ESKIN-MOZES (SURVEY)

ESKIN-MARGULIS (RANDOM WALKS ...)

ESKIN-MARGULIS-MOZES

ESKIN-MIRZAKHANI-RAFI

ATHREYA

KLEINBOCK-MIRZADEH.

MEYN-TWEEDIE "MARKOV CHAINS & STOCH. STABILITY"

$X_1, X_2, \dots, X_n, \dots \in S$

Markov chain.

$V : S \to [1, \infty)$

$$(*) \quad E\left(V(X_1) \mid X_0 = x\right) \le c\,V(x) + b$$

$\underline{0 < c < 1}$, $b > 0$.

$$\left[\, l > \frac{b}{1-c} \,\right]$$

<u>Strong</u> <u>Recurrence</u>

$$P_x\left(V(X_n) \geq l \;:\; 1 \le n \le N\right) \le \frac{V(x)}{l}\left(c + \frac{b}{l}\right)^N$$

$$P_x\Big(\underbrace{V(X_n) \geq l \;:\; 1 \leq n \leq N}_{B_N}\Big) := p_N$$

$$B_N \subset B_{N-1} \subset \cdots \qquad p_N = P(B_N).$$

$$D_N = E\big(V(X_N)\,\chi_{B_N}\big)$$

$$\underbrace{l}_{\text{lower bd value}}\;\underbrace{p_N}_{\text{meas of set}} \leq \underbrace{D_N}_{\text{integral of function}} \qquad [\text{b/c } V(X_N) \geq l \text{ on } B_N].$$

$$(*)$$

$$\leq c\,\underbrace{E\big(V(X_{N-1})\,\chi_{B_{N-1}}\big)}_{D_{N-1}} + \underbrace{b\,E(\chi_{B_{N-1}})}_{b\,p_{N-1}}$$

$$\ell p_N \le D_N \le c D_{N-1} + b p_{N-1} \qquad \ell > b/_{1-c}.$$

$$\left(p_{N-1} \le \frac{D_{N-1}}{\ell} \right) \qquad \le \left(c + \frac{b}{\ell} \right) D_{N-1}$$

$$\cdots \qquad \le \left(c + \frac{b}{\ell} \right)^N D_0$$

$$= \left(c + \frac{b}{\ell} \right)^N V(x)$$

$$p_N \le \frac{V(x)}{\ell} \left(c + \frac{b}{\ell} \right)^N$$ as desired. $\square$

$S = \mathbb{R}^2 / \{ \binom{0}{0} \}$ $\{ \theta_n \}$ iid $(0, 2\pi)$ unif r.v.'s.

$t > 0$

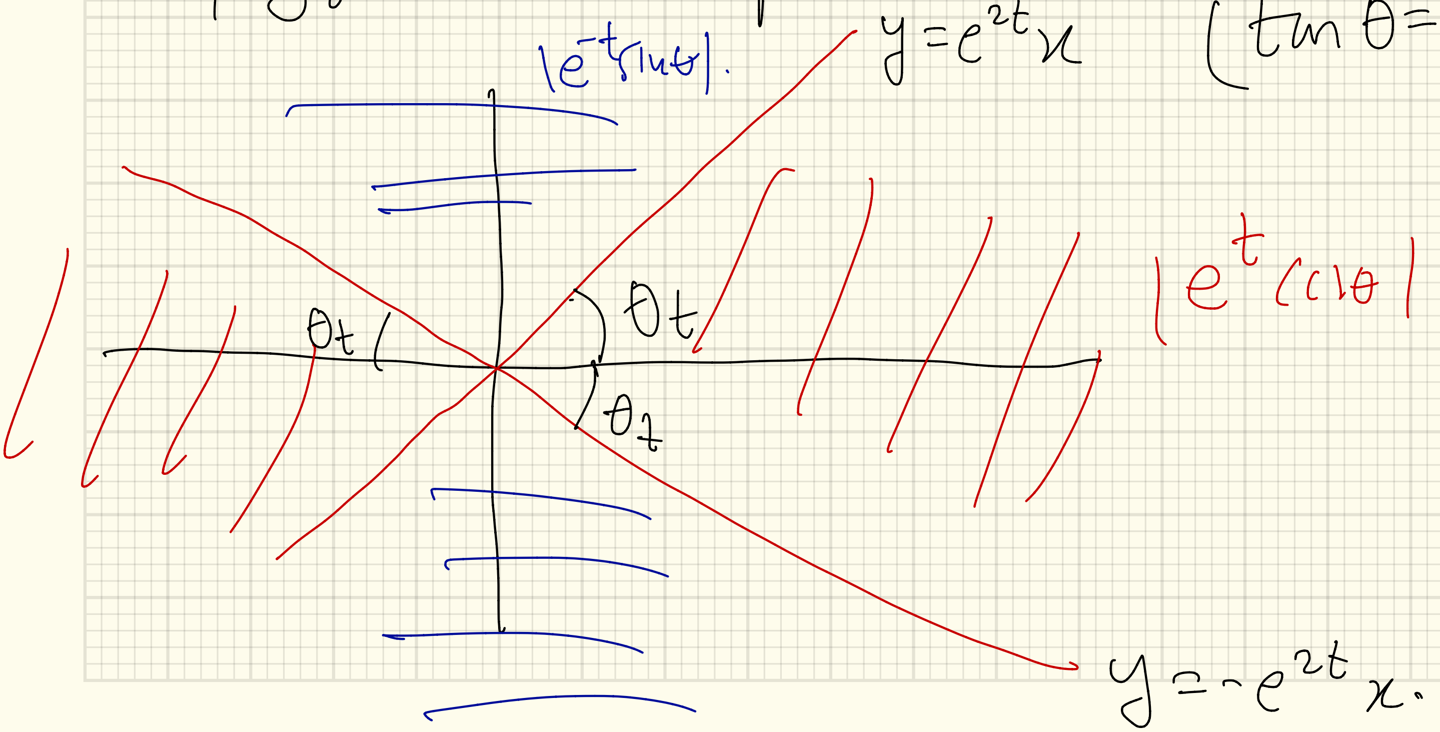
$$X_{n+1} = \underbrace{\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}}_{g_t} \underbrace{\begin{pmatrix} \cos\theta_n & -\sin\theta_n \\ \sin\theta_n & \cos\theta_n \end{pmatrix}}_{r_{\theta_n}} \underbrace{X_n}_{X_n}.$$

$$V(X) = \frac{1}{|X|_{sup}}$$

$X \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$$\left\| g_t r_\theta \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_{\sup} = \left\| \begin{pmatrix} e^{t} \cos\theta \\ e^{-t} \sin\theta \end{pmatrix} \right\|$$

$(\tan\theta = e^{2t})$



$e^{t}\cos\theta$

$e^{-t}\sin\theta$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\left|g_t r_\theta \binom{1}{0}\right|_{\sup}}\,d\theta$$

$$=\frac{1}{2\pi}\left(2\cdot e^{-t}\int_{-\theta_t}^{\theta_t}\sec\theta\,d\theta+2e^{t}\int_{\theta_t}^{\pi-\theta_t}\csc\theta\,d\theta\right)$$

$$\theta_t=\arctan\left(e^{2t}\right)$$

$$\int\sec\theta\,d\theta=\log|\sec\theta+\tan\theta|$$

$$\int\csc\theta\,d\theta=-\log|\cot\theta+\csc\theta|$$

$\sec(\arctan x)=\sqrt{x^2+1}$

$\csc(\arctan x)=\sqrt{1+\frac{1}{x^2}}$

$\cot(\arctan x)=\frac{1}{x}.$

$$= \frac{1}{\pi}\left(e^{-t} \log\left(\frac{e^{2t} + \sqrt{1+e^{4t}}}{-e^{2t} + \sqrt{1+e^{4t}}} \right) \right.$$

$$\left. - e^{t} \log\left(- \frac{\left(e^{-2t} - \sqrt{1+e^{-4t}}\right)}{\left(e^{-2t} + \sqrt{1+e^{-4t}}\right)} \right) \right)$$

$$= \frac{2}{\pi}\left(e^{-t} \log\left(e^{2t} + \sqrt{1+e^{4t}} \right) + \right.$$

$$\left. - e^{t} \log\left(e^{-2t} + \sqrt{1+e^{-4t}} \right) \right)$$

$$\frac{2}{\pi}\left(e^{-t}\log\left(e^{2t}+\sqrt{1+e^{4t}}\right) + \right.$$

$$\left. - e^{t}\log\left(e^{-2t}+\sqrt{1+e^{-4t}}\right)\right)$$

$$\sqrt{1+x} \sim 1+\frac{x}{2} \qquad \log(1+x)\sim x.$$

$$\simeq \frac{2}{\pi}\left(e^{-t}\log\left(2e^{2t}\right) - e^{t}\log\left(e^{-2t}+1+\frac{e^{-4t}}{2}\right)\right)$$

$$\simeq \frac{2}{\pi}\left(e^{-t}\left(\log 2+2t\right) - e^{t}\left(e^{-2t}\right)\right)$$

$$\le \frac{2}{\pi} e^{-t} \left((\lg 2 - 1) + 2t \right)$$

$\mathbb{H}^2 / SL_2\mathbb{Z}$ $\longrightarrow$ the space of lattices (or space of flat tori) in $\mathbb{R}^2$

$$SO(2) \backslash SL_2\mathbb{R} / SL_2\mathbb{Z}$$

$$SO(2)\, g\, SL_2\mathbb{Z} \longleftrightarrow g\mathbb{Z}^2 =: \Lambda_g$$

$\alpha(\Lambda_g) = 1/$ length of shortest vector in the lattice.

$SL_n\mathbb{R} / SL_n\mathbb{Z}$ $K = SO(n)$

$\{k_n\}$ g_t diag.

$$x_{n+1} = g_t k_n x_n.$$

Eskin-Margulis.

Eskin-Margulis:

Borel-Harishchandra: $H(\mathbb{R}) \supset H(\mathbb{Z})$

lattice [H s.s. $\mathbb{Q}$-group].

$\mathcal{H}$ = stratum of area 1 translation surfaces

$\ell^{-1} : \mathcal{H} \to \mathbb{R}^+$ 1/length of shortest saddle connection.

Eskin-Masur.

A. $\longrightarrow$ recurrence.

Foster-Lyapunov drift functions.

Exercice :

U_n iid uniform r.v. $[0,1]$.

$$X_{n+1} = \begin{pmatrix} 1 & e^{2i\pi U_n} \\ 0 & 1 \end{pmatrix} \cdot X_n$$

PROJECT : BUILD MARGULIS FUNCTION on SPACE of $SL_2\mathbb{R} \ltimes \mathbb{R}^2 / SL_2\mathbb{Z} \ltimes \mathbb{Z}^2$

using interpretation as space of Heisenberg lattice

Hopf Ratio Ergodic Thm $T \circlearrowleft (X,\mu)$ m.p. erg.

μ ∞ meas. $f, g \in L^1(\mu)$ $g > 0$.

$$\frac{\sum_{n=0}^{N} f(T^n x)}{\sum_{n=0}^{N} g(T^n x)} \xrightarrow[\text{a.e. } x]{} \frac{\int_X f \, d\mu}{\int_X g \, d\mu}$$

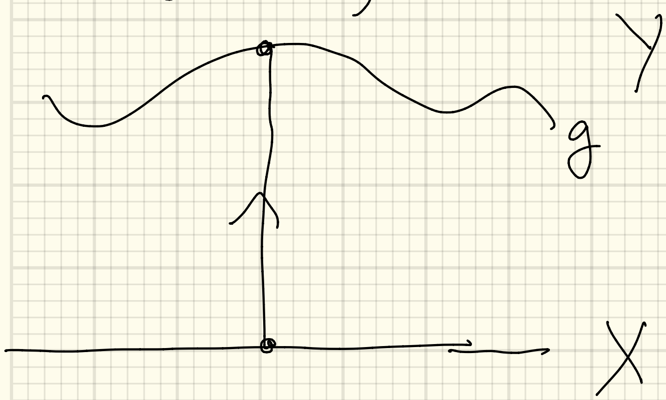
Birkhoff for flows (Y, η) $T_t \circlearrowleft (Y, \eta)$

$F \in L^1(\eta)$ $\displaystyle \int_0^T F(T_t y)\, dt \xrightarrow[\text{a.e. } y]{} \int F \, d\eta$.

(X, μ, T)

$f \in L^1, \quad g \in L^1$

$g > 0.$



$T_t : (Y, \eta) \circlearrowleft$

$dy = d\mu\, dt.$

$(x, g(x)) \sim (Tx, 0)$

$$F(x, t) = \frac{f(x)}{g(x)}$$

$$\frac{1}{T_n} \int_0^{T_n} F(x,t)\, dt = \frac{\sum_0^{n-1} f(T^i x)}{\sum_0^{n-1} g(T^i x)}$$

$$T_n^{(x)} = \sum_{i=0}^{n-1} g(T^i x)$$

$$\int F \, d\nu = \int f \, d\mu.$$

$$\int 1 \, d\nu = \int g \, d\mu.$$

$\square$.