

Setting

(X, d) a σ -compact metric space, \mathcal{B} its Borel σ -algebra, μ , a probability measure on (X, \mathcal{B}) .

Mixing for \mathbb{Z} -actions

Definition

Let (X, \mathcal{B}, μ, T) be a Borel probability measure preserving system. We say that it is mixing if for every $A, B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B)$$

Equivalently:

- ▶ for every $f, g \in L^2(\mu)$ we have

$$\lim_{n \rightarrow \infty} \int_X f(x)g(T^n x) d\mu(x) = \int f d\mu \int g d\mu.$$

- ▶ U_T^n converges in the weak operator topology to integration against constant functions
- ▶ The sequence of measures $(id \times T^n)_* \mu$ converge in the weak-* topology to $\mu \otimes \mu$.

Definition

We say (X, \mathcal{B}, μ, T) is *mixing of order k* if for every $A_1, \dots, A_k \in \mathcal{B}$

$$\lim_{n_i - n_j \rightarrow \infty} \mu(T^{-n_1} A_1 \cap \dots \cap T^{-n_k} A_k) = \mu(A_1) \cdot \dots \cdot \mu(A_k).$$

Question

(Rokhlin) *Does 2-mixing imply 3-mixing?*

Partial progress

1. True for *Rank 1 systems* (Kalikow)
2. True for *finite rank systems* (Ryzhikov)
3. True for systems with singular (with respect to Lebesgue) spectral type (Host)
4. Follows from the Hopf argument (Coudène-Hasselblatt-Troubetzkoy).

Mixing for group actions

Let G be a completely metrizable topological group and for each $g \in G$ let T_g be a measure preserving map of (X, μ) . Further assume $T_{g_1} T_{g_2} = T_{g_1 g_2}$.

We suppress the T from now on.

Definition

We say $\mathbb{X} = (X, \mathcal{B}, \mu, G)$ is mixing if for every $A, B \in \mathcal{B}$ we have

$$\lim_{g \rightarrow \infty} \mu(A \cap g^{-1}B) = \mu(A)\mu(B).$$

Definition

We say $\mathbb{X} = (X, \mathcal{B}, \mu, G)$ is mixing of order k if for every $A_1, \dots, A_k \in \mathcal{B}$ we have

$$\lim_{g_i g_j^{-1} \rightarrow \infty} \mu(g_1 A_1 \cap \dots \cap g_k A_k) = \mu(A_1) \dots \mu(A_k).$$

Theorem

(Ledrappier) When $G = \mathbb{Z}^2$, 2-mixing does not imply 3-mixing.

Theorem

(Mozes) When $G = SL(2, \mathbb{R})$, 2-mixing implies mixing of all orders and in particular 3-mixing.

Mozes proved this result in much larger generality. We present this special case for concreteness

Idea of proof that 2-mixing implies 3-mixing

- 1) It suffices to show that if $\vec{g}_n = (id, \alpha_n, \beta_n) \in G^3$ is a sequence so that $\alpha_n, \beta_n, \alpha_n^{-1}\beta_n \rightarrow \infty$ and $(\vec{g}_n)_*\mu$ weak-*converges to a measure σ then $\sigma = \mu^3$.
- 2) Let $(Y, \nu), (Z, \eta)$ be probability measure spaces, and τ be coupling of them. If (Z, η, T) is ergodic and τ is $(id \times T)$ -invariant then $\tau = \nu \times \eta$.
- 3) Let σ be as in 1). Either σ is (id, ϕ, ψ) -invariant where $T : X \times X$ by $T(x, y) = (\phi x, \psi y)$ is $\mu \times \mu$ -ergodic OR σ is (id, id, ψ) invariant where $T = \psi$ acts ergodically on (X, μ) .

Proof of theorem

- By our assumption that the action of G is 2-mixing, the projection of σ onto any two coordinates is $\mu \times \mu$.
- If σ is (id, id, ψ) -invariant, then applying 2) to $(Y, \nu) = (X \times X, \mu \otimes \mu)$ and $(Z, \eta) = (X, \mu)$ gives the theorem.
- If σ is (id, ϕ, ψ) -invariant then applying 2) to $(Y, \nu) = (X, \mu)$ and $(Z, \eta) = (X \times X, \mu \otimes \mu)$ gives the theorem.

Justification of 1)

- Assume we have a sequence \vec{g}_n as in 1).
- By the compactness of measures with total variation at most 1, we may choose a subsequence where $(\vec{g}_{n_i})_*\mu$ converges to something.
- G is mixing iff this is automatically $\mu \otimes \mu \otimes \mu$.

Justification of 2)

Proposition

Let (Y, ν) , (Z, η) be probability measure spaces, and τ be coupling of them. If (Z, η, T) is ergodic and τ is $(id \times T)$ -invariant then $\tau = \nu \times \eta$.

By disintegration of measures applied to projection onto Y , there are probability measure τ_y so that $\tau_y(\{y\} \times Z) = 1$ and $\int_Y \tau_y d\nu = \tau$.

We may identify τ_y with measures $\tilde{\tau}_y$ on Z and by assumption these are T -invariant.

Because the projection onto Z of τ is η , we have $\eta = \int_Y \tilde{\tau}_y d\nu$ where the $\tilde{\tau}_y$ are T -invariant.

But η is an extreme point in the convex set of T -invariant probability measures.

Thus $\tilde{\tau}_y = \eta$ for ν -almost every y .

Finally $\tau = \int_Y \tau_y d\nu = \int_Y (\delta_y \otimes \eta) d\nu = \nu \otimes \eta$.

Justification of 3) invariance prelimit

Lemma

$(\vec{g}_n)_*\mu$ is $(h, \alpha_n h \alpha_n^{-1}, \beta_n h \beta_n^{-1})$ -invariant for all n .

$$\int_{X^3} f d((\vec{g}_n)_*\mu) = \int_X f(x, \alpha_n x, \beta_n x) d\mu \quad (1)$$

$$= \int_X f(hx, \alpha_n hx, \beta_n hx) d\mu. \quad (2)$$

Observe that $(h, \alpha_n h \alpha_n^{-1}, \beta_n h \beta_n^{-1})(id, \alpha_n, \beta_n) = (hx, h\alpha_n x, h\beta_n x)$.
Giving invariance: $(h, \alpha_n h \alpha_n^{-1}, \beta_n h \beta_n^{-1})$ changes one description of $(g_n)_*\mu$ to another.

Justification of 3) invariance in the limit

Lemma

If $(h_n, \alpha_n h_n \alpha_n^{-1}, \beta_n h_n \beta_n^{-1})$ converges to (θ, ϕ, ψ) then σ is (θ, ϕ, ψ) -invariant.

Let $f \in C_c(X^3)$ and $F(x, y, z) = f(\theta x, \phi y, \psi z) \in C_c(X)$.

$$\int_{X^3} f d\sigma = \lim_{n \rightarrow \infty} \int_X f(x, \alpha_n x, \beta_n x) d\mu \quad (3)$$

$$= \lim_{n \rightarrow \infty} \int_X F(x, \alpha_n x, \beta_n x) d\mu \quad (4)$$

$$= \int_{X^3} F d\sigma \quad (5)$$

Looking for a limit A

Proposition

Assume that whenever $g_n \in SL(2, \mathbb{R})$ goes to infinity we have that for any neighborhood of id , U and bounded set B ,

$$g_n U g_n^{-1} \not\subset B$$

for all large n , then there is a sequence $h_n \in SL(2, \mathbb{R})$ so that

1. $h_n \rightarrow id$
2. $\max\{\|\alpha h_n \alpha_n^{-1}\|, \|\beta_n h_n \beta_n\|^{-1}\} = 2$

The proposition says that after passing to a subsequence, we may assume σ is (θ, ϕ, ψ) -invariant with $\theta = id$ and at least one of ϕ, ψ not equal to the identity.

Proof of Proposition

- Let $\Phi : SL(2, \mathbb{R}) \rightarrow [1, \infty)$ by
 $\Phi_n(h) = \max\{\|\alpha h_n \alpha_n^{-1}\|, \|\beta_n h_n \beta_n\|^{-1}\}.$
- Φ_n is continuous.
- Under our assumption $[1, 2] \subset \Phi_n(U)$ for all large n .
- So for all large n we may choose $h_n \in U$ so that
 $\max\{\|\alpha h_n \alpha_n^{-1}\|, \|\beta_n h_n \beta_n\|^{-1}\} = 2.$
- Choosing shrinking U we may assume $h_n \rightarrow id$.

A little bit more

Lemma

ϕ and ψ have both eigenvalues 1 and in particular any non-identity element has to be μ -mixing.

- $h_n \rightarrow id$ means the eigenvalues of h_n converge to 1.
- The eigenvalues of h_n are the eigenvalues of $\alpha_n h_n \alpha_n^{-1}$ and $\beta_n h_n \beta_n^{-1}$.
- Because eigenvalues change continuously the eigenvalues of ϕ and ψ are 1.
- Because G is mixing, any element of G that generates an unbounded subgroup is mixing.

Looking for a limit B

Proposition

Whenever $g_n \in SL(2, \mathbb{R})$ goes to infinity we have that for any neighborhood of id , U and bounded set B ,

$$g_n U g_n^{-1} \not\subset B$$

for all large n .

This is a computation.

Doing the computation

Let $g_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Observe $g_n \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} g_n^{-1} =$

$$\begin{pmatrix} a_n d_n - a_n c_n s - b_n c_n & -a_n b_n + a_n s^2 + a_n b_n \\ c_n d_n - c_n^2 s - c_n d_n & -b_n c_n a + a_n c_n s + a_n d_n \end{pmatrix} =$$
$$\begin{pmatrix} 1 - a_n c_n s & a_n s^2 \\ -c_n^2 s & 1 + a_n c_n s \end{pmatrix}.$$

For this to be bounded for all small s we need that a_n and c_n are bounded (in n).

Doing the computation II

Similarly, $g_n \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} g_n^{-1} =$

$$\begin{pmatrix} 1 + b_n d_n & b_n^2 s \\ d_n^2 s & 1 - b_n d_n s \end{pmatrix}.$$

For this to be bounded for all small s we need that b_n and d_n are also bounded.

This contradicts that g_n is unbounded.

Recap of 3)

So σ is (id, ϕ, ψ) -invariant with at least one of $\phi, \psi \neq id$. If only one is non-identity, it is mixing and thus ergodic and so we have one option for 3). Otherwise they are both mixing, so there product is ergodic and we have the other option for 3).