Disjointness in Ergodic Theory, Minimal Sets, and a Problem in Diophantine Approximation[1]

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0. Summary. The objects of ergodic theory—measure spaces with measure-preserving transformation groups—will be called *processes,* those of topological dynamics—compact metric spaces with groups of homeomorphisms—will be called *flows.* We shall be concerned with what may be termed the "arithmetic" of these classes of objects. One may form *products* of processes and of flows, and one may also speak of *factor processes* and *factor flows.* By analogy with the integers, we may say that two processes are *relatively prime* if they have no non-trivial factors in common. An alternative condition is that whenever the two processes appear as factors of a third process, then their product too appears as a factor. In our theories it is unknown whether these two conditions are equivalent. We choose the second of these conditions as the more useful and refer to it as *disjointness.*

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ELEMENTARY PROOF OF FURSTENBERG'S DIOPHANTINE RESULT

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ABSTRACT. We present an elementary proof of a diophantine result (due to H. Furstenberg) which asserts (in a special case) that for every irrational α the set $\{2^m 3^n \alpha | m, n \geq 0\}$ is dense modulo 1. Furstenberg's original proof employs the theory of disjointness of topological dynamical systems.

1. INTRODUCTION

Throughout the paper by a *semigroup* we mean an infinite subset of positive integers which is closed under multiplication. Two integers p, q are called multiplicitively independent if both are ≥ 2 and the ratio of their logarithms $(\log p)/(\log q)$ is irrational. The equivalent requirement is that p and q should not be integral powers of a single integer.

Furstenberg's topological ×2 ×3 theorem
(following Boshernitzan)

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Thm (Furstenberg '67)

$\mathbb{T} = \mathbb{R}/\mathbb{Z}$ $\quad m_2 : \mathbb{T} \to \mathbb{T} \quad m_2(x) = 2x \bmod \mathbb{Z}$

$m_3 : \mathbb{T} \circlearrowleft \quad m_3(x) = 3x \bmod \mathbb{Z}$

If $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ then $\{ m_2^{\ell} \circ m_3^{k}(x) : k, \ell \in \mathbb{N} \}$

is dense.

Example ① $\alpha = \frac{1}{5}$

$\left\{ \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5} \right\}$ is m_2, m_3-inv.

② $\left\{ \frac{1}{15}, \frac{2}{15}, \frac{4}{15}, \frac{8}{15}, \frac{1}{5}, \frac{2}{5}, \frac{4}{5}, \frac{3}{5} \right\}$

is m_2, m_3 -inv.

(3) $S \subset \mathbb{N}$ is a <u>semigroup</u> if

$\alpha, \beta \in S \Rightarrow \alpha\beta \in S$.

(4) Suppose $S = \mathbb{N}$

$S\alpha$ dense if $\alpha \notin \mathbb{Q}$

<u>Def</u> $p, q \in \mathbb{N}$ are called <u>multiplicatively ind.</u> if $\frac{\log p}{\log q} \notin \mathbb{Q}$

$$\left(\frac{\log p}{\log q} = \frac{r}{s} \iff \log p^r = \log q^s \right.$$

$$\left. \iff p^r = q^s \right)$$

2, 3 are mult. ind.

6, 3 " " " .

<u>Def</u> A semigroup $S \subset \mathbb{N}$ is <u>non-lacunary</u> if S contains two multiplicatively independent numbers

<u>Thm</u> If $S \subset \mathbb{N}$ non-lacunary and $K \subset \mathbb{T}$ is closed, S-invariant, infinite then $K = \mathbb{T}$.

<u>Prop</u> The following are equivalent

(1) S is non-lacunary

(2) there is no $r \in \mathbb{N}$ s.t.

$$S \subset \{r^k : k \in \mathbb{N}\}$$

(3) If we write $S = \{s_1, s_2, \dots\}$

with $\dots < s_i < s_{i+1} < \dots$

$$\frac{s_{i+1}}{s_i} \xrightarrow[i \to \infty]{} 1$$

<u>Pf</u> (Ex with hints)

Idea for (1) $\Rightarrow$ (3).

WNLG $S = \{p^k q^\ell : k, \ell \in \mathbb{N}\}$

where $\frac{\log p}{\log q} \notin \mathbb{Q}$

Given $\varepsilon>0$ want to find i_0 s.t.

$\forall i \geq i_0 \quad \frac{s_{i+1}}{s_i} < 1+\varepsilon$

Find $r, s \in \mathbb{N}$ so that
r', s'

$$\left|\frac{r}{s} - \frac{\log p}{\log q}\right| < \frac{\varepsilon}{2s \log q}$$
$\frac{r'}{s'}$

$$\frac{r'}{s'} < \frac{\log p}{\log q} < \frac{r}{s}$$

$s_i = p^k \cdot q^l \qquad p^s < q^r \quad \frac{q^r}{p^s} < 1+\varepsilon$

$$s_{i+1} \leq \frac{p^k}{p^s} q^l \cdot q^r = q^{l+r} p^{k-s} < (1+\varepsilon) s_i$$

From now on $S \subset \mathbb{N}$ non-lacunary semigp

$K \subset \mathbb{T}$ is <u>S-inv</u> if $\forall s \in S$, $s(K) \subset K$.

<u>Lemma 1</u> If $K \subset \mathbb{T}$ closed, infinite, S invariant, and contains a non-isolated rational number then $K = \mathbb{T}$.

<u>Lemma 2</u> If $K \subset \mathbb{T}$ closed, S-invariant, nonempty, then K contains a rational point.

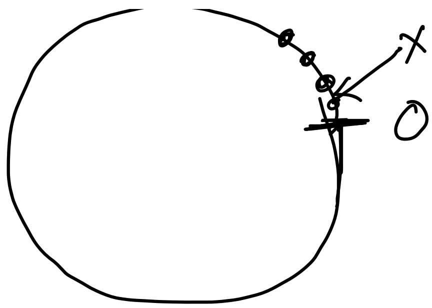
<u>Pf</u> of thm assuming lemmas 1 and 2.

Let K be infinite closed S-inv.

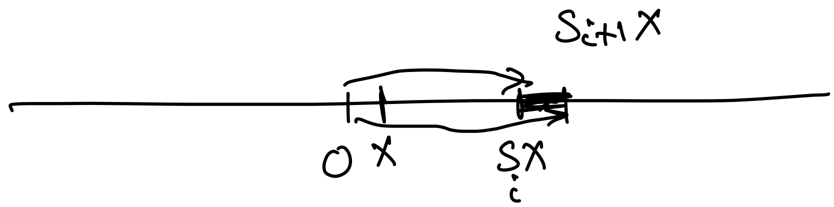
Want to show that $K = \mathbb{T}$.

Use Lemma 2 with K' instead of K, where K' is the set of accumulation points of K. By Lemma 2 K' contains a rational point. Applying Lemma 1 we get the theorem.

<u>Pf of Lemma 1</u>. Suppose 0 is an accumulation point of K.



If $s \in S$, $x < \frac{1}{s}$, sx is obtained by multiplying in $\mathbb{R}$.



$\frac{s_{i+1}}{s_i} < 1+\varepsilon$ $\quad s_{i+1}x - s_i x =$

$< s_i x(1+\varepsilon-1)$

$= s_i x \varepsilon < \varepsilon$

Pf of Lemma 2 By contradiction assume K contains no rational points, is nonempty and S-inv. We will show K is dense to get a contradiction. Let $\varepsilon > 0$. Let $p, q \in S$ mult. ind. in S. Let t be an integer satisfying

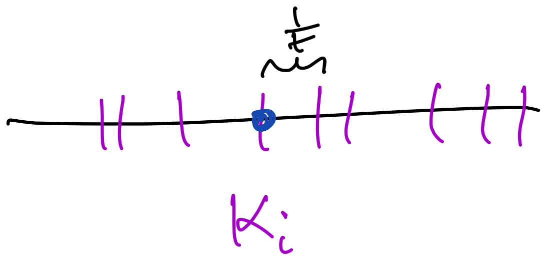
$t\varepsilon > 1$ $\quad \gcd(t,p) = 1 = \gcd(t,q)$

There is $u \in \mathbb{N}$ s.t. $p^u \equiv q^u \equiv 1 \pmod{t}$.

Define inductively

$$K = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_{t-1}$$

where $K_{i+1} = \{ x \in K_i : x + \frac{1}{t} \pmod{\mathbb{Z}} \text{ in } K_i \}$



Then: • each K_i is inv. under both p^u and q^u.

By induction. $\quad \frac{p^u}{t} - r = \frac{1}{t}$ for some r

• each K_i is closed.

• if $K_i \neq \emptyset$ then K_i is infinite.

(If $K_i \neq \emptyset$ then K_i contains an irrational x_i by contradiction assumption,

$(p^u)^m (q^u)^n \cdot x$ is infinite and in K_i.

- ~~?~~ $K_{i+1} \neq \emptyset$.

~~?~~ To see this, consider $D_i = K_0 - K_i$

D_i closed (since K_i compact).

D_i has an accumulation at 0 because K_i is infinite. D_i is inv. under S = semigroup generated by p^u, q^u

$\Longrightarrow$ (Lemma 1) $D_i = \mathbb{T}$

$\Longrightarrow \frac{1}{t} \in D_i \Longrightarrow K_{i+1} \neq \emptyset$.

So $K_{t-1} \neq \emptyset$.

Let $y \in$ ~~K_i~~ K_{t-1}. Then

$$\underset{K_{t-1}}{\underset{\cap}{y}}, \underset{K_{t-2}}{\underset{\cap}{y - \tfrac{1}{t}}}, y - \tfrac{2}{t}, \dots, y - (1 - \tfrac{1}{t}) \in K_0$$

$\Longrightarrow K_0$ is $\frac{1}{t}$ dense $\Longrightarrow K_0$ is

ε-dense.

Furstenberg x2 x3 conjecture

Let S be a non-lacunary semigroup.
Any S-inv. measure is either atomic or Lebesgue.

atomic: $\exists\, x_0$ s.t. $\mu(\{x_0\}) > 0$.

invariant: $\forall s \in S$, $\forall A \subset \mathbb{T}$ Borel

$$\mu(A) = \mu(s^{-1}(A)).$$