

# A construction of Bowen-Margulis measure

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# Anosov flows: Definition

Fix  $M$  to be a  $C^1$  Riemannian manifold and  $\{g_t\}$  a differentiable flow on  $M$ . We say  $g_t$  is an **Anosov flow** if there is a splitting of tangent bundle  $TM$ ,

$$T_x M = \underbrace{E_x^s}_{\text{stable}} \oplus \underbrace{E_x^u}_{\text{unstable}} \oplus \mathbb{R}v_x, \quad \text{for } x \in M$$

such that

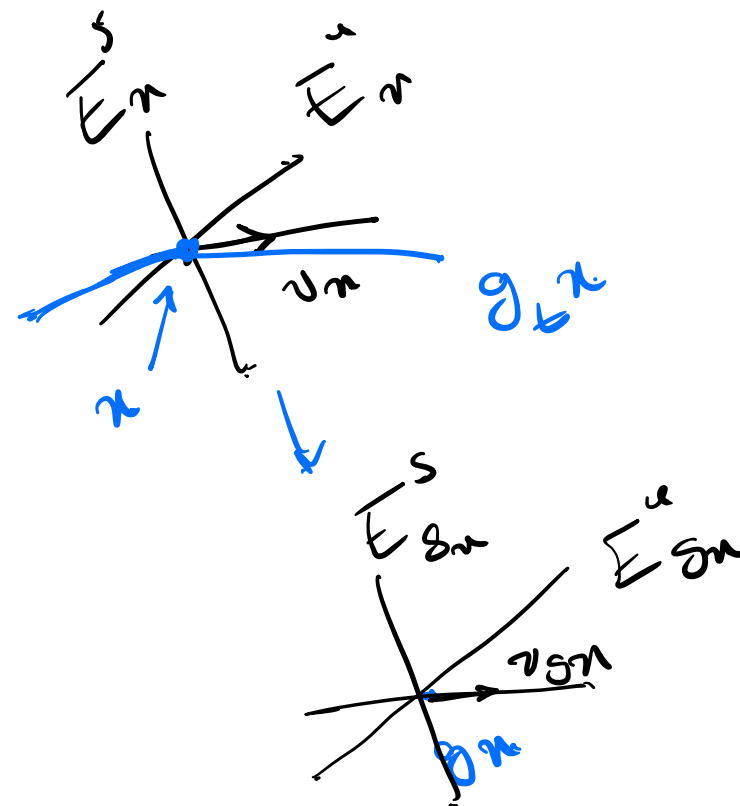
①  $v_x = \frac{dg_t x}{dt} \big|_{t=0}$  is the flow direction.

②  $E^s$  and  $E^u$  are  $g_t$ -invariant.

and there is a constant  $\kappa > 0$  such that

③  $\|D_x g_t(v)\| \leq e^{-\kappa t} \|v\|$  for  $v \in E_x^s$  and  $t \geq 0$ .

④  $\|D_x g_t(v)\| \geq e^{\kappa t} \|v\|$  for  $v \in E_x^u$  and  $t \geq 0$ .



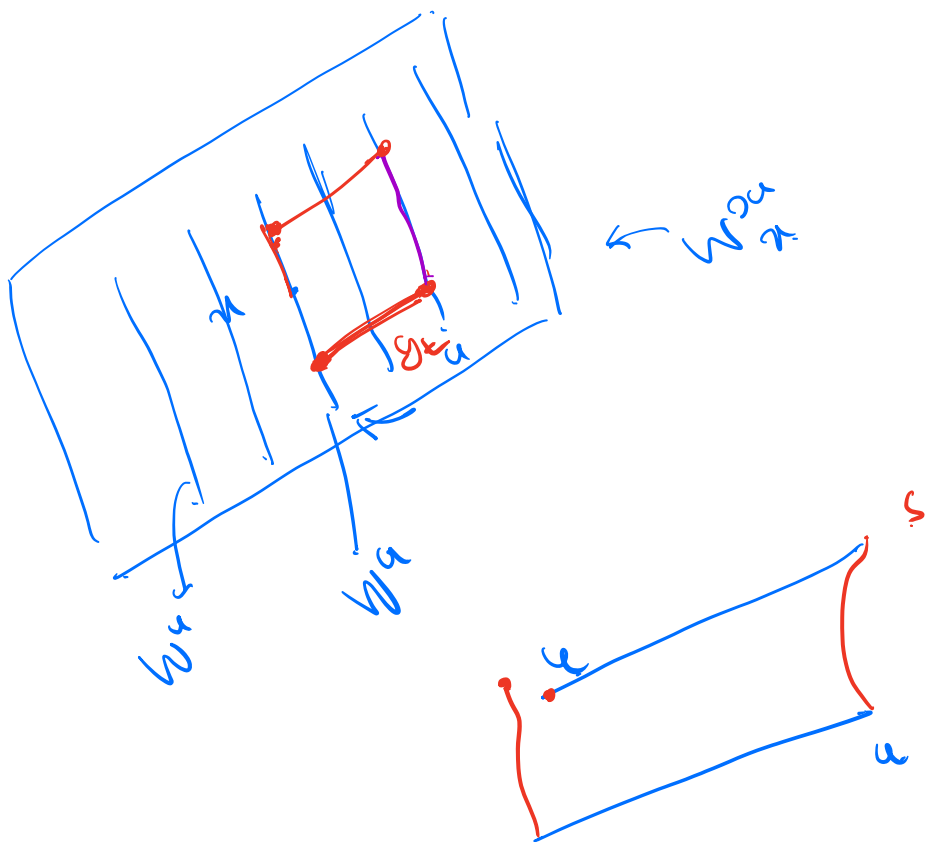
$$D_x g_t \in E_x^s = E_{g_t x}^s$$

# Anosov flows: Properties

- ① The distribution  $E^u$  is integrable, and the corresponding foliation is called the *unstable* foliation. The leaf that passes through  $x \in M$  is denoted by  $W_x^u$ . strongly unstable  $W_x^u$
- ② The distribution  $W^u \oplus \mathbb{R}v$  is integrable, and the corresponding foliation is called the *center-unstable* foliation. The leaf that passes through  $x \in M$  is denoted by  $W_x^{0u}$ .
- ③ The distribution  $E^s$  is integrable, and the corresponding foliation is called the *stable* foliation. The leaf that passes through  $x \in M$  is denoted by  $W_x^s$ .
- ④ The distribution  $E^s \oplus \mathbb{R}v$  is integrable, and the corresponding foliation is called the *center-stable* foliation. The leaf that passes through  $x \in M$  is denoted by  $W_x^{cs}$ . ↖
- ⑤ The leaves of unstable foliation are  $C^1$ , and they vary in a Hölder-continuous way. ↖



$$T_x W_x^u = E_x^u$$



$[\cdot, \cdot]$ ,  $\theta(\cdot, \cdot)$ , and flow box

## Definition

The unstable metric  $d^u$  on  $W^u$  is defined by considering the restriction of Riemannian metric to unstable leaf  $W^u$ . This induces a topology on  $W^u$  that is different from the topology induced from  $M$ .

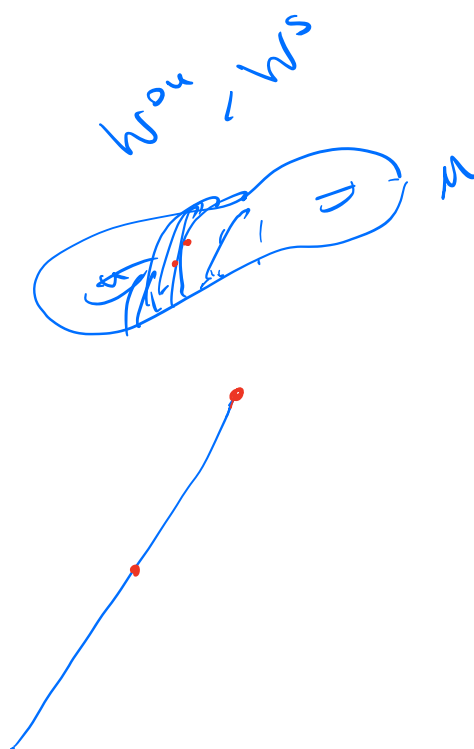
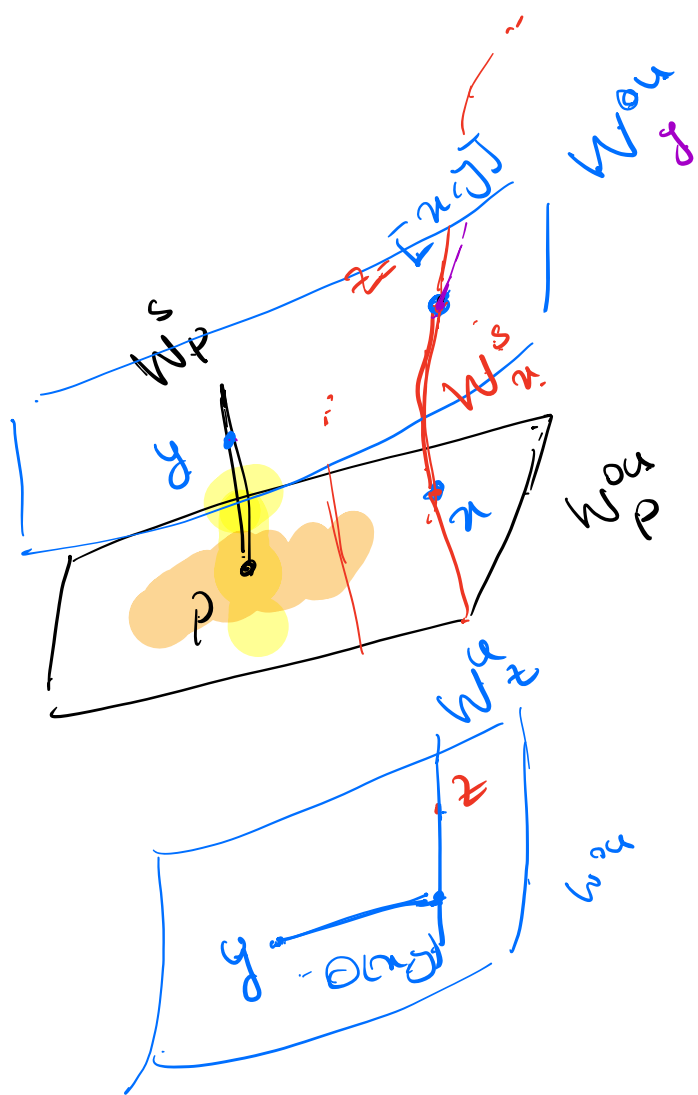
For  $p \in M$  and  $x \in W_x^{0u}$ ,  $y \in W^s$  close to  $p$ ,

- 1 Set  $z := [x, y] := W_x^{s, \text{loc}} \cap W_y^{0u, \text{loc}}$ .
- 2 Set  $\theta(x, y) \in \mathbb{R}$  such that  $g_t(y) \in W_z^u$  for  $t := \theta(x, y)$ .

## Definition

A set of the form  $[U, V]$  for small neighborhoods  $p \in U \subset W^{0u}$  and  $p \in V \subset W^s$  is called a *flow box*.

$$\{[x, y] : x \in U, y \in V\}$$



# Invariant measures

- 1 There exists a unique  $g_t$ -invariant measure in the Lebesgue class, called the *smooth invariant measure*.
- 2 Denoting the topological entropy of  $g_t$  by  $\delta$ , there exists a unique  $g_t$ -invariant measure with entropy  $\delta$ . This measure is called the Bowen-Margulis measure, and is denoted by  $\mu_{\text{BM}}$  (or just  $\mu$ ).

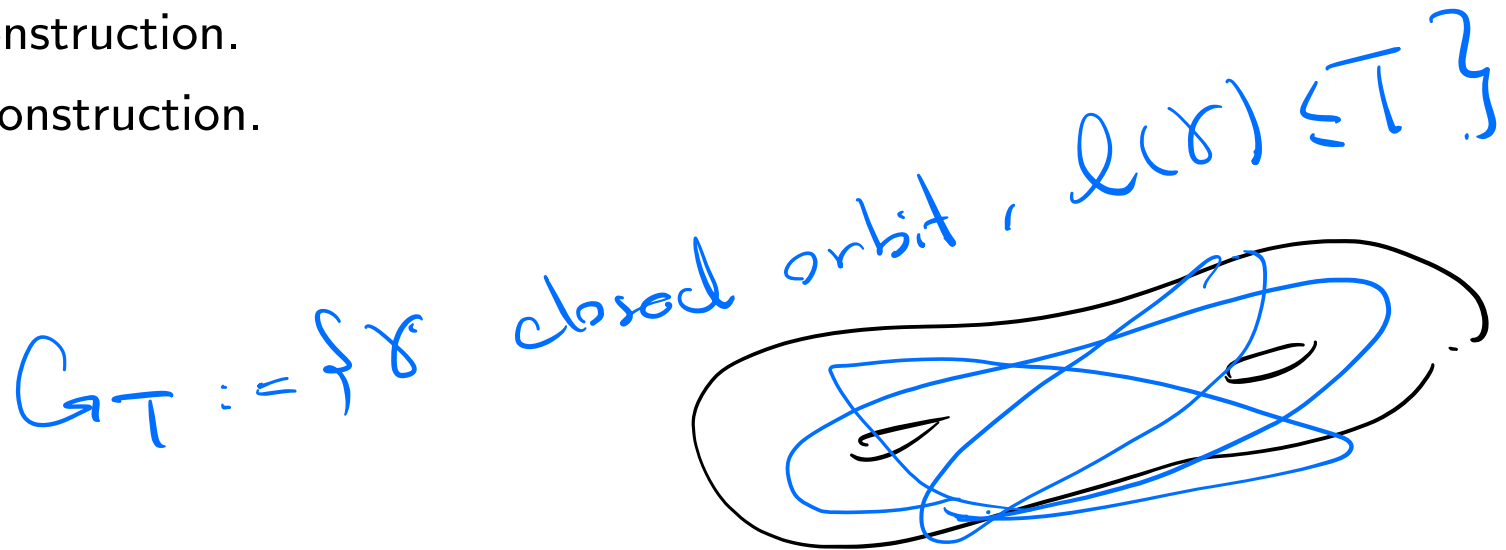
$$(X, \mu, T)$$

Gibbs measure

$$\mu, g_t, \mu$$

# Two constructions of Bowen-Margulis measure

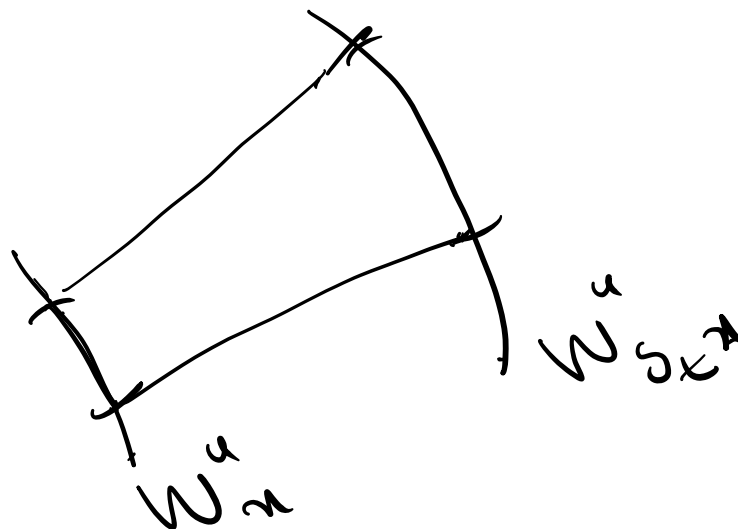
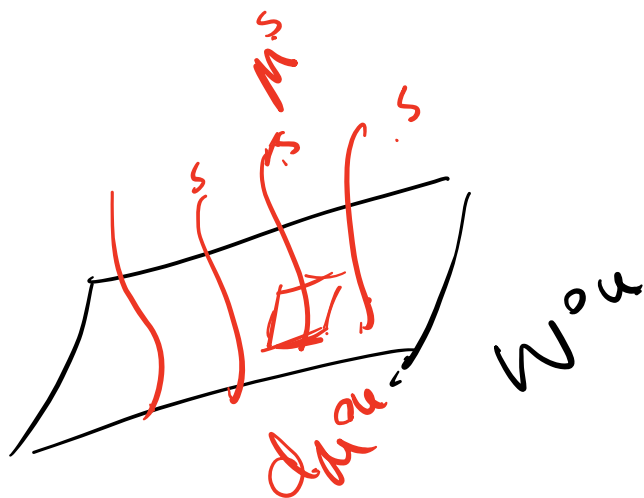
- 1 Bowen's construction.
- 2 Margulis' construction.



# Bowen-Margulis measure: Properties

There are a family of measures  $\mu_x^u$  (resp.  $\mu_x^{0u}, \mu_x^s$ ) supported on  $W_x^u$  (resp.  $W_x^{0u}, W_x^s$ ) for  $x \in M$  such that

- ①  $d\mu^u(g_t x) = e^{\delta t} \mu^u(x)$ .
- ②  $d\mu^{0u}(y) = e^{\delta t} d\mu^u(x) dt$  for  $y = (x, t)$ .
- ③ Locally,  $d\mu_{\text{BM}} = d\mu^{0u} d\mu^s$ .
- ④ In a flow box  $[U, V]$ ,  $\mu_{\text{BM}}(z) = e^{-\delta\theta(x,y)} d\mu^{0u}(x) d\mu^s(y)$  for  $z = [x, y]$ .



# The Presentation

We present the paper "A new description of the Bowen-Margulis measure" by Ursula Hamenstädt. The paper is written for  $g_t$  the geodesic flow on a manifold of (variable) negative curvature; however, the arguments carry over to general Anosov flows. We present the argument for Anosov flows.

To simplify the notation, we assume  $\kappa = 1$  in the definition of Anosov flow. We also assume  $\dim E^s = \dim E^u = 1$ .

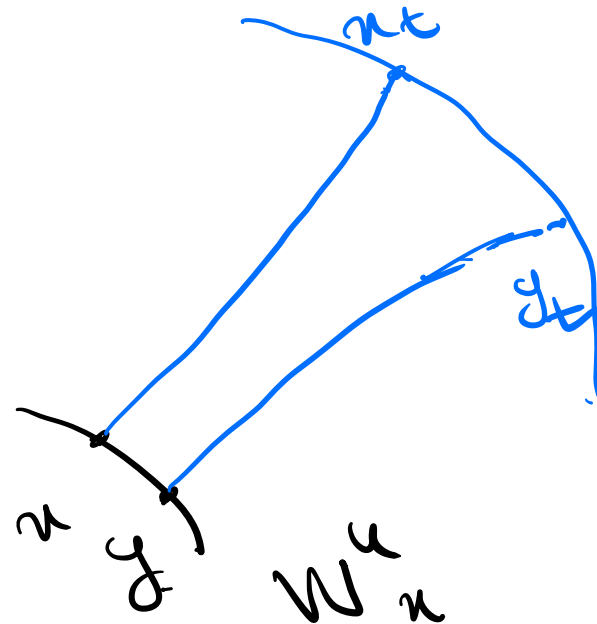
**Notation.** For functions  $f(x), g(x)$  we write  $f(x) \prec g(x)$  if there exists a constant  $C > 0$  such that  $f(x) \leq Cg(x)$  for all  $x$  in the domain. We write  $f(x) \asymp g(x)$  if  $f(x) \prec g(x)$  and  $f(x) \succ g(x)$ . We may extend this notation to measures.

$\mu, \nu$  on  $X$ ,  $\mu \prec \nu$  if  $\exists C > 0$  for every  $A \in \mathcal{B}(X)$ ,  $\mu(A) \leq C\nu(A)$ .  
 $\mu \asymp \nu$

# Hamenstädt distance

Hamenstädt distance  $d_H^u(\cdot, \cdot)$  is defined on unstable foliation  $W^u$ .

- 1 The definition.
- 2 Verifying the triangle inequality.
- 3 For  $r < 1, x \in M$ ,  $\mu_{BM}^u(B_H^u(x, r)) \asymp r^\delta = e^{-T\delta}$ .



$$d_H^u(x, y)$$

$$\text{Fix } x, y \in W^u,$$

$$x_t := g_t x, y_t := g_t y$$

$$d_t(x_t, y_t) := d^u(x_t, y_t)$$

$$\exists T = T(x, y) \text{ s.t. } d_T(x_T, y_T) = 1,$$

$$\text{we define } d_H^u(x, y) := e^{-T}$$

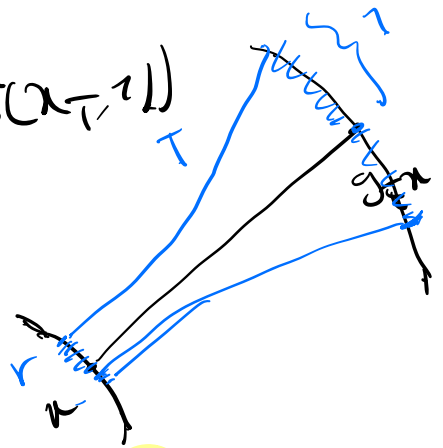
Let  $T$  be such that  $r = e^{-T}$

then  $B^u_{Ham}(x, e^{-T}) = g_{-T}(B^u(x_T, 1))$

$$\mu^u_{BM}(B^u_{Ham}(x, e^{-T})) =$$

$$e^{-\delta T} \mu_{BM}(B^u(x_T, 1)) \approx 1$$

$$\{\mu(B^u(y, 1))\}_{y \in M}$$



# Hausdorff measure

For a metric space  $(X, d)$ , the  $\delta$ -dimensional Hausdorff measure  $\mu_H$  is defined by

$$\mu_H(A) := \sup_{\epsilon \rightarrow 0} \inf \left\{ \sum_{i=0}^{\infty} r_i^\delta \mid r_i \leq \epsilon, \text{ and } A \subset \bigcup B(x_i, r_i) \right\}$$

for  $A$  a Borel subset of  $X$ .

## Definition

We denote the  $\delta$ -dimensional Hausdorff measure on  $\underline{W^u}$  with respect to  $\underline{d_H^u}$  by  $\underline{\mu_H^u}$ .

Theorem.  $\mu_H^u = c \mu_{BM}^u.$

$$\underline{\mu_{BM}^u \asymp \mu_H^u}$$

$\exists C$

①  $\mu_{BM}^u \prec \mu_H^u.$  

②  $\mu_H^u \prec \mu_{BM}^u.$

$$C^{-1} \mu_H^u(A) \leq \mu_{BM}^u(A) \leq C \mu_H^u(A)$$

$$* \mu_{BM}^u \stackrel{?}{\leq} \mu_H^u$$

$$\mu_{BM}(B_H^u(x, r)) \leq r^s$$

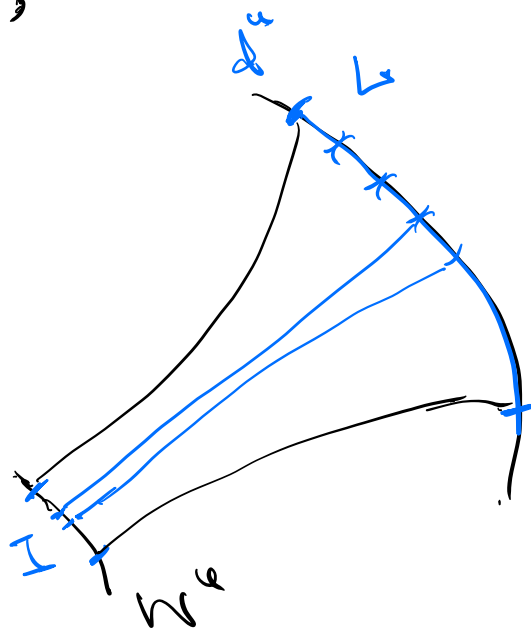
$$A \in \mathcal{B}(W_x^u), \quad A \subset \bigcup_i B_H^u(x_i, r_i)$$

$$\begin{aligned} \mu_{BM}^u(A) &\leq \sum_i \mu_{BM}(B_H^u(x_i, r_i)) \\ &\leq \sum_{i=1}^{\infty} r_i^s \end{aligned} \quad \swarrow W_x^u$$

$$\mu_H^u \prec \mu_{BM}^u$$

Construct "good" covers of  $A \subset \mathcal{B}(W^e)$

Fix an interval  $I \subset W^u$ , let  $T$  be large enough.

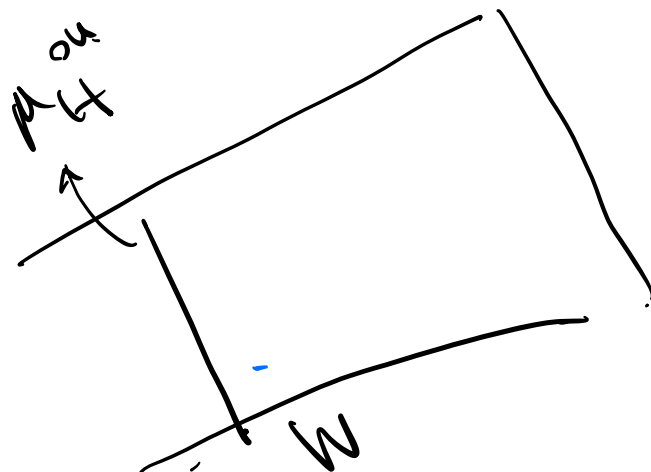
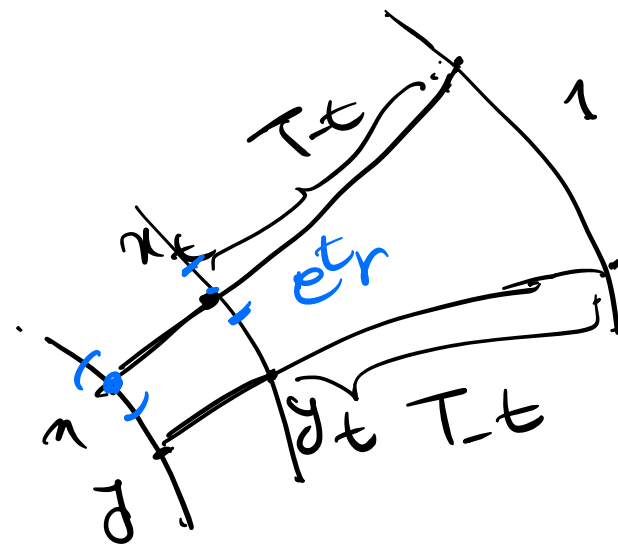


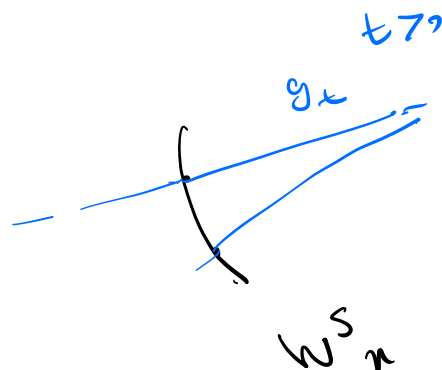
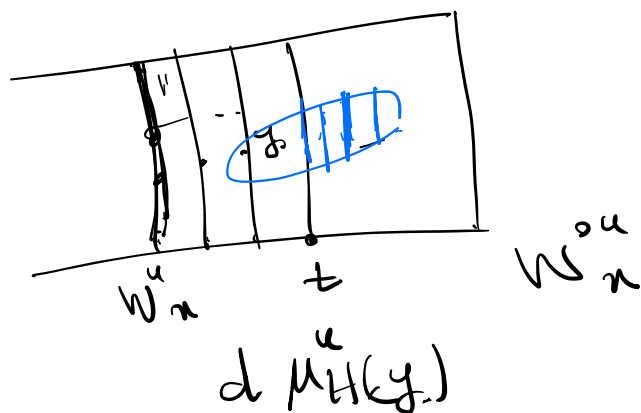
# Hausdorff measure on $M$

- 1 For  $x, y \in W^u$ ,  $d_H^u(g_t x, g_t y) = e^t d_H(x, y)$ . Hence  $d\mu_H^u(g_t x) = e^{\delta t} d\mu_H^u(x)$ .
- 2 Define  $\mu_H^{0u}$  on  $W^{0u}$  by  $d\mu_H^{0u}(y) = e^{\delta t} d\mu_H^u(x) dt$ .
- 3 Define  $\mu_H^s$ .
- 4 Define  $\mu_H$  on  $M$  so that locally,  $d\mu_H = d\mu_H^{0u} d\mu_H^s$ . Then, in a flow box  $[U, V]$ ,  $\mu_H(z) = e^{-\delta\theta(x,y)} d\mu_H^{0u}(x) d\mu_H^s(y)$  for  $z = [x, y]$ .

$$d_H(x, y) = e^{-T}$$

$$d_H^t(x_t, y_t) = e^{-(T-t)}$$





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$$\mu_H^u \preceq \mu_{BM}^u, \quad \mu_H^{ou} \preceq \mu_{BM}^{ou}$$

$$\mu_H^s \preceq \mu_{BM}^s$$

$$\mu_H \preceq \mu_{BM} \quad \rightsquigarrow \quad \frac{d\mu_H}{d\mu_{BM}} \text{ is}$$

$$g_t\text{-invariant} \rightarrow \mu_H = c \mu_{BM}$$

$\mu_{\text{BM}}$  and  $\mu_H$  are the same, up to a constant

- 1 There exists a constant  $C > 0$  such that  $\mu_H = C\mu_{\text{BM}}$ .
- 2 There are constants  $C^u$  and  $C^s$  such that  $\mu_H^u = C^u \mu_{\text{BM}}^u$  and  $\mu_H^s = C^s \mu_{\text{BM}}^s$ .

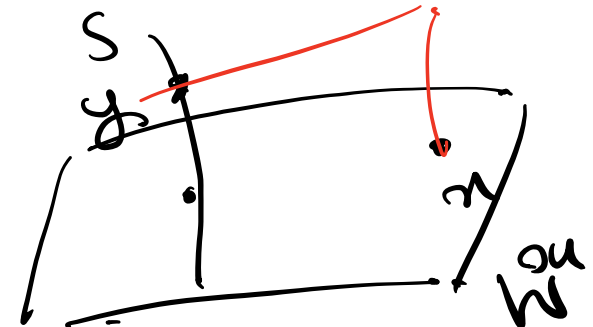
## Question

Find  $C^u, C^s$ !

in a flow box we have

$$d\mu_{\text{BM}}(z) = e^{\theta(x,y)} d\mu_{\text{BM}}^u(x) d\mu_{\text{BM}}^s(y)$$

$$d\mu_H(z) = e^{\theta(x,y)} d\mu_H^u(x) d\mu_H^s(y)$$



$$\mu \otimes v = c \mu' \otimes v' \rightarrow \begin{aligned} \mu &= c_1 \mu' \\ v &= c_2 v' \end{aligned}$$