Smooth Realization and Conjugation By Approximation

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Motivation

von Neumann (1932) asked whether the models of classical ergodic theory had analogs in the smooth category. In modern terms,

Question

Is every measure preserving transformation isomorphic to diffeomorphism of a compact manifold preserving a measure equivalent to the volume?

Obstructions

Thanks to Kushnirenko (1965) we know that every volume preserving diffeomorphism of a compact manifold has finite entropy. Thus, we must augment our central question:

Question

Is every finite entropy measure preserving transformation isomorphic to diffeomorphism of a compact manifold preserving a measure equivalent to the volume?

There are obstructions to realization on low dimensional manifolds:

1. A circle diffeomorphism preserving a smooth volume is isomorphic to a rotation.
2. A weakly mixing surface diffeomorphism of positive entropy is Bernoulli.

Weaker Requirements

If we weaken either of the requirements then the answer is in the affirmative.

Theorem (Lind-Thouvenot (1978))

1. *Every finite-entropy transformation is isomorphic to a linear automorphism of the 2-torus preserving a Borel probability measure.*
2. *Every finite-entropy transformation is isomorphic to a homeomorphism of the 2-torus preserving Lebesgue measure.*

Overview of Conjugation by Approximation Method

We consider a manifold M admitting a smooth non-trivial action of $\mathbb{T}$, $\{S_\alpha\}_{\alpha\in\mathbb{T}}$, that preserves a smooth Riemannian measure λ.
For most purposes it suffices to consider $M = \mathbb{T} \times [0,1]^{d-1}$.
Our desired transformation T will be the limit in $\mathrm{Diff}^\infty(M)$ of the sequence of periodic diffeomorphisms T_n given by

$$T_n = B_n^{-1} \circ S_{\alpha_{n+1}} \circ B_n$$

where $\alpha_n : \frac{p_n}{q_n} \in \mathbb{Q}$ and $B_n \in \mathrm{Diff}^\infty(M)$.

$$\begin{tikzcd} (M,\lambda) \arrow[r, "T_n"] \arrow[d, "B_n"'] & (M,\lambda) \arrow[d, "B_n"] \\ (M,\lambda) \arrow[r, "S_{\alpha_n}"'] & (M,\lambda) \end{tikzcd}$$

Overview of Conjugation by Approximation Method

We assume that B_n preserves the measure λ.
We need to impose conditions on our conjugating maps B_n and on the numbers α_n if we are to hope to have T_n converge. We define our maps B_n inductively

$$B_n = A_n \circ B_{n-1} = A_n \circ \cdots \circ A_1$$

where we require

$$A_n \circ S_{\alpha_{n-1}} = S_{\alpha_{n-1}} \circ A_n.$$

This condition constitutes the **sole restriction** on the choice of A_n.
Typically, the sequence of conjugating maps B_n **does not** converge.

Smooth Convergence

Notice

$$\begin{aligned} d_{C^n}\big(T_n, T_{n-1}\big) &= d_{C^n}\left(B_n^{-1} \circ S_{\alpha_n} \circ B_n, B_{n-1}^{-1} \circ R_{\alpha_{n-1}} \circ B_{n-1}\right) \\ &= d_{C^n}\left(B_n^{-1} \circ S_{\alpha_n} \circ B_n, B_{n-1}^{-1} \circ S_{\alpha_{n-1}} \circ A_n^{-1} \circ A_n \circ B_{n-1}\right) \\ &= d_{C^n}\left(B_n^{-1} \circ S_{\alpha_n} \circ B_n, B_{n-1}^{-1} \circ A_n^{-1} \circ S_{\alpha_{n-1}} \circ A_n \circ B_{n-1}\right) \\ &\le C(n) \cdot \left(\left\| DB_n^{-1} \right\|_{C^{n+1}}\right)^{n+1} \cdot |\alpha_{n+1} - \alpha_n| \end{aligned}$$

where $C(n)$ depends only on the order of the derivatives and the geometry of the manifold.

This guarantees that, regardless of B_n, we can make $d_{C^n}\big(T_n, T_{n-1}\big)$ as small as we like by taking α_n sufficiently close to α_{n-1}

Smooth Convergence

We define

$$\alpha_{n+1} = \alpha_n + \beta_n \quad \text{where } \beta_n = \frac{1}{k_n\, l_n\, q_n^2}$$

we will have to chose k_n and l_n large enough to satisfy particular properties at later stages of the construction. This choice means that

$$\begin{aligned} p_{n+1} &= p_n k_n l_n q_n + 1 \\ q_{n+1} &= k_n\, l_n\, q_n^2 \end{aligned}$$

and $\gcd(p_{n+1}, q_{n+1}) = 1$.

Measurable Partitions

Here we follow notation of Kunde that makes explicit what was implicit in the original paper of Anosov-Katok and makes the combinatoric nature of future constructions clearer.
For simplicity we consider the two dimensional case where we may consider $M = \mathbb{T} \times [0,1]$ with the action $S_\alpha = (x + \alpha, y)$.
Let $\xi_{kq,s} = \{\Delta^{i,j}_{kq,s} : 0 \leq i \leq kq - 1, 0 \leq j \leq s - 1\}$ be a partition of M with

$$\Delta^{i,j}_{kq,s} = \Big[\frac{i}{kq}, \frac{i+1}{kq}\Big) \times \Big[\frac{j}{s}, \frac{j+1}{s}\Big)$$

Note that as $k, s \to \infty$ we have $\mathrm{diam}(\Delta^{i,j}_{kq,s}) \to 0$.

Iterative Construction

Fix a decreasing sequence $\epsilon_n > 0$ such that $\sum \epsilon_n < \infty$.
Suppose that we have $A_1, \ldots, A_n$ and $\alpha_n = \frac{p_n}{q_n}$, with p_n, q_n relatively prime, defined

1. We choose k_n, s_n sufficiently large that

$$\operatorname{diam}(B_n^{-1} \Delta^{i,j}_{k_n q_n, s_m}) < \frac{1}{2^n}$$

   and we choose an S_{1/q_n}-equivariant permutation of $\xi_{k_n q_n, s_n}$, π_{n+1}.
   Note:

   1. $\{B_n^{-1} \xi_{k_n q_n, s_m}\}$ is a generating sequence of partitions.
   2. In many cases, we require that

$$\operatorname{diam}(B_n^{-1} \Delta^{i,0}_{k_n q_n, 1}) < \frac{1}{2^n}$$

Iterative Construction II

We would like to define A_{n+1} such that A_{n+1}^{-1} induces π_{n+1} on $\xi_{k_n q_n, s_n}$ however this is clearly impossible for a smooth map. Let us define

$$\tilde{\Delta}^{i,j}_{kq,s} = \Big[\frac{i + \frac{\epsilon_n}{4}}{kq}, \frac{i + 1 - \frac{\epsilon_n}{4}}{kq}\Big) \times \Big[\frac{j + \frac{\epsilon_n}{4}}{s}, \frac{j + 1 - \frac{\epsilon_n}{4}}{s}\Big)$$

. which has the property that

$$\lambda\big(\bigcup_{i,j} \tilde{\Delta}^{i,j}_{kq,s}\big) > 1 - \epsilon_n$$

and instead require that

2. A_{n+1} induces the permutation π_{n+1} on $\{\tilde{\Delta}^{i,j}_{kq,s}\}$. This is possible by the Moser Trick.
3. Finally l_n remains available to ensure that α_{n+1} is sufficiently close to α_n.

The properties of the limiting diffeomorphism depend on the choices of k_n, π_{n+1}, and l_n.

To prove a property using conjugation by approximation it is necessary to

Spaces

We may achieve T is an prescribed neighborhood of the initial element S_{α_1} or by applying a fixed diffeomorphism B at the outset in a neighborhood of any diffeomorphism conjugated to an element of the action. Thus, the natural space for these constructions is

$$\mathcal{A}(M) = \overline{\{B^{-1} \circ S_t \circ B : t \in \mathbb{T}, B \in \mathrm{Diff}^\infty(M, \lambda)\}}$$

where the closure is in $\mathrm{Diff}^\infty(M, \lambda)$.
One may also consider the closure of all diffeomorphism that are conjugated to a specific element of the action:

$$\mathcal{A}_\alpha(M) = \overline{\{B^{-1} \circ S_\alpha \circ B : B \in \mathrm{Diff}^\infty(M, \lambda)\}}$$

where the closure is in $\mathrm{Diff}^\infty(M, \lambda)$.

Anosov-Katok

New Examples in Smooth Ergodic Theory. Ergodic Diffeomorphisms. Trans. Moscow. Math. Soc., Vol 23, 1970

The paper is 34 pages long and was apparently written in a weekend.

It proves the following results:

1. The set of weakly mixing diffeomorphisms is residual (i.e. it contains a dense G_δ-set) in $\mathcal{A}(M)$ in the $\mathrm{Diff}^\infty(M)$-topology.
2. There is an ergodic diffeomorphism $T \in \mathrm{Diff}^\infty(M, \lambda)$ that is measure-theoretically isomorphic to the circle rotation by $\lambda = \lim_{n\to} \alpha_n$.
3. For $h \in \mathbb{Z}^+$ there is an ergodic diffeomorphism $T \in \mathrm{Diff}^\infty(M, \lambda)$ that is measure-theoretically isomorphic to some ergodic translation on $\mathbb{T}^h$.
4. There is an ergodic diffeomorphism $T \in \mathrm{Diff}^\infty(M, \lambda)$ measure-theoretically isomorphic to some ergodic translation on $\mathbb{T}^\infty$.

Before this paper it was unknown whether there was an ergodic area-preserving diffeomorphism of the disk!

Quantitative Anosov-Katok

If one examines the arguments in Anosov and Katok one sees that the set of weakly mixing diffeomorphisms is residual in $\mathcal{A}_\alpha(M)$ for a G_δ set of α. Unfortunately, neither the paper nor the proof enables us to precisely describe the set of possible αs.

Theorem (Fayad-Saprykina, 2005)

If $\alpha \in \mathbb{T}$ is Liouville, i.e for every $C > 0$ and every $n \in \mathbb{Z}^+$ there exist infinitely many pairs with

$$|\alpha - \frac{p}{q}| < \frac{C}{q^n}.$$

then the set of weak mixing diffeomorphisms in residual in $\mathcal{A}_\alpha(M)$. If the manifold M is the disk then the restriction of the diffeomorphism to the boundary is the rotation R_α

Quantitative Anosov-Katok

The comment about the boundary is there because of "Herman's Last Geometric Theorem" which says that a diffeomorphism T of the disk that has a Diophantine rotation on the boundary has the boundary accumulated by a positive measure of T-invariant curves (and consequently is not ergodic).

Theorem

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$ is Diophantine if and only if there is no ergodic diffeomorphism of the disk whose restriction to the boundary has rotation number α.

The key is an estimate on the size of $\|DB_n^{-1}\|_{C^k}$ (

$$\|DB_n^{-1}\|_{C^k} < C \cdot q_{n-1}^{2k}$$

Quantitative Anosov-Katok

Using the same argument in Anosov-Katok for measure-theoretic isomorphism and the estimates in Fayad-Saprykina

Theorem (Fayad-Saprykina-W., 2007)

For every Liouville $\alpha \in \mathbb{T}$ there exists an ergodic $T \in \mathrm{Diff}^\infty(M, \lambda)$ that is measure-theoretically isomorphic to the rotation R_α. If $M = \mathbb{T}^d$ for $d \geq 2$ then the result can be strengthened to a uniquely ergodic diffeomorphism.

We call these non-standard realizations of R_α.
The existence of a non-standard realization for R_α when $\alpha \in \mathbb{T}$ is a Diophantine number remains open.

Analytic Anosov-Katok

One has to be very careful when working in the analytic category. The fact that the existence of an ergodic real-analytic diffeomorphism of the disk is unknown makes it clear that the full scope of the Anosov-Katok machinery will not work.
Using the AK machinery and explicit formulas for A_n

Theorem (Saprykina, 2003)

There exist real-analytic, uniquely ergodic, area preserving diffeomorphisms of $\mathbb{T}^2$ that are not conjugate to a linear automorphisms.

Analytic Anosov-Katok

More general real-analytic constructions on tori awaited Banerjee's *block-slide* maps.

Theorem (Banerjee, 2017)

There exists uniquely ergodic real-analytic diffeomorphisms of the two dimensional torus T^2 preserving the Lebesgue measure that are metrically isomorphic to some irrational rotations of the circle.

Theorem (Kunde 2017)

Let $\rho > 0$, $m \geq 2$ and $\mathbb{T}^m$ be the torus with Lebesgue measure λ. There exists a weak mixing real-analytic diffeomorphism $T \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^m, \lambda)$ preserving a measurable Riemannian metric.

Analytic Anosov-Katok

Theorem (Banerjee-Kunde, 2019)

For any $\rho > 0$, $h \geq 1$ and $d \geq 2$ there exists an ergodic real-analytic diffeomorphism $T \in \mathrm{Diff}^{\omega}_{\rho}(\mathbb{T}^d, \lambda)$ which is measure-theoretically isomorphic to a translation on $\mathbb{T}^h$.