Measure Rigidity of Cartan Actions

Setting: $\mathbb{Z}^k \curvearrowright X \leftrightsquigarrow f_1, \dots, f_k : X \to X$ commuting transformations

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Observation 1: Examples can be constructed by taking products

$$\left.\begin{array}{l} g_1 : Y_1 \to Y_1 \\ g_2 : Y_2 \to Y_2 \\ \vdots \\ g_k : Y_k \to Y_k \end{array}\right\} \qquad \begin{array}{l} X = Y_1 \times \dots \times Y_k \\ f_i : X \longrightarrow X \\ (y_1, \dots, y_i, \dots, y_k) \longmapsto (y_1, \dots, g_i(y_i), \dots, y_k) \end{array}$$

Observation 2: Examples are special

Problem / Conjecture (Smale)
The set of diffeomorphisms of a smooth manifold X which have trivial C^1-centralizer are residual in the C^r-topology.

[C^1-case: Bonatti - Crovisier - Wilkinson]

Assumptions

I. The action is by toral automorphisms.

↳ Useful Consequences

- $T\mathbb{T}^n = \mathbb{T}^n \times \mathbb{R}^n$
- If $a \in \mathbb{Z}^k$, da is the same matrix at every point of $\mathbb{T}^n$

Associated Decomposition

Let $A_i = df_i$ = derivative of the i^{th} generator

Then $A_i \in SL(n, \mathbb{Z})$ are commuting matrices

Let $E_1, \dots, E_m \subset \mathbb{R}^n$ be the joint eigenspaces* for $A_1, \dots, A_k$

$$A_i = \begin{array}{c|c} & \begin{array}{cccc} E_1 & E_2 & \cdots & E_m \end{array} \\ \begin{array}{c} E_1 \\ E_2 \\ \vdots \\ E_m \end{array} & \begin{pmatrix} B_i^{(1)} & 0 & \cdots & 0 \\ 0 & B_i^{(2)} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_i^{(m)} \end{pmatrix} \end{array}, \quad \lambda_i^{(j)} = \text{e-val of } A_i \text{ on } E_j = \text{eval of } B_i^{(j)}$$

II. $\dim(E_i)=1$, no two functionals χ_{j_1} and χ_{j_2} are proportional. $\Big\}$ Cartan

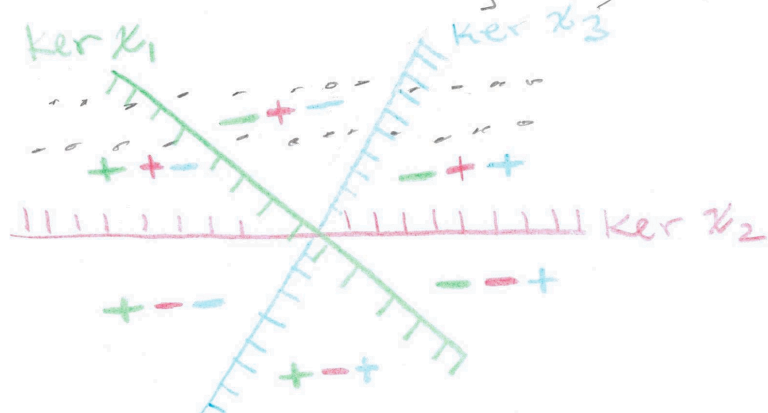
And E_i is dense in $\mathbb{T}^n$ $\forall i$ (Not a product)

$\boxed{\text{Ex}}$ $\mathbb{Z}^2 \curvearrowright \mathbb{T}^3$

f_1 induced by $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 3 & -2 \end{pmatrix}$

f_2 induced by $\begin{pmatrix} 0 & 2 & 7 \\ 7 & 21 & -12 \\ -12 & -29 & 45 \end{pmatrix}$

$\Delta = \{\chi_1, \chi_2, \chi_3\}$



$\mathbb{Z}^2 \cap \subset \mathbb{R}^2$

- Each component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^{m} \ker \chi_j$ is called a Weyl Chamber
- The Weyl chambers describe the possible stable/unstable foliation combinations.

- That no two χ_{j_1} and χ_{j_2} are proportional implies that each Oseledets space is an intersection of stable distributions
  $\rightarrow$ (Hence Hölder and integrable)

- That $\dim(E_j) = 1$ implies that the derivative is diagonalizable

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GOAL Classify invariant measures.

Compare w/ $\mathbb{Z}$-actions: Lots of invariant measures!
Put any measure on the SFT for a Markov partition.

Theorem (Rudolph, Katok, Spatzier, Einsiedler, Lindenstrauss, Wang, Kalinin, Rodriguez-Hertz)
If μ is an invariant measure for an irreducible action on a torus, and $h_\mu(a) > 0$, then μ is Haar measure.

$a = (a_1, \dots, a_k) \in \mathbb{Z}^k$

$$a \cdot x = f_1^{a_1} f_2^{a_2} \cdots f_k^{a_k}(x)$$

$$d_a(x) = A_1^{a_1} A_2^{a_2} \cdots A_k^{a_k}$$

$$= \begin{pmatrix} B^{(1)}(a) & & \\ & \ddots & \\ & & B^{(m)}(a) \end{pmatrix} \qquad B^{(j)}(a) = (B_1^{(j)})^{a_1} (B_2^{(j)})^{a_2} \cdots (B_k^{(j)})^{a_k}$$

Only 1 eigenvalue,

$$\lambda^{(j)}(a) = (\lambda_1^{(j)})^{a_1} \cdots (\lambda_k^{(j)})^{a_k}$$

$$\chi_j(a) = \log \lambda^{(j)}(a) = \sum_{i=1}^{k} a_i \log \lambda_i^{(j)} \quad \text{is a linear functional}$$

Compare: Oseledet's Decomposition, Lyapunov exponents.

Action by $\mathbb{Z}$: $\chi_j(e_1) = j^{\text{th}}$ Lyapunov exponent

Conjecture (Furstenberg)
If $h_\mu(a)=0$ for every $a\in\mathbb{Z}^k$, then μ is supported on a periodic orbit.

Main Idea
Transitive isometries are uniquely ergodic.

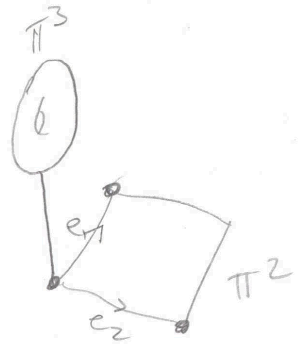
Tool: $\mathbb{Z}^k$-suspensions.

$$\overline{X} = \mathbb{R}^k \times X$$

$$(t, x) \sim (t-a, a\cdot x)$$

$\tilde{X} = \overline{X}/\sim$ = Suspension space for the $\mathbb{Z}^k$-action.

- $\tilde{X}$ is an X-fiber bundle over $\mathbb{T}^k$
- $\tilde{X}$ has a canonical $\mathbb{R}^k$ action: translation in the $\mathbb{R}^k$-coordinate
- If $a \in \mathbb{Z}^k$, then a preserves every fiber, and acts by the initial $\mathbb{Z}^k$-action

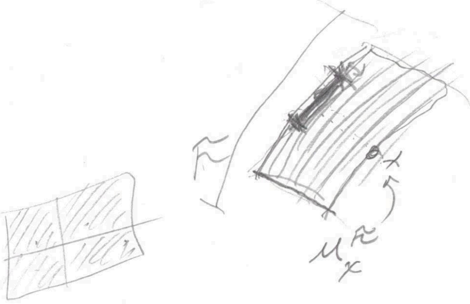


Advantage of working on $\tilde{X}$: Can move in <u>any</u> direction of $\mathbb{R}^2$, not just the rational ones

Tool 2: Conditional measures.

Theorem (Margulis, Sinai, Katok–Spatzier)

If $\tilde{\mathcal{F}}$ is an unstable foliation for some element $a \in \mathbb{Z}^k$, then there exists a disintegration of μ along $\tilde{\mathcal{F}}$. They are unique up to a constant depending only on the leaf.

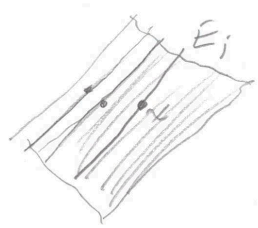


Trick: Sinai Partitions

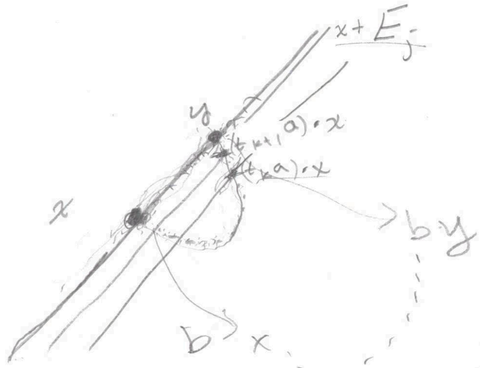
- Every atom is a piece of a leaf of $\tilde{\mathcal{F}}$
- Measurable, well-defined disintegration
- Apply expanding dynamics to make the atom huge, then normalize.

Observe: If μ is Haar on $\mathbb{T}^n$, $\mu_x^{W^{\chi_j}}$ = Haar on the space $\boxed{x+E_j}$. $\boxed{\text{This characterizes the Haar measure}}$

Requires E_j is dense.



The Higher Rank Trick



$t_k a b x \to b y$

$x \mapsto \mu_x^{\tilde{x}}$

$a \in \ker \chi_j$

$(t_k a)\cdot x \longrightarrow y \in x + E_j$

$t_k a : W^{\chi_j}(x) \longrightarrow W^{\chi_j}((t_k a)\cdot x)$

is an isometry

$\Downarrow$

$(t_k a)_* \mu_x^{W^{\chi_j}} = c\,\mu^{W^{\chi_j}}_{(t_k a)\cdot x}$ (and this passes to limits on larger & larger Lusin sets)

$\Downarrow$

$\mu_x^{W^{\chi_j}}$ satisfies

$T_{v*}\mu_x^{W^{\chi_j}} = \sigma_x(v)\,\mu_x^{W^{\chi_j}}, \quad v \in E_j$

$\sigma_x : E_j \to \mathbb{R}^\times$ is a homomorphism

$\Downarrow$

$\sigma_x(v) = e^{cv}$ for some c.

- $\underline{\sigma_x(v) = \sigma_x(e^{\chi(b)} v)} \Rightarrow \underline{\sigma_x \equiv 1}$ (choose b expanding)
- Therefore $\underline{(T_v)_* \mu_x^{w_j} = \mu_x^{w_j}}$, and $\mu_x^{w_j} =$ ~~Haar~~. Lebesgue.

<u>Things we swept under the rug</u> / <u>Strong assumptions made</u>

- <u>Upgrading</u> measurable to continuous (Choose Luzin sets)
- Ergodicity of <u>$\ker \chi_j$-action</u> (Irreducibility / No rank-one factors)
- E_j is dense in $\mathbb{T}^n$ (<u>Measure</u> is Haar on a subtorus)
- Generating enough translations along E_j ($\underline{h_\mu > 0} \Rightarrow$ nonatomic conditional measures)
- $\dim(E_j) = 1$ (Returns are isometries, how to deal w/ <u>Jordan</u> blocks?)
- <u>Toral Automorphisms</u> (How to deal w/ only subexponential growth?)

THANK YOU