Fubini Foiled: Pathological foliations from symbolic dynamics

1. Fubini
2. Its foiling
3. The example
4. "In nature"

Today: $I=[0,1]$, Leb $\qquad I^2 = \square$, Leb_2

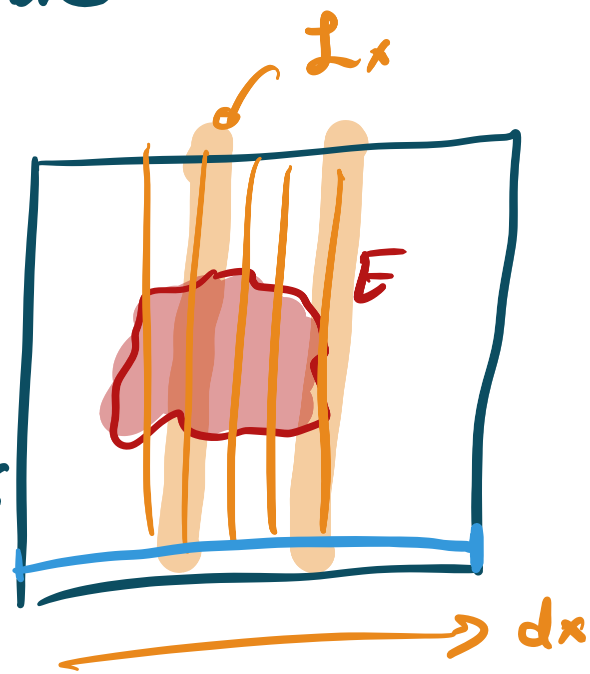
1. Fubini

If $E \subset I^2$ measurable

$$\mathrm{Leb}_2(E) = \int_{I_x} \left(\int_{I_y} \mathbb{1}_E \, dy \right) dx$$

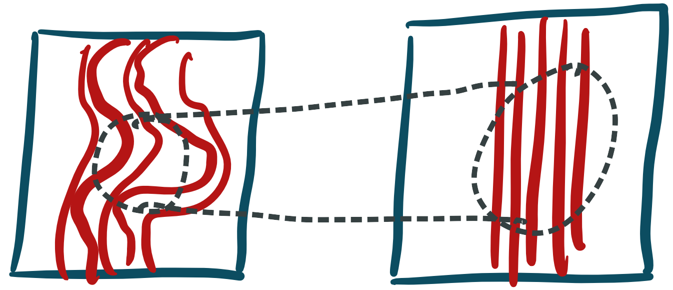
$$= \int_{\text{transversal}} \int_{\mathcal{L}_x} \mathbb{1}_E \, d\left(\mathrm{Leb}_x(y)\right) dx$$

leafwise measure.

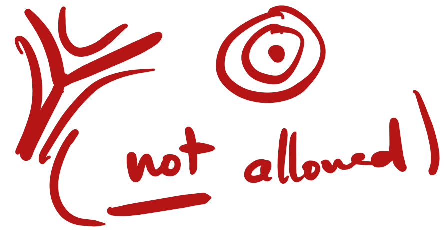


More generally:

A SMOOTH foliation of I^2 is a set of leaves filling out I^2 s.t.



I^2 has local SMOOTH charts to I^2 taking each leaf to $\{x\}\times I$

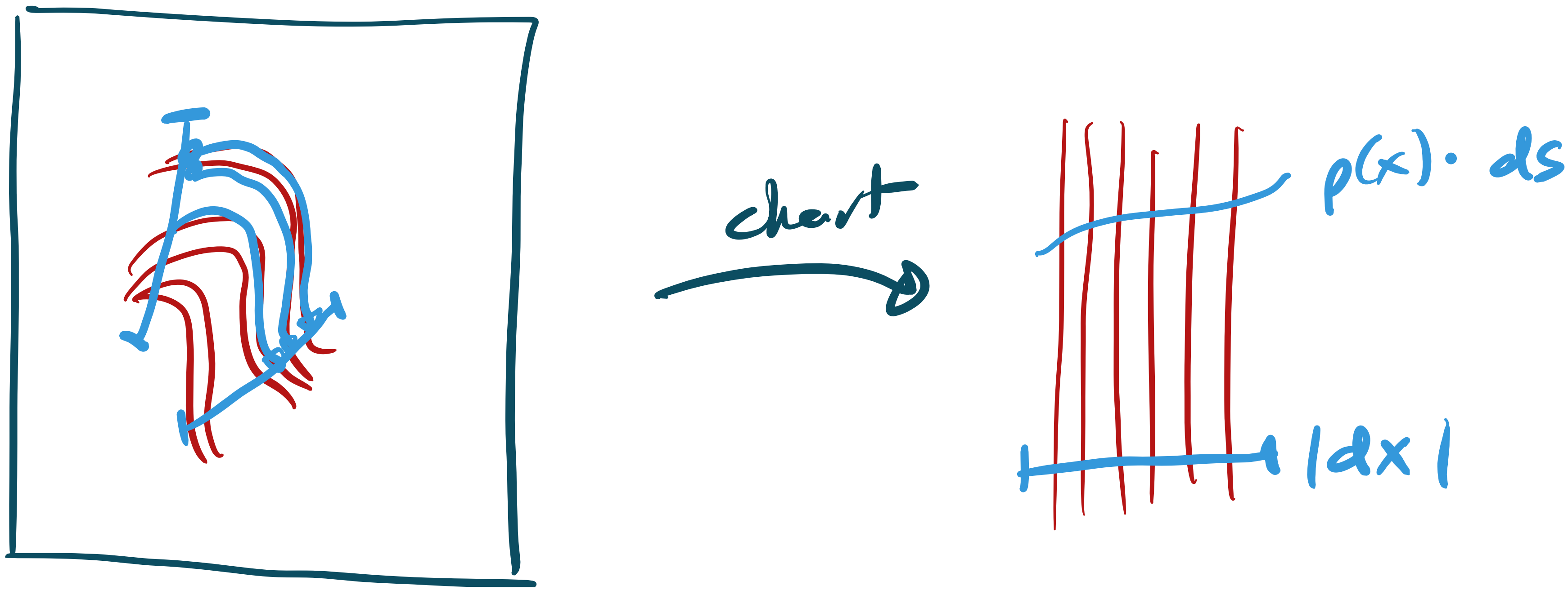


For any smooth (enough) foliation of I^2,

Fubini holds locally, i.e.

$$\mathrm{Leb}_2(E) = \int_{\text{transversal}} \int_{L_x \cap E} \underbrace{\rho_x(y)}_{\text{density function}} \mathbb{1}_E \, d\,\underbrace{\mathrm{Leb}_x}_{\text{leafwise}}(y) \, d\,\mathrm{Leb}(x)$$

Fubini & holonomy:



get "holonomy map" between transversals

Fubini holds $\iff$ holonomy map is a.c.

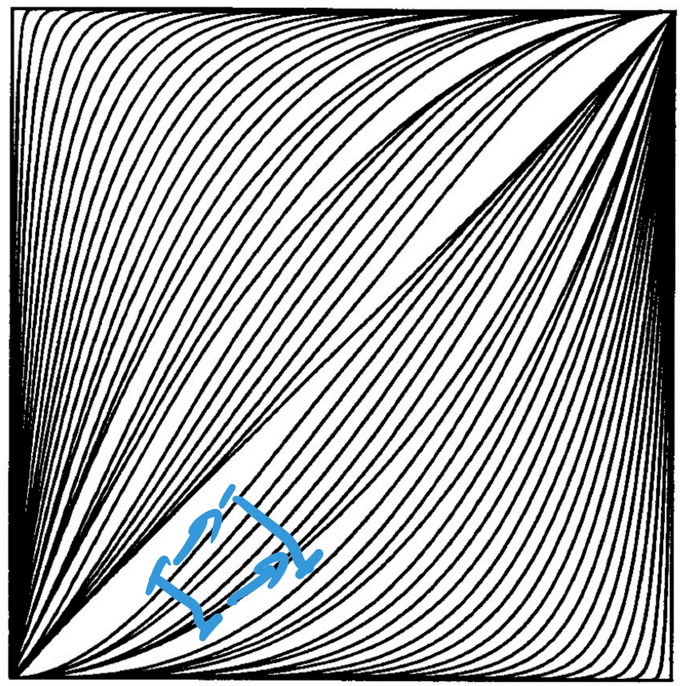
2. Foiled! [Katok, via Milnor]

$\exists$ foliation of I^2 w/ smooth leaves

& $E \subset I^2$ w/ $Leb_2(E) = 1$ s.t.

$$\forall x \qquad \#(E \cap \mathcal{L}_x) \leq 1.$$

⚠ smooth foliations $\subsetneq$ foliation w/ smooth leaves



must not be "smooth in transverse direction"

$\Updownarrow$

holonomy maps must not be a.c.

(eg: holonomy maps ~ Cantor function)

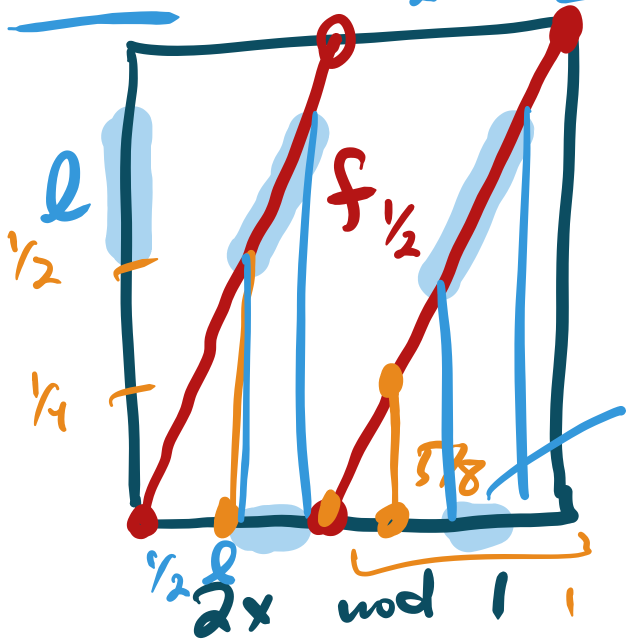
3. The example

Recall: binary representations

$3 = 2 + 1$	$\frac{5}{8} = \frac{1}{2} + \frac{0}{4} + \frac{1}{8}$	$\pi = 3.1415\ldots$
binary 11	$.1\ 0\ 1$	$11.0010\ldots$

Equivalently, look at $2^n \cdot x \bmod 1$ & record if $\geq$ or $< \frac{1}{2}$

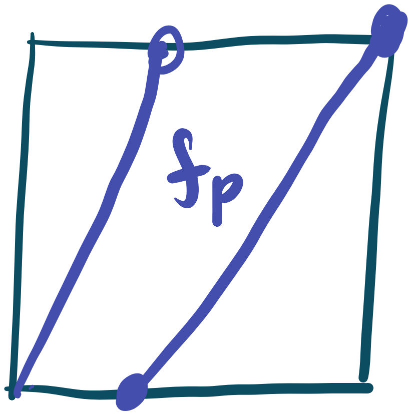
note: $f_{1/2}$ preserves leb measure.



$\frac{5}{8} \geq \frac{1}{2}$	$\frac{1}{4} < \frac{1}{2}$	$\frac{1}{2} \geq \frac{1}{2}$
1	0	1

code $x \in [0,1]$ by $a_0, \ldots a_n$

$$a_n = \begin{cases} 0 & \text{if } f_{1/2}^n(x) < \frac{1}{2} \\ 1 & \geq \frac{1}{2} \end{cases}$$



$p \in I$

f_p preserves Leb

<u>code</u> $x \in [0,1]$ by $a_0 \dots a_n$

$$C_p(a_0) = \begin{cases} 0 & \text{if } f_p^n(x) < p \\ 1 & \phantom{\text{if }} f_p^n(x) \ge p \end{cases}$$

C_p defines a (measurable) map $I \longrightarrow \{0,1\}^{\mathbb{N}} =: \Sigma$

$$\begin{tikzcd} I \arrow[r] \arrow[d, "f_p"'] & \{0,1\}^{\mathbb{N}} \arrow[d, "\text{shift}"] \\ I \arrow[r] & \Sigma \end{tikzcd}$$

C_p is a <u>conjugacy</u> between

f_p & shift.

takes Leb on I to $\boxed{\mu_p}$ on Σ p-Bernoulli measure

$\mu([0,\dots]) = p \qquad \mu([1,\dots]) = 1-p$

$\mu([0,1,\dots]) = p \cdot (1-p)$

Strong Law of Large #s: for Leb a.e. $x \in I$, ratio (0s in $C_p(x)$) $\to p$

ratio of 0s in f_p-coding of x is p.

Set:

$$E := \{(x,p) \mid \text{SLLN holds}\} \subset [0,1]^2$$

1. measurable. (fun exercise)
2. $\mathrm{Leb}_2(E) = 1$ use Fubini!

SLLN $\Rightarrow$ $\forall p$ $\mathrm{Leb}_p(E \cap (\{p\} \times I)) = 1$

Foliation:

$$[0,1] \xrightarrow{c_p} \Sigma \ni \alpha$$

$$\Gamma_\alpha := \{(c_p^{-1}(\alpha), p)\}$$

"things w/ same coding"

1. Everything has a coding $\Rightarrow$ $\bigcup_\alpha \Gamma_\alpha = I^2$
2. α has ≤ 1 limiting proportion of 0s. $\Rightarrow$ $|\Gamma_\alpha \cap E| \leq 1$
3. Smoothness: coding gives $c_p^{-1}(\alpha)$ as an analytic function of p.

Fubini doesn't hold

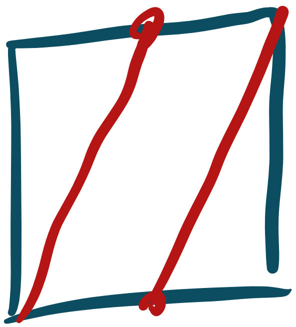
$\Rightarrow$ holonomy maps shouldn't be a.c.

$\Gamma_\alpha := \{(p, c_p^{-1}(\alpha))\}$

"things w/ same coding"

$$\Sigma \xrightarrow{c_{1/3}^{-1}} [0,1] \xrightarrow{c_{1/2}} \Sigma$$

Why should we believe this?

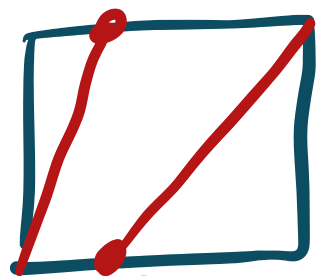


$f_{1/2}$

transversal

transversal

$$\begin{tikzcd} {[0,1] = \{1/2\} \times I} \arrow[r, "c_{1/2}"] & \Sigma & {[0,1]} \arrow[l, "c_{1/3}"'] \arrow[ll, bend left] \end{tikzcd}$$



$f_{1/3}$

Analogy w/ Cantor function (not a.c.)

binary

ternary

$[0,1] \longleftarrow X = \{x \in I \mid \text{ternary expansion has no 1s}\}$

$0_2 \longleftarrow 0_3$

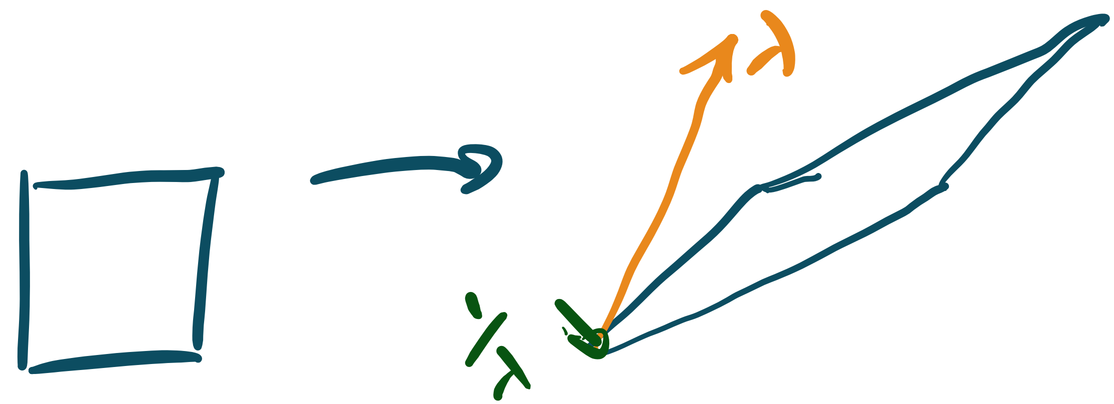
1_3

$1_2 \longleftarrow 2_3$

4. In nature (Why was Katok looking for this?)

Anosov / uniformly hyperbolic dynamics:

cat map $\begin{pmatrix}2 & 1\\ 1 & 1\end{pmatrix} : \mathbb{T}^2 \circlearrowleft$



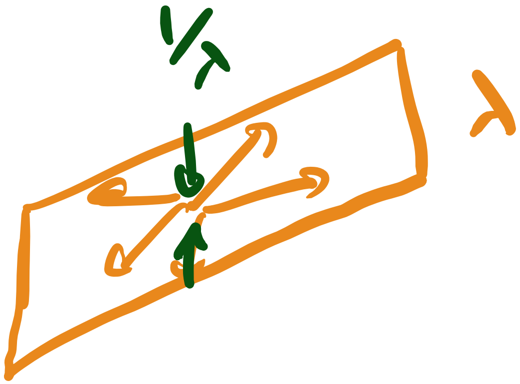
Has "expanding" & "contracting" foliations (eigenspaces)

Fubini holds for these foliations!

More generally:

Anosov diffeos = maps like cat map $f : M \circlearrowleft$

$\forall x$ $T_x M$ splits into "expanding" & "contracting" direction



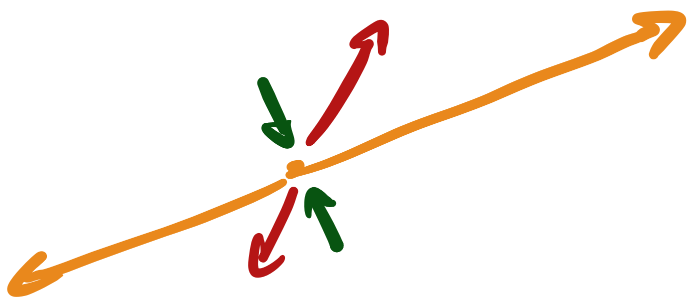
Classical fact: [Anosov & Sinai] if f Anosov ($\&$ C^2) then can do Fubini w/ expanding / contracting foliations.

Oseledets: $f: M \circlearrowleft$ smooth, measure preserving. Then

$\exists \lambda_1 > \lambda_2 > \ldots > \lambda_n$ s.t. for a.e. $x \in M$, $\exists$ subspaces

$$T_x M = V_1 \supset V_2 \supset \ldots \supset V_n \supset 0$$

s.t. $\forall v \in V_i \setminus V_{i+1}$, $\|f^n(v_i)\|$ grows (shrinks) exp. w/ rate λ_i.



non-uniformly hyp. so long as $\lambda_i \neq 0 \ \forall i$.

(might not get foliation)

Q: does Fubini hold for foliations coming from V_i?

A: NO, & this is generic.

Thm: [Shub-Wilkinson, Ruelle-Wilkinson]

Take $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \times \mathrm{id}$ on $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{T}^1$.

$\exists$ open set $f \in \mathcal{U} \subset \mathrm{Diff}^2_{\mathrm{Leb}}(\mathbb{T}^3)$ s.t. $\forall g \in \mathcal{U}$

SW

- g is ergodic & NUH
- g preserves a fibration $\mathbb{T}^3 \xrightarrow{\pi} \mathbb{T}^2$ & $\pi_* g = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

  leaves are C^2 circles $\mathcal{L}_x$
- $\exists \lambda$ s.t. $\forall v$ tangent to $\mathcal{L}_x$, $\|g^n_* v\| \sim \lambda^n$
- $\exists$ set $E \subset \mathbb{T}^3$ of full measure s.t.

  $\forall x \quad \#(E \cap \mathcal{L}_x) < \infty.$

SW: measure 0

RW: finite

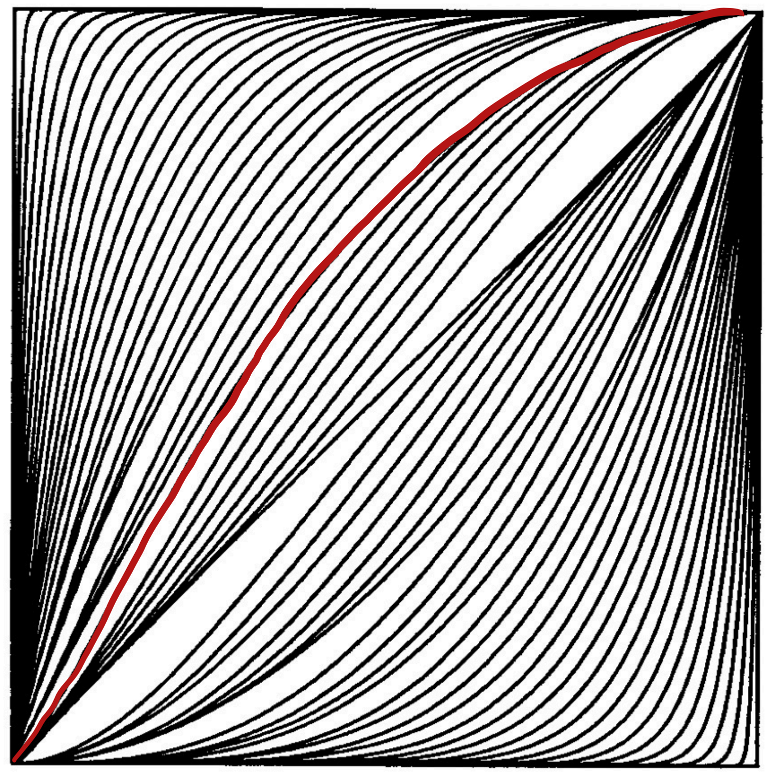
$\exists$ foliation of I^2 w/ smooth leaves

& $E \subset I^2$ w/ $Leb_2(E) = 1$ s.t.

$$\forall x, \quad \#(E \cap \mathcal{L}_x) \leq 1.$$

smooth foliations $\subsetneq$ foliations w/ smooth leaves

$\boxed{\text{Thanks!}}$



$\Gamma_\alpha := \{(p, C_p^{-1}(\alpha))\}$

"things w/ same coding"