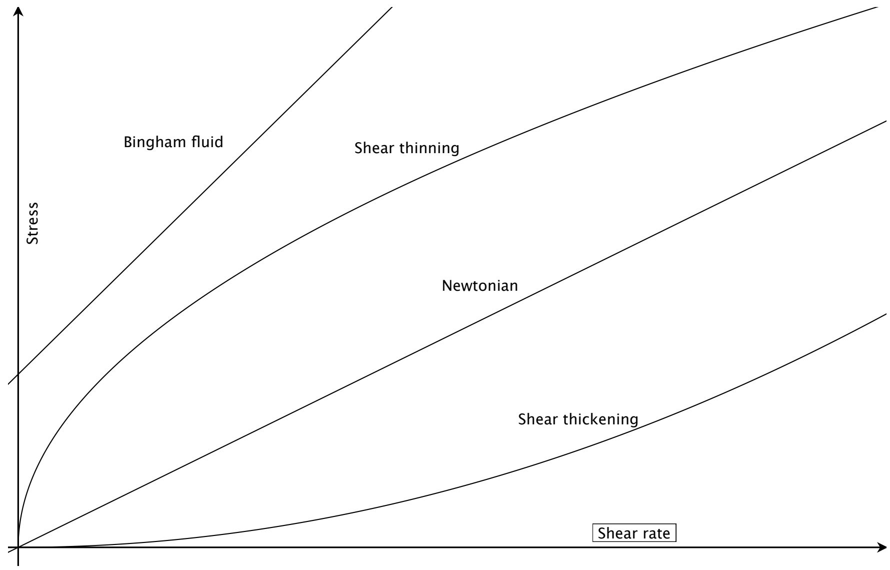

Figure 1: Functional behavior of the stress in different fluids as a function of the shear rate.

We will consider two specific examples of flow of GNF: a power law fluid in a finite length pipe and a Bingham fluid in a Couette device.

2 Power law fluid in pipe flow

Definition 2. A power law fluid is a fluid that obeys the constitutive relationship (stress-rate of strain relationship) $\sigma = \kappa |\dot{\gamma}|^n$.

Remark 1. If $0 < n < 1$, then the fluid is shear thinning. In practice (polymer melt), $n \approx 0.3 - 0.7$. If $n = 1$, then the fluid is Newtonian.
The constitutive equation for a power law fluid is sometimes called Ostwald de Waele equation.

We want to find the unidirectional steady state flow profile in a pipe of length L and radius R subject to a pressure drop ΔP. The flow is unidirectional and flowing along the axis of the pipe, it only depends on the radial distance from the centerline. On the boundary of the pipe, we impose a no-slip boundary condition (no flow). If $n = 1$, then we know that the profile is parabolic, with maximum speed in the center of the pipe.

This problem is best described using cylindrical coordinates with z-axis along the direction of the pipe. Therefore, we seek a steady unidirectional flow of the form $\mathbf{u}(r, \theta, z) = u_z(r)\mathbf{e}_z$

satisfying the Navier Stokes equations, i.e

$$\nabla_c \cdot \mathbf{u} = 0 \quad 0 = -\nabla_c p + \nabla_c \cdot \underline{\underline{\sigma}}.$$

In the above, $\nabla_c = (\mathbf{e}_r \partial_r + \frac{1}{r}\mathbf{e}_\theta \partial_\theta + \mathbf{e}_z \partial_z)$ is the gradient operator in cylindrical coordinates with basis vectors $\mathbf{e}_r = (\cos\theta, \sin\theta, 0)^T$, $\mathbf{e}_\theta = (-\sin\theta, \cos\theta, 0)^T$ and $\mathbf{e}_z = (0,0,1)^T$ and each vector/tensor is given in cylindrical coordinates. For example, $\underline{\underline{\sigma}} = \sigma_{ij}\mathbf{e}_i\mathbf{e}_j$ with $i, j \in \{r, \theta, z\}$.

It is easy to see that the incompressibility condition is satisfied. In fact, it is the condition that allows us to write u_z as an independent function of z. Because we are looking for axisymmetric solutions, all derivatives with respect to θ are zero and there is no θ dependence. Finally, the pressure drop is in the z-direction so that $\nabla_c p = \frac{\Delta P}{L}\mathbf{e}_z$.

The components of the rate of strain tensor $\underline{\underline{E}}$ in cylindrical coordinates are

$$E_{rr} = \partial_r u_r \quad E_{\theta\theta} = \frac{1}{r}\partial_\theta u_\theta + \frac{u_r}{r} \quad E_{zz} = \partial_z u_z$$
$$2E_{\theta z} = \frac{1}{r}\partial_\theta u_z + \partial_z u_\theta \quad 2E_{zr} = \partial_z u_r + \partial_r u_z \quad 2E_{r\theta} = r\partial_r\left(\frac{u_\theta}{r}\right) + \frac{1}{r}\partial_\theta u_r$$

Plugging in $\mathbf{u} = u_z(r)\mathbf{e}_z$, we have

$$E_{rr} = E_{\theta\theta} = E_{zz} = E_{\theta z} = E_{r\theta} = 0 \quad 2E_{zr} = \partial_r u_z(r).$$

In other words, the only non-zero component is $E_{r\theta}$ and the corresponding non-zero component of $\underline{\underline{\sigma}}$ is $\sigma_{r\theta}$. To see this, we let $\partial_r u_z(r) = \dot{\gamma}$ in the constitutive equation of a power law fluid.

Putting everything together, most of the entries in the vector Navier-Stokes equation are zero and we are only left with the PDE (out of the $\mathbf{e}_z$ component)

$$0 = -\partial_z p + \frac{1}{r}\partial(r\sigma_{rz}).$$

Since the pressure drop is given, the above equation can be integrated and we find

$$\sigma_{rz} = \frac{r}{2}\frac{\Delta P}{L} + \frac{C}{r},$$

where C is the constant of integration. Typically, we would find C using the boundary conditions, but the only boundary condition we have are in terms of u_z and not σ_{rz}. However, since we are looking for axisymmetric solutions, they have to be bounded at $r = 0$. Because $1/r$ blows up at the origin, we must therefore have $C = 0$ and $\sigma_{rz} = r\frac{\Delta P}{2L}$. It will be useful to introduce a new constant σ_w as the stress at the wall, i.e. $\sigma_w = \frac{R\Delta P}{2L}$. With this notation, the stress becomes

$$\sigma_{rz} = \sigma_w \frac{r}{R}. \tag{1}$$

To obtain, the flow profile we will make use of the specific constitutive equation for power law fluid. We recall that $\dot{\gamma} = \partial_r u_z$. Since u_z is maximum at $r = 0$ and $u_z(R) = 0$, we know that $\partial_r u_z < 0$, in other words $|\dot{\gamma}| = -\dot{\gamma} = -\partial_r u_z$.

Plugging the constitutive equation in Eq. (1), we have

$$\kappa|\partial_r u_z|^n = \frac{\sigma_w}{R} r$$

or taking the nth root and the above observation about the sign of $\dot{\gamma}$

$$-\partial_r u_z = \left(\frac{\sigma_w}{R\kappa}\right)^{1/n} r^{1/n}.$$

The solution to the previous ODE is

$$u_z(r) = \left(\frac{\sigma_w}{R\kappa}\right)^{1/n} (-1)\frac{r^{1/n+1}}{1/n+1} + \tilde{C},$$

where $\tilde{C}$ is the constant of integration. Using the BC $u_z(R) = 0$, we can solve for $\tilde{C}$ and the final solution is

$$u_z(r) = \left(\frac{\sigma_w}{R\kappa}\right)^{1/n} \frac{R^{1/n+1} - r^{1/n+1}}{1/n+1}. \tag{2}$$

It is easy to check that if $n = 1$, then u_z is parabolic as expected.

3 Bingham fluid in a Couette device

Definition 3. A Bingham fluid is a fluid that obeys the constitutive relationship (stress-rate of strain relationship) $\dot{\gamma} = 0$ if $\sigma < \sigma_y$ and $\sigma = \sigma_y + \mu\dot{\gamma}$ if $\sigma > \sigma_y$. σ_y is called the yield stress.

We consider the problem of finding the flow of a Bingham fluid between two concentric cylinders of length L, inner radius a and outer radius b. The inner cylinder is held fixed, the outer radius is submitted to a torque T and as a result is rotating with angular velocity Ω. The boundary conditions are $u_\theta(a) = 0$ and $u_\theta(b) = \Omega b$. We look for steady axisymmetric steady solution of the form $\mathbf{u} = (0, u_\theta(r), 0)$. The corresponding only non-zero component of $\underline{\underline{E}}$ in cylindrical coordinates is $2E_{r\theta} = r\partial_r\left(\frac{u_\theta}{r}\right)$, so $\dot{\gamma} = 2E_{r\theta}$. Since there is no pressure drop, the $\mathbf{e}_\theta$ component of the Navier Stokes equation gives

$$0 = \frac{1}{r^2}\frac{d}{dr}(r^2\sigma_{r\theta}).$$

Solving it, we have $\sigma_{r\theta} = \frac{C}{r^2}$ or $\sigma_{r\theta} = \frac{\sigma_w b^2}{r^2}$, where σ_w is the stress at the outer cylinder.

We define the yield radius as the radius at which the yield radius equals $\sigma_{r\theta}$. Substituting, we find $r_y = \sqrt{\frac{\sigma_w}{\sigma_y}}b$. If $r_y < r < b$, then $\dot{\gamma} = 0$ and $u_\theta = \Omega r$ solid body rotation. On the other hand, if $a < r < r_y$, then

$$\dot{\gamma} = r\partial_\theta\left(\frac{u_\theta}{r}\right) = \frac{\sigma_{r\theta} - \sigma_y}{\mu} = \frac{\sigma_y}{\mu}\left[\frac{r_y^2}{r^2} - 1\right]$$

Solving the ODE, we obtain

$$\frac{u_\theta}{r} = \frac{-1}{2}\frac{\sigma_y}{\mu}\frac{r_y^2}{r^2} - \frac{\sigma_y}{\mu}\ln r + C.$$

Plugging in the boundary condition at $r = a$, $u_\theta(a) = 0$, yields

$$C = \frac{\sigma_y}{\mu} \ln a + \frac{\sigma_y}{2\mu} \frac{r_y^2}{a^2}.$$

Therefore, the solution in $a < r < r_y$ is

$$\frac{u_\theta}{r} = \frac{\sigma_y}{\mu} \left[\frac{1}{2} \left(\frac{r_y^2}{a^2} - \frac{r_y^2}{r^2} \right) - \ln \frac{r}{a} \right].$$

4 General strategy for solving isothermal flow problems

From first principles, the following three PDEs are always true

$$\begin{aligned} \text{Conservation of mass} &\quad \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{u} = 0 \\ \text{Conservation of linear momentum} &\quad \rho(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = \mathbf{F}_b + \nabla \cdot \underline{\underline{T}} \\ \text{Conservation of angular momentum} &\quad \underline{\underline{T}} = \underline{\underline{T}}^T. \end{aligned}$$

If the flow is incompressible, then the first equation is replaced by $\nabla \cdot \mathbf{u} = 0$ and ρ is constant.

Based on the geometry of the specific problem, the above equations can be expressed in a particular coordinate system: cartesian, cylindrical, spherical. Boundary conditions on the domain are imposed on the flow, usually in the form of a no-slip and no-penetration condition.

Major simplifications of the PDEs can be obtained using assumptions or a-priori knowledge about the flow. For example, when looking for steady solutions, then $\partial_t \mathbf{u} = 0$. Another common example is unidirectional incompressible flow, $\mathbf{u} = u\mathbf{e}$ in some prescribed direction $\mathbf{e}$. In this case, the scalar u is still a function of the three coordinates, but it is usually the case that one can assume no dependence on one of the direction. Incompressibility can then be used to reduce the dependence of u to a single coordinate, which is known as 1d flow.

To solve the PDEs, we need constitutive relations between $\underline{\underline{T}}$ and some other variables. These equations are usually not derived from first principles. From the static problem, we can always write $\underline{\underline{T}} = -p\underline{\underline{I}} + \underline{\underline{\sigma}}$ with $\text{tr}(\underline{\underline{\sigma}}) = 0$. From the discussion of a Newtonian fluid, we argue that $\underline{\underline{\sigma}}$ is a function of $\underline{\underline{E}} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right)$ and higher derivatives of the rate-of-strain tensor and not of the antisymmetric $\underline{\underline{W}}$ which corresponds to rigid body rotations for small displacements. Mathematically, we write $\underline{\underline{\sigma}} = f(\underline{\underline{E}}, \text{higher derivatives of } \underline{\underline{E}})$. We note that by writing such a functional relationship, we are assuming that the state of $\underline{\underline{E}}$ at time t is independent from past history.

In the case of a unidirectional and 1d incompressible flow, the strategy for finding analytical solution can be summarized as:

1. Since $\mathbf{u} = u(y)\mathbf{e}_1$ ($y, \mathbf{e}_1$ arbitrary), $\underline{\underline{E}}$ has only two equal non-zero components denoted $\dot{\gamma} = \partial_y u$ and so does $\underline{\underline{\sigma}}$. The constitutive relationship is a scalar law, $\sigma = f(\dot{\gamma}, \ldots)$.

2. Solve the Navier-Stokes equation in terms of σ keeping the integration constant.

3. Plug σ into the constitutive relationship to get an equation for $\dot{\gamma}$.

4. Integrate and solve for u. Apply the boundary conditions to find the two constants of integration.