Limiting amplitude principle for a two-layered medium composed of a dielectric material and a metamaterial

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Abstract

For wave propagation phenomena, the limiting amplitude principle (LAP) holds if the time-harmonic regime represents the large time asymptotic behavior of the solution of the evolution problem with a time-harmonic excitation. Considering a two-layered medium composed of a dielectric material and a Drude metamaterial separated by a plane interface, we prove that the LAP holds except for a critical situation related to a surface resonance phenomenon.

Keywords: Maxwell’s equations, metamaterials, spectral theory.

1 Introduction

In the frequency domain, the permittivity and permeability of a non-dissipative dispersive material \( \varepsilon(\omega) \) and \( \mu(\omega) \) are real-valued functions of the frequency \( \omega \). For metamaterials, these coefficients may become negative in particular frequency ranges, which raises theoretical and numerical difficulties. In [1], the authors proved that for a transmission problem between a dielectric material and a metamaterial separated by a smooth interface, the time-harmonic problem is well-posed except when both ratios of \( \varepsilon \) and \( \mu \) across the interface are equal to \(-1\) (which is the case of the “perfect lens” [3]). Nevertheless, the associated time-dependent problem remains well-posed. What is the link between both problems, in particular when the harmonic problem is ill-posed? We answer here the question in the case of a planar transmission problem which involves a Drude metamaterial.

2 Formulation of the problem

We consider a two-layered medium composed of a standard dielectric material and a Drude material, both homogeneous and non-dissipative, which fill respectively the half planes \( \mathbb{R}_-^3 = \{ x = (x, y, z) \in \mathbb{R}^3 \mid x < 0 \} \) and \( \mathbb{R}_+^3 = \{ x = (x, y, z) \in \mathbb{R}^3 \mid x > 0 \} \). \((e_x, e_y, e_z)\) will refer to the canonical basis of \( \mathbb{R}^3 \). We denote by \( E \) and \( H \) the electric and magnetic fields and by \( D \) and \( B \) the electric and magnetic inductions. In the presence of a source current density \( J_s \), the evolution of \((E, D, H, B)\) is governed by Maxwell’s equations:
\[
\begin{align*}
\partial_t D - \text{Curl} \, H &= -J_s \\
\partial_t B + \text{Curl} \, E &= 0,
\end{align*}
\]
(where the usual transmission conditions at the interface \( x = 0 \) are implicitly understood). These equations must be supplemented by the constitutive laws of each material. In the dielectric material, they are simply expressed by
\[
D = \varepsilon_0 \, E \quad \text{and} \quad B = \mu_0 \, H,
\]
for two positive constants \( \varepsilon_0 \) and \( \mu_0 \). In a dispersive medium, these laws involve two additional unknowns, the electric and magnetic polarizations \( P \) and \( M \):
\[
D = \varepsilon_0 \, E + P \quad \text{and} \quad B = \mu_0 \, H + M.
\]
For the Drude model, the fields \( P \) and \( M \) are related to \( E \) and \( H \) through
\[
\begin{align*}
\partial_t P &= J \\
\partial_t M &= K \quad \text{and} \quad \partial_t J = \varepsilon_0 \, \Omega_e^2 \, E \\
\partial_t K &= \mu_0 \, \Omega_m^2 \, H,
\end{align*}
\]
where \( \Omega_e \) and \( \Omega_m \) are positive parameters. By eliminating \( D, B, P \) and \( M \) in the above equations, we obtain
\[
(P) \quad \left\{ \begin{array}{l}
\varepsilon_0 \, \partial_t E - \text{Curl} \, H + \Pi \, J = -J_s \quad \text{in} \ \mathbb{R}_3^3, \\
\mu_0 \, \partial_t H + \text{Curl} \, E + \Pi \, K = 0 \quad \text{in} \ \mathbb{R}_3^3, \\
\partial_t J = \varepsilon_0 \, \Omega_e^2 \, E \quad \text{in} \ \mathbb{R}_3^3, \\
\partial_t K = \mu_0 \, \Omega_m^2 \, H \quad \text{in} \ \mathbb{R}_3^3,
\end{array} \right.
\]
where \( \Pi \) denotes the operator of extension by 0 of a vectorial field defined on \( \mathbb{R}_+^3 \) to \( \mathbb{R}^3 \).
When looking for time-harmonic solutions of (P): \((E(x), \mathcal{H}(x), J(x), K(x)) e^{-i\omega t}\) for a periodic current density \(J_s(x) e^{-i\omega t}\), we can eliminate \(J\) and \(K\). In the half-plane \(\mathbb{R}^3_+\) filled by the Drude material, we obtain

\[
\begin{align*}
    i\omega \varepsilon(\omega) E + \text{Curl} \mathcal{H} &= J_s, \\
    -i\omega \mu(\omega) \mathcal{H} + \text{Curl} E &= 0,
\end{align*}
\]

where

\[
\varepsilon(\omega) = \varepsilon_0 \left(1 - \frac{\Omega_e^2}{\omega^2}\right) \quad \text{and} \quad \mu(\omega) = \mu_0 \left(1 - \frac{\Omega_m^2}{\omega^2}\right).
\]

In the half-plane \(\mathbb{R}^3_-\) filled with the dielectric material, we obtain the same equations with \(\varepsilon(\omega)\) and \(\mu(\omega)\) replaced by \(\varepsilon_0\) and \(\mu_0\). Note that in the Drude material, \(\varepsilon(\omega)\) and \(\mu(\omega)\) become negative at low frequencies (which justifies the word “metamaterial”). Moreover, both ratios \(\varepsilon(\omega)/\varepsilon_0\) and \(\mu(\omega)/\mu_0\) are simultaneously equal to \(-1\) at the same frequency and only if \(\Omega_e = \Omega_m\) (\(\Omega^*\)) and \(\omega = \pm \Omega^*/\sqrt{2} \ (\pm \omega^*)\), where \(\omega^*\) is called the plasmonic frequency.

### 3 Main results

We are interested in the long-time behavior of the solution of the transverse magnetic (TM) version of (P) for a time-harmonic source term \(J_s(x,y) = J_s(x,y) e^{-i\omega t} e_z\) with \(\omega > 0\) and zero initial conditions. In this case, we have \(E = (0,0,E_z)\) and \(H = (H_x, H_y, 0)\) where \(E_z, H_x\) and \(H_y\) do not depend on \(z\), as well as the same properties for \(J\) and \(K\). We express below our main result in terms of the electrical field \(E_z\), but the same results hold for the other unknowns \(H_x, H_y, J_z, K_x, K_y\).

**Theorem 1**

(i) If \(\Omega_e \neq \Omega_m\), the LAP holds at all frequencies, in the sense that for all \(\omega > 0\), there exists a function \(E_z\) (related to the time-harmonic problem) such that

\[
E_z(\cdot, t) = E_z(\cdot) e^{-i\omega t} + o(1) \quad \text{as} \quad t \to +\infty,
\]

where \(o(1)\) stands for a function which tends to 0 in \(L^1_{\text{loc}}(\mathbb{R}^2)\).

(ii) If \(\Omega_e = \Omega_m\), the LAP never holds. More precisely, with the same notations as above,

- If \(\omega \neq \omega^*\), then there exists functions \(E_z^*\) and \(E_z\) such that

\[
E_z(\cdot, t) = \sum_{\pm} E_z^{*\pm}(\cdot) e^{i\omega t} + E_z(\cdot) e^{-i\omega t} + o(1);
\]

- If \(\omega = \omega^*\), then there exists functions \(E_z^*\) and \(E_z^\pm\) such that

\[
E_z(\cdot, t) = t E_z^*(\cdot) e^{-i\omega^* t} + \sum_{\pm} E_z^{\pm}(\cdot) e^{i\omega t} + o(1).
\]

### 4 Method of Analysis

The (very technical) proof follows from standard arguments (see, e.g., [4]). The main difficulty here is related to the dependence of \(\varepsilon(\omega)\) and \(\mu(\omega)\) with respect to \(\omega\) (see [2] for details). We first rewrite the original problem (P) as an abstract Schrödinger equation

\[
dU/dt + i\mathbb{A} U = F e^{-i\omega t} \quad \text{with} \quad U(0) = 0,
\]

where \(\mathbb{A}\) is an unbounded self-adjoint operator in an appropriate Hilbert space \(\mathcal{H}\). The key of the analysis is the spectral theory of the operator \(\mathbb{A}\). This permits a quasi-explicit representation of the solution via the (generalized) diagonalization of \(\mathbb{A}\). This is achieved by combining a partial Fourier transform along the interface with Sturm-Liouville type techniques in the orthogonal direction. For \(\Omega_e = \Omega_m\), the resonance phenomenon is linked to the fact that \(\mathbb{A}\) admits at the plasmonic frequency \(\omega^*\) an eigenvalue of infinite multiplicity.

### References


