1. Ergodic Theory of rotations of the circle

1.1. Preliminaries. Let $\alpha \in (0,1) \setminus \mathbb{Q}$. We define a dynamical system, rotation by α, $R_\alpha : [0,1) \to [0,1)$ by $R_\alpha(x) = x + \alpha - \lfloor x + \alpha \rfloor$. Observe that R preserves Lebesgue measure, meaning $R_* \lambda = \lambda$. That is $\lambda(A) = \lambda(R^{-1}A)$ for all measurable sets A. In this note we think of $[0,1)$ as being a circle of radius 1 by considering the following distance: $d(x,y) = \min\{|x-y|, 1-|x-y|\}$. With this metric R_α is an isometry.

Defintion 1. *Let $T : X \to X$ preserve μ. T is* ergodic with respect to μ *if $\mu(T^{-1}A \Delta A) = 0$ implies $\mu(A)\mu(A^c) = 0$.*

Often authors will not require that T is μ-measure preserving. In this case, it is just a condition on the measure class.

Theorem 1. *Irrational rotations are ergodic with respect to Lebesgue measure.*

Proof. This is your homework! □

We also have the following:

Theorem 2. *Let $f \in C_c(S^1)$, then for any irrational α and $x \in [0,1)$*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=0}^{N-1} f(e^{2\pi i j x}) = \int f d\lambda.$$

Where λ is uniform measure on S^1 normalized so that $\lambda(S^1) = 1$.

To prove this we use the so called Weyl Critertion:

Lemma 1. *(Weyl's Criterion) Let $\{p_j\}_{j=0}^{\infty} \subset S^1$. $\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} e^{2in\pi p_k x} = 0$ for every $n \in \mathbb{Z} \setminus 0$ (equivalently $n \in \mathbb{N}$) iff $\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} f(e^{2in\pi p_k x}) = \int f d\lambda$ for all $f \in C(S^1)$.*

Corollary 1. If α is irrational, $x \in [0,1)$ and $a < b$ then $\lim_{N \to \infty} \sum_{j=0}^{N-1} \chi_{[a,b]}(R_\alpha^j x) = b - a$.

Theorems 1 and 2 suggest the so called Birkhoff ergodic theorem:

Theorem 3. *(Birkhoff) Let $(X, \mathcal{B}, \mu)$ be a probability space. Let T be μ ergodic and $f \in L^1(\mu)$. Then*

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} f(T^i x) = \int f d\mu$$

for μ a.e. x. Moreover, let $s_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$. Then s_n converges in $L^1(\mu)$ to $\int f d\mu$.

Note: If T is just measure preserving then the limit exists for almost every x. The resulting function is T invariant.

Corollary 2. Let $(X, \mathcal{B}, \mu)$ be a probability space. Let T be μ ergodic and moreover assume (X, d) is a σ-compact metric space and μ is a Borel measure then there exists G_μ with $\mu(G_\mu) = 1$ and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(T^i x) = \int f d\mu$$

for all $f \in C_c(X)$ and $x \in G_\mu$.

G_μ is the set of *μ-generic points*.

1.2. **Chebyshev's Theorem.** Observe that there exists a so that $R^i(0)$ is further from 0 than $R0$ for $1 < i < a$ $R^a(0)$ is closer to 0 then $R(0)$. In this case $R^a(0) \in (1-\alpha, 1)$ and $R^{a+1}(0)$ crosses 1. Now $\frac{1}{a+1} < \alpha < \frac{1}{a}$. Also if $\delta = 1 - a_1\alpha$ then $R^{i+a}0 = R^i(0) - \delta$. So on an interval of size α we act by rotation by $-\delta$. Rescaling we have rotation by $-(\frac{1}{\alpha}(1 - a_1\alpha)) = -(\frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor)$.

Defintion 2. *Let $G(\alpha) = \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor$. This is the Gauss map of α.*

The Gauss map does not preserve Lebesgue but it does preserve a mutually absolutely continuous measure: $\mu(A) = \int_A \frac{dx}{x+1}$.

Let us consider $R|_{[0,\alpha)}$, the first return map of R to $[0, 1)$:

Defintion 3. *If $T : X \to X$ be μ-measure preserving and A be measurable then the* Poincaré first return map *to A is defined by*

$$T|_A : A \to A \text{ by } T(x) = T^{n_x}(x)$$

where $n_x = \min\{n > 0 : T^n x \in A\}$.

The first return map is defined almost everywhere by the Poincaré Recurrence Theorem:

Lemma 2. *Let $(X, \mathcal{B}, \mu)$ be a finite measure space. If $T : X \to X$ is μ-measure preserving then for any measurable set A we have that for μ almost every $x \in A$ there exists $n_x > 0$ so that $T^{n_x}(x) \in A$. That is, there exists $A' \subset A$ with $\mu(A \setminus A') = 0$ so that for all $x \in A'$ there exists $n_x > 0$ with $T^{n_x} x \in A$.*

Proof. Step 1: It suffices to show that for any set A with $\mu(A) > 0$ we have that there exists $n < 0$ with $\mu(T^n A \cap A) > 0$.

To see this observe $\{x \in A : T^n x \notin A \text{ for all } n > 0\} = A \setminus \cup_{i=1}^{\infty} T^{-i} A$ and $A \setminus \cup_{i=1}^{\infty} T^{-i} A$ is measurable. Now if $T^j x \in A \setminus \cup i = 1^{\infty} T^{-1} A$ then $T^j x \in A$.

Step 2: The sufficient condition holds.

Let $\frac{\mu(X)}{n} < \mu(A)$. Since T is measure preserving we have that $\mu(T^{-i}A) \geq \frac{\mu(X)}{n}$. X can not contain n disjoint (up to sets of measure 0) sets of measure greater than $\frac{\mu(X)}{n}$. So there exists $0 \leq i < j < n$ with $\mu(T^{-i}A \cap T^{-j}A) > 0$. Now $T^{-i}A \cap T^{-j}A = T^{-i}(A \cap T^{-(j-i)}A)$. □

Lemma 3. *The return times R to $[0, \alpha)$ are $a + 1$ if $x \in [0, \delta)$ and a otherwise, where a is the first term in the continued fraction expansion.*

This is because if $x \in [0,\alpha)$ then $R^a(x) \notin [0,\alpha)$ iff $x \in [0,\delta)$ and so $R^a(x) \in [1-\delta,\delta)$.

Now let $\hat{R} = R|_{[0,\alpha)}$ and we can consider $\hat{R}|_{[\alpha-\delta,\alpha)}$. Now there exists b so that the first return map for $\hat{R}$ to $[0,\delta)$ is either b or $b+1$. There exists $\epsilon = \alpha - b\delta < \delta$ [1] so that $[\alpha-\epsilon,\alpha)$ has return time $b+1$ and the rest has return time b.

It is natural to consider the return times for R to $[0,\delta)$. We have b or $b+1$ terms. They each have size either a or $a+1$.

Claim 1. Exactly one of them has size $a+1$.

There is 1 interval in $[0,\alpha)$ of size δ responsible for the return time of $a+1$. Until first return to $[\alpha-\delta,\delta)$ the orbit is δ separated. So we hit this interval at most once. Now we start in $[\alpha-\delta,\alpha)$ and we move by δ to the right, so have to hit an interval of size δ before first return. So we hit it exactly once.

Corollary 3. The return time of R to $[\alpha-\delta,\alpha)$ is either $ab+1$ or $ab+a+1$.[2]

Remark 1. Inductively let $q_{-1}=0, q_0=1, q_{i+1}=a_{i+1}q_i+q_{i-1}$. Let $\alpha_n = (-1)^n G^n(\alpha)$. Let $I_n = [0,\prod_{i=0}^{n-1}|\alpha_i|)$. Before rescaling $R|_{I_n}$ is rotation by $\alpha_n \prod_{i=1}^{n-1}|\alpha_i|$. After rescaling by the (inverse of the) size of the interval we can consider this as rotation by α_n on $[0,1)$. Inductively we can see that the return times to I_n are q_{n+1} and $q_{n+1}+q_n$.

Theorem 4. *(Chebyshev) If R is rotation by $\alpha \notin \mathbb{Q}$ then $|R^m(x)-y| < \frac{2}{m}$ for infinitely many m.*

1.3. **Problems.**

(1) Prove Corollary 1.
(2) Show that the return time to an interval under a circle rotation is at most 3 valued.
(3) Improve Chebyshev's theorem to show that if R is rotation by $\alpha \notin \mathbb{Q}$ then $|R^m(x)-y| \leq \frac{1}{m}$ for infinitely many m.
(4) Show that if $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}+\alpha\mathbb{Q}$ then $T_{\alpha,\beta} : [0,1)\times[0,1)$ by $(x,y) = (x+\alpha-\lfloor x+\alpha\rfloor, y+\beta-\lfloor y+\beta\rfloor)$ is ergodic.
(5) Use Chebyshev's theorem to show that if $\alpha \notin \mathbb{Q}$ and $x \in [0,1)$ then for every $\epsilon > 0$ we have
$$\{y : \exists\infty\, m \text{ so that } |R_\alpha^m(x)-y| < \frac{\epsilon}{m}\}$$
has full measure.
Hint: Use Chebyshev's theorem to show that $\cup_{i=N}^{\infty} B(R_\alpha^i(x),\frac{\epsilon}{i})$ has no Lebesgue density points.
(6) Show that any Gauss measure invariant subset with positive measure has full measure.
(7) * Show that $(X,\mathcal{B},\mu,T)$ is a probability measure preserving system, $A \in \mathcal{B}$ and $\mu(A) > 0$ then for any $\epsilon > 0$ there exists n so that $\mu(T^{-n}A\cap A) > \mu(A)^2 - \epsilon$.

[1]Note that this implies $\frac{\alpha}{b+1} < \delta < \frac{\alpha}{b}$ and so $\dfrac{1}{a+\dfrac{1}{b}} < \alpha = \frac{1}{a+\delta} < \dfrac{1}{a+\dfrac{1}{b+1}}$

[2]It is perhaps best to think of these numbers as $1(a+1)+(b-1)a$ and $1(a+1)+(b+1-1)a$.