Final prep notes

Homotopy

Maps $f,g\colon X\to Y$ are *homotopic* (and we write $f\sim g$) if there is a continuous map

$$F\colon X\times[0,1]\to Y$$

with $f(x)=F(x,0)$ and $g(x)=F(x,1)$. The map F is the *homotopy*. A map $f\colon X\to Y$ is a *homotopy equivalence* if there is map $g\colon Y\to X$ with $f\circ g$ and $g\circ f$ both homotopic to the identity. Then the spaces X and Y are *homotopy equivalent*. For a homotopy F we write $f_t(x)=F(x,t)$.

A space X is *contractible* if the identity map is homotopic to a constant map or, equivalently, X is homotopy equivalent to a point.

If $A\subset X$ then $r\colon X\to A$ is a *retraction* if the restriction of r to A is the identity. A *deformation retract* from X to A is a homotopy F from the identity map on X to a retraction to A with the restriction of f_t to A the identity for all $t\in[0,1]$.

Lemma 0.1 *If F is a deformation retract from X to $A\subset X$ then the retraction f_1 is a homotopy equivalence.*

If $f,g\colon[0,1]\to X$ are paths then f and g are *path homotopic* if they are homotopic via a homotopy F with $F(0,t)=f(0)=g(0)$ and $F(1,t)=f(1)=g(1)$ for all $t\in[0,1]$.

Spaces

If X and Y are spaces, $B\subset Y$ is a subspace and $f\colon B\to Y$ is a continuous map then we form then $X\sqcup_f Y$ is the quotient space obtained from the disjoint union of X and Y with the equivalence relation $\sim$ given by $b\sim f(b)$ for $b\in B$. Lots of spaces can be constructed in this way.

The *wedge sum* of X and Y is $X\vee Y=X\sqcup_f Y$ where B is a point. In most cases the homotopy type of $X\vee Y$ doesn't depend on $B=\{b\}$ or $f(b)$ which is why it is not included in the notation.

The *cone* of Y is $CY=\{1\}\sqcup Y\times[0,1]$ where $B=Y\times\{1\}$ and the *suspension* of X is $SY=\{-1,1\}\sqcup_f Y\times[-1,1]$ with $f(y,\pm1)=\pm1$. Note for the cone CY we don't need to describe the map f as for any space there is only one map to a space with one element.

The most important spaces for algebraic topology are CW complexes. They are constructed inductively. A 0-dimensional CW complex X^0 is a discrete set. An n-dimensional CW complex X^n is given by $X^n=X^{n-1}\sqcup_f Y$ where Y is a disjoint union of n-dimensional balls and B is the boundary of the balls. In particular B is a disjoint union of $(n-1)$-dimensional spheres. If X^n is an n-dimensional CW complex then we have a nested family of k-dimensional CW complexes

$$X^0\subset X^1\subset\cdots\subset X^k\subset\cdots\subset X^n=X.$$

Then X^k is the *k-skeleton* of X.

Some useful notation: We have $X^k=X^{k-1}\sqcup Y$ where Y is a disjoint union $\{e^k_\alpha\}$ of k-dimensional balls indexed by α and B is the disjoint union of the boundary $(k-1)$-dimensional spheres $\{\partial e^k_\alpha\}$. We denote the restriction of f to each ∂e^k_α by $\varphi_\alpha\colon\partial e^k_\alpha\to X^{k-1}$. The *characteristic map*

$$\Phi_\alpha\colon e^k_\alpha\to X$$

is the composition of the quotient map $X^{k-1}\sqcup_{\varphi_\alpha} e^k_\alpha$ with the inclusion map $X^k\hookrightarrow X$.

Given a $k+1$-cell e_α^{k+1} and k-cell e_β^k in a CW complex X we have a map between k-spheres defined as follows. Let X' be the k-dimensional subcomplex of X^k that contains all cells except for e_β^k. Then the quotient X^k/X' is a k-sphere. The composition of the boundary map $\varphi\colon \partial e_\alpha^{k+1} \to X^k$ and the quotient map $X^k \to X^k/X'$ is then a continuous map between k-spheres. We label this map $\varphi_{\alpha\beta}$.

A pair (X,A) has the *homotopy extension property* if for all spaces Y and continuous maps from $X \times \{0\} \sqcup A \times [0,1] \to Y$ there is an extension $X \times [0,1] \to Y$. That is if we have a map from X to Y and a homotopy of the restriction of that map to A then the homotopy extends to a homotopy on all of X.

Lemma 0.2 *If X is a CW complex and A is a sub-complex then (X,A) has the homotopy extension property.*

Proposition 0.3 *If (X,A) has the homotopy extension property and A is contractible then the quotient map $q\colon X \to X/A$ is a homotopy equivalence.*

Covering spaces and the fundamental group

If $p\colon \tilde{X} \to X$ is a continuous map then $U \subset X$ is *evenly covered* if for every component V of $p^{-1}(U)$ the restriction of p to V is a homeomorphism to U. The $p\colon \tilde{X} \to X$ is a *covering space* if every $x \in X$ has an evenly covered neighborhood.

If $f,g\colon [0,1] \to X$ are paths with $f(1) = g(0)$ define

$$f * g\colon [0,1] \to X$$

by

$$f * g(t) = \begin{cases} f(2t) & 0 \le t \le 1/2 \\ g(2t-1) & 1/2 \le t \le 1. \end{cases}$$

The fundamental group of X with basepoint $x_0 \in X$, denoted by $\pi_1(X,x_0)$ is the set of path homotopy classes $f\colon ([0,1],\{0,1\}) \to (X,x_0)$ with the operation $[f]\cdot[g] = [f*g]$. This gives $\pi_1(X,x_0)$ the structure of group.

Theorem 0.4 *Let $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a covering space and let $f\colon (Y,y_0) \to (X,x_0)$ be a continuous map with Y path connected and locally path connected. Then there is a lift $\tilde{f}\colon (Y,y_0) \to (\tilde{X},x_0)$ if and only if $f_*(\pi_1(Y,y_0)) \subset p_*(\pi_1(\tilde{X},\tilde{x}_0))$. If the lift exists it is unique.*

Theorem 0.5 *Let X be path connected, locally path connected and semilocally simply connected. Then for every subgroup G of $\pi_1(X,x_0)$ there is a covering space $p_G\colon (X_G,x_G) \to (X,x_0)$ with $(p_G)_*(\pi_1(X_G,x_G)) = G$. Furthermore if $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ is another covering space with $p_*(\pi_1(\tilde{X},\tilde{x}_0)) = G$ then there is a homeomorphism $\phi\colon (\tilde{X},\tilde{x}_0) \to (X_G,x_G)$ with $p = \phi \circ p_G$.*

Free amalgamations

Let G_0 and G_1 be groups and let $\tilde{\mathscr{W}}$ be words in G_0 and G_1 - that is $w \in \tilde{\mathscr{W}}$ is finite sequence $x_0 \cdots x_n$ with each x_i in G_0 or G_1. A word w is reduced if consecutive letters are in distinct groups and no x_i is the identity. The set of reduced words is $\mathscr{W}$.

If w is a word and there are consecutive letters x_i and x_{i+1} both in the same group G_j we can replace the two letters x_i and x_{i+1} with the single letter $(x_i x_{i+1})$. If one of the x_i is the identity in either G_0 or G_1 we can remove it. This defines an equivalence relation on the set of words.

Theorem 0.6 *Each word is equivalent to a unique reduced word. The binary operation on reduced words w_0 and w_1 obtained by first concatenating the words and then reducing gives $\mathscr{W}$ a group structure where the empty word is the identity.*

We write this group as $G_0 * G_1$. It is the *free amalgamation* of G_0 and G_1.

Theorem 0.7 (Van Kampen) *Let (X,x_0) be path connected and X_0, X_1 open subspaces such that $X_0 \cap X_1$ is path connected and contains the basepoint x_0. Then $\pi_1(X,x_0)$ is the quotient of $\pi_1(X_0,x_0) * \pi_1(X_1,x_0)$ by the normal subgroup generated by elements of the form $[f]_i[f]_{1-i}^{-1}$ where $[f]_i$ is the image of the element $[f] \in \pi_1(X_0 \cap X_1, x_0)$ in $\pi_1(X_i,x_0)$.*

Note that to understand the Van Kampen theorem you need to understand the definition of a free amalgamation of groups. You should also be be able to compute the fundamental group as generators and relations of basic examples (a CW complex with one vertex).

Homology

A *chain complex* $C = \{C_n, \partial_n\}$ is a family of abelian groups C_n and homomorphisms $\partial_n \colon C_n \to C_{n-1}$ with $\partial_n \circ \partial_{n+1} = 0$. Then *cycles* are $Z_n(C) = \ker \partial n$ and *boundaries* are $B_n(C) = \mathrm{im} \partial_{n+1}$. Note that $B_n(C) \subset Z_n(C)$ and the homology groups are $H_n(C) = Z_n(C)/B_n(C)$.

A *chain map* between chain complexes B and C and a family of homomorphisms $\phi_n \colon B_n \to C_n$ with $\partial_n \circ \phi_n = \phi_{n-1} \circ \partial_n$.

Lemma 0.8 *A chain map induces a homomorphism*

$$(\phi_n)_* \colon H_n(B) \to H_n(C).$$

Two chain maps ϕ and ψ are *chain homotopic* if there are homomorphisms $P_n \colon B_n \to C_{n+1}$ with

$$\partial_{n+1} P_n = \phi_n - \psi_n - P_{n-1} \partial_n.$$

Then P is a *chain homotopy.*

Lemma 0.9 *Chain homotopic maps induces the same homomorphisms on homology.*

A sequence of groups C_n and homomorphisms $\phi_n \colon C_n \to C_{n-1}$ is *exact* if $\mathrm{im} \phi_{n+1} = \ker \phi_n$. In other words an exact sequence is a chain complex where the homology groups are zero. A *short exact sequence* is an exact sequence of length 5 where the first and last groups are zero. That is we have

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

where j is surjective, A is the kernel of j and i is the inclusion map. One example of this when B is isomorphic to $A \oplus C$ and $i(a) = (a,0)$ while $j(a,c) = c$. This is a *split* short exact sequence.

Lemma 0.10 *Let*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

be a short exact sequence. Then the following are equivalent:

1. *The sequence is split.*
2. *There exists a homomorphism $p \colon C \to B$ with $j \circ p$ the identity.*
3. *There exists a homomorphism $q \colon B \to A$ such that $q \circ i$ is the identity.*

Not all short exact sequences split. The most important example is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{k} \mathbb{Z} \longrightarrow \mathbb{Z}_k \longrightarrow 0$$

where the map between $\mathbb{Z}$'s is multiplication by a positive integer $k \geq 2$.

If A, B, and C are chain complexes and i and j are chain maps then

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is a *short exact sequence of chain complexes* if for each n the sequence

$$0 \longrightarrow A_n \xrightarrow{i_n} B_n \xrightarrow{j_n} C_n \longrightarrow 0$$

is short exact.

Lemma 0.11 (Snake Lemma) *If*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is a short exact sequence of chain complexes then

$$\cdots \longrightarrow H_n(A) \xrightarrow{(i_n)_*} H_n(B) \xrightarrow{(j_n)_*} H_n(C) \xrightarrow{\partial_n} H_{n-1}(A) \longrightarrow \cdots$$

where the boundary map is defined as follows: If $c \in Z_n(C)$ is a cycle representing a homology class $[c] \in H_n(C)$ then there is a $b \in B_n$ with $j_n(b) = c$ (since j_n is surjective). Then $j_{n-1}(\partial_n b) = \partial_n(j_n(b)) = \partial_n(c) = 0$ and since $\mathrm{im}\, i_n = \ker j_{n-1}$ we have $a \in A_{n-1}$ with $i_{n-1}(a) = \partial_n(b)$. Then $\partial_n([c]) = [a]$.

The subscript n is usually clear from context so we will drop it.

The *standard n-simplex* is

$$\Delta^n = \left\{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} | \sum x_i = 1 \text{ and } x_i \geq 0\right\}.$$

A *singular n-simplex* is a continuous map $\sigma\colon \Delta^n \to X$. An *$n$-chain* is a finite sum

$$a_1\sigma_1 + \cdots + a_k\sigma_k$$

where the σ_i are singular n-simplices and $a_i \in \mathbb{Z}$. If σ is a singular n-simplex we let $\sigma_{[v_0 \cdots \hat{v}_i \cdots v_n]}$ be the restriction of σ to the *face* of Δ^n obtained by setting the ith coordinate to be zero. There is a canonical linear map from Δ^{n-1} to this face so $\sigma_{[v_0 \cdots \hat{v}_i \cdots v_n]}$ is a singular $(n-1)$-simplex. We define

$$\partial\sigma = \sum_{i=0}^{n} (-1)^i \sigma_{[v_0 \cdots \hat{v}_i \cdots v_n]}$$

and for an n-chain $\alpha = a_1\sigma_1 + \cdots a_k\sigma_k$ we define

$$\partial\alpha = a_1\partial\sigma_1 + \cdots + a_k\partial\sigma_k.$$

Then $C(X) = \{C_n(X), \partial\}$ is the *singular chain complex* for X. The homology groups for this chain complex are written $H_n(X)$. These are the *singular homology groups* for X.

If $f\colon X \to Y$ the $f_\#\colon C_n(X) \to C_n(Y)$ is defined as follows. If σ is a singular k-simplex in X then $f_\#(\sigma) = f \circ \sigma$ is a singular k-simplex in Y and we extend $f_\#$ to chains by $\alpha = a_1\sigma_1 + \cdots + a_m\sigma_m$ by

$$f_\#(\alpha) = a_1 f_\#(\sigma_1) + \cdots + a_m f_\#(\sigma_m).$$

Then $f_\#$ is a chain map and induces homomorphisms $f_*\colon H_n(X) \to H_n(Y)$.

Lemma 0.12 *If $f\colon X \to Y$ and $g\colon Y \to Z$ then $(g \circ f)_* = g_* \circ f_*$.*

Theorem 0.13 *If $f, g\colon X \to Y$ are homotopic then $f_* = g_*$. If f is a homotopy equivalence then f_* is an isomorphism.*

If $\alpha = a_1\sigma_1 + \cdots + a_m\sigma_m$ is a 0-simplex we define

$$\varepsilon(\alpha) = \sum a_i.$$

This defines a homomorphism $\varepsilon\colon C_0(X) \to \mathbb{Z}$ and for any $\alpha \in C_1(X)$ we have $\varepsilon(\partial\alpha) = 0$ so

$$\cdots C_2(X) \xrightarrow{\partial} C_1(X) \xrightarrow{\partial} C_0(X) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a chain complex. The homology of this chain complex is the *reduced* homology of X and it is denoted $\tilde{H}_n(X)$.

Proposition 0.14 *For $n \geq 1$ we have $H_n(X) \cong \tilde{H}_n(X)$ while for $n = 0$ we have $H_n(X) \cong \tilde{H}_n(X) \oplus \mathbb{Z} \cong \mathbb{Z}^k$ where k is the number of path components of X. If X is path connected then $H_0(X) \cong \mathbb{Z}$ (and ε is an isomorphism) and $\tilde{H}_0(X) \cong 0$.*

A Δ-*complex* is a *CW* complex complex with extra structure. In particular we identify the k-cells with the standard k-simplex and require that the attaching maps restricted to each $(k-1)$-dimensional face is linear map, vertex respecting map to a $(k-1)$-cell in the k-skeleton. The characteristic map for each k-cell is then a singular k-simplex. Then $C_k^\Delta(X)$ is the subgroup of $C_k(X)$ of chains of the singular k-simplices coming from these characteristic maps. The restriction on the attaching maps makes $\{C_n^\Delta(X), \partial\}$ a subcomplex of $\{C_n(X), \partial\}$.

Theorem 0.15 *The inclusion $C_n^\Delta(X) \hookrightarrow C_n(X)$ induces isomorphisms on homology.*

If $A \subset X$ (or if X is Δ-complex and $A \subset X$ is sub-complex) then $C_k(X,A) = C_k(X)/C_k(A)$ (or $C_k^\Delta(X,A) = C_k^\Delta(X)/C_k^\Delta(A)$). Then

$$0 \longrightarrow C(A) \longrightarrow C(X) \longrightarrow C(X,A) \longrightarrow 0$$

is a short exact sequence of chain complexes so we have

Theorem 0.16

$$\cdots \longrightarrow H_n(A) \longrightarrow H_n(X) \longrightarrow H_n(X,A) \longrightarrow H_{n-1}(A) \longrightarrow \cdots$$

is a long exact sequence.

Theorem 0.17 (Excision) *Let (X,A) be a pair and assume that $B \subset A$ with $\bar{B}$ contained in the interior of A. Then the inclusion map $(X \smallsetminus B, A \smallsetminus B) \hookrightarrow (X,A)$ induces and isomorphism $H_n(X \smallsetminus B, A \smallsetminus B) \to H_n(X,A)$.*

Theorem 0.18 *If X is a CW complex and A is a subcomplex the quotient map $q\colon (X,A) \to (X/A, A/A)$ induces isomorphisms $q_*\colon H_n(X,A) \to H_n(X/A, A/A)$. If $n \geq 1$ the inclusion map $H_n(X/A) \to H_n(X/A, A/A)$ is an isomorphism.*

Theorem 0.19 (Mayer-Vietoris) *If $X = U \cup V$ with U and V open then*

$$\cdots \longrightarrow H_n(U \cap V) \xrightarrow{\Phi} H_n(U) \oplus H_n(V) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial} H_{n-1}(U \cap V) \longrightarrow \cdots$$

is a long exact sequence with $\Phi([\alpha]) = ([\alpha], -[\alpha])$ and $\Psi([\alpha],[\beta]) = [\alpha] + [\beta]$. If $\alpha + \beta$ represents a cohomology class in $H_n(X)$ with $\alpha \in C_n(A)$ and $\beta \in C_n(B)$ the $\partial([\alpha + \beta]) = [\partial\alpha] = -[\partial\beta]$.

The same result holds if we replace homology with reduced homology.

CW homology

If X is a *CW* complex then the homology groups $H_n(X^n, X^{n-1})$ are free abelian groups generated by $H_n(e^n_\alpha, \partial e^n_\alpha) \hookrightarrow H_n(X^n, X^{n-1})$ where the homomorphism is induced by the characteristic map $\Phi_\alpha \colon (e^n_\alpha, \partial e^n_\alpha) \to (X^n, X^{n-1})$. We also have homomorphisms

$$d_n \colon H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

obtained by composing the boundary map $H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1})$ with the inclusion map $H_{n-1}(X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$. Then $d_n \circ d_{n+1} = 0$ so $\{H_n(X^n, X^{n-1}), d_n\}$ is the *cellular chain complex* with cellular homology groups $H^{CW}_n(X)$.

Theorem 0.20

$$H^{CW}_n(X) \cong H_n(X)$$

If $f \colon S^n \to S^n$ then $f_* \colon H_n(S^n) \to H_n(S^n)$ is multiplication by an integer k. Then the degree of f is $\deg f = k$. When we defined *CW* complexes above for each $k+1$-cell e^{k+1}_α and k-cell e^k_β we described a map $\varphi_{\alpha\beta}$ between k-spheres. Let $d_{\alpha\beta}$ be the degree of this map.

Theorem 0.21

$$d_n(e^n_\alpha) = \sum_\beta d_{\alpha\beta} e^{n-1}_\beta$$

Theorem 0.22 *Assume X is path connected. The natural map from $\pi_1(X, x_0)$ to $H_1(X)$ is surjective and the kernel is the commutator subgroup of $\pi_1(X, v)$. (The commutator subgroup of a group is products of elements of the form $[x,y] = xyx^{-1}y^{-1}$.)*

Homology with coefficients

The chain complex $C_n(X;G)$ is chain $\alpha = a_1\sigma_1 + \cdots + a_k\sigma_k$ where the σ_i are singular simplices and the $a_i \in G$ where G is an abelian group.

Cohomology

If $C = \{C_n, \partial\}$ is a chain complex and G an abelian group then C^*_n is the group of homomorphisms from C_n to G and

$$\delta \colon C^*_n \to C^*_{n+1}$$

is the homomorphism given by $\delta\varphi(\alpha) = \varphi(\partial\alpha)$ with $\varphi \in C^*_n$ and $\alpha \in C_{n+1}$. We have $\delta^2 = 0$ so $C^* = \{C^*_n, \delta\}$ is chain complex. The homology groups of C^* are the *cohomology groups* $H^n(C;G)$.

A *free resolution* of an abelian group H is a long exact sequence

$$\cdots \to F_1 \to F_0 \to H \to 0$$

where the F_i are free abelian groups. Note that the free resolution is an exact sequence so has trivial homology. However, if we take the dual sequence $F^*_i = \mathrm{Hom}(F_i, G)$ (and $H^* = \mathrm{Hom}(H, G)$) we get a chain complex that may not be exact so the free resolution may have non-trivial cohomology.

Proposition 0.23 *If $\{F_i\}$ and $\{F'_i\}$ are free resolutions of abelian groups H and H' then any homomorphism $\phi \colon G \to G'$ extends to a chain map between the free resolutions and the induced maps on cohomology only depend on ϕ. In particular, if we let $H = H'$ and assume that ϕ is the identity we see that the cohomology of the free resolution only depends on H.*

Theorem 0.24 (Universal Coefficient Theorem) *If C is a chain complex of free abelian groups then*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C),G) \longrightarrow H^n(C;G) \xrightarrow{h} \mathrm{Hom}(H_n(C),G) \longrightarrow 0$$

is a split short exact sequence.

The homomorphism h can be described follows: If $[\varphi] \in H^n(C;G)$ is represented by a co-cycle φ then for all $\alpha = \partial\beta \in B_n(C)$ we have $\varphi(\alpha) = \varphi(\partial\beta) = \delta(\beta) = 0$ so φ descends to a homomorphism $\tilde{\varphi}$ from $H_n(C)$ to G and $h(\varphi) = \tilde{\varphi}$.

For $\mathrm{Ext}(H_{n-1}(C),G)$ we have the free resolution

$$0 \longrightarrow B_{n-1}(C) \longrightarrow Z_{n-1}(C) \longrightarrow H_{n-1}(C) \longrightarrow 0$$

which dualizes to the chain complex

$$0 \longleftarrow B^*_{n-1}(C) \longleftarrow Z^*_{n-1}(C) \longleftarrow H^*_{n-1}(C) \longleftarrow 0.$$

Therefore $\mathrm{Ext}(H_{n-1}(C),G)$ is the quotient of $B^*_{n-1}(C)$ by the image of $Z^*_{n-1}(C)$. For $\varphi \in B^*_{n-1}(C)$ we note that $\delta\varphi \in Z^n(C;G)$ so $[\delta\varphi]$ represents a cohomology class in $H^n(C;G)$.

To calculate Ext we can use the following facts:

- $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H,G) \oplus \mathrm{Ext}(H',G)$;
- If H is free then $\mathrm{Ext}(H,G) = 0$;
- $\mathrm{Ext}(\mathbb{Z}_k, G) = G/kG$.

For example $\mathrm{Ext}(\mathbb{Z}_k,\mathbb{Z}) = \mathbb{Z}_k$ while if n and m are relatively prime integers $\mathrm{Ext}(\mathbb{Z}_n,\mathbb{Z}_m) = 0$.

Induced maps on cohomology

If $i \colon B \to C$ is a chain map then we get induced homomorphisms $i^* \colon H^n(C;G) \to H^n(B;G)$ defined as follows: We first define i^* on the level of co-chains. If $\varphi \in C^*_n$ and $\alpha \in B_n$ we define $i^*(\varphi)(\alpha) = \varphi(i(\alpha))$. Then $i^*(\varphi)$ is a co-chain in B^*_n and the map i^* descends to homomorphism from $H^n(C;G) \to H^n(B;G)$. Note that the order of the homomorphism has been reversed from the order of i.

Singular cohomology

If X is a topological space then $C^k(X;G) = C_k(X;G)^*$. Then $\{C^k(X;G),\delta\}$ is the singular co-chain complex. We can also define simplicial and cellular cohomology. All of the theorems in homology have corresponding theorems in homology (excision, Mayer-Vietoris, simplicial and cellular equivalences with singular). However, the direction of all of the homomorphisms is reversed.

The group of relative co-chains $C^k(X,A;G)$ is the subgroup $C^k(X;G)$ consisting of co-chains that are zero on any chain contained in A. Note that in cohomology this is a subgroup rather than a quotient group as in homology. Just as in homology we get a long exact sequence in cohomology except that once again the direction of the homomorphisms is reversed.

If $f \colon X \to Y$ are induced maps on cohomology are also reversed. Namely we have $f^* \colon H^k(Y;G) \to H^k(X;G)$ with $f^*([\varphi])([\alpha]) = \varphi(f_*(\alpha))$.

Proposition 0.25 *If $A \subset X$ and $r \colon X \to A$ is a retraction then $r_* \colon H_k(X) \to H_k(A)$ is surjective and $r^* \colon H^k(A;G) \to H^k(X;G)$ is injective. If $\iota \colon A \hookrightarrow X$ is the inclusion map then ι_* is injective and ι^* is surjective.*

Cup product

We now work in a commutative ring R instead of an abelian group G. If $\varphi \in C^k(X;R)$ and $\psi \in C^\ell(X;R)$ then define

$$\varphi \smile \psi(\sigma) = \varphi(\sigma_{[v_0 \cdots v_k]}) \cdot \psi(\sigma_{[v_k \cdots v_{k+\ell}]})$$

on a singular $(k+\ell)$-simplex σ and extend it to arbitrary $(k+\ell)$-chains. We have

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^k \varphi \smile \delta\psi.$$

From this formula we get a well defined map

$$H^k(X;R) \times H^\ell(X;R) \to H^{k+\ell}(X;R)$$

defined by

$$[\varphi] \smile [\psi] = [\varphi \smile \psi].$$

Manifolds and orientation

A topological space M is a *manifold* if every point in $x \in M$ has a neighborhood U with U homeomorphic to $\mathbb{R}^n$. We usually assume that manifolds are Hausdorff and second countable. If M is connected then, by invariance of domain, the dimension n is constant and this is the dimension of M. By excision we have that

$$H^n(M, M-x) \cong H^n(U, U-x) \cong H^n(\Delta^n, \partial\Delta^n).$$

These groups are all isomorphic to $\mathbb{Z}$ and if $\sigma\colon \Delta^n \to M$ is an embedding with x contained in the interior of the image then σ is a generator of $H^n(M, M-x)$.

Define

$$\tilde{M} = \{\mu_x | x \in M \text{ and } \mu_x \text{ is a generator of } H^n(M, M-x)\}.$$

To give $\tilde{M}$ a topology we let σ be a singular simplex as above and let

$$U_\sigma = \{\mu_x = [\sigma]\}.$$

This is a basis for a topology on $\tilde{M}$ and we give $\tilde{M}$ this topology. Then the map $\mu_x \mapsto x$ is a 2-to-1 covering map from $\tilde{M}$ to M. If $\tilde{M}$ has two components then M is *orientable* and a choice of component is an *orientation*. Otherwise M is non-orientable.

Theorem 0.26 *If M is orientable and compact then there for each choice of orientation there is an $[M] \in H_n(M)$ such that for each $x \in M$ the homomorphism from $H_n(M) \to H_n(M, M-x)$ takes $[M]$ to the orientation.*

For any compact manifold M we have $H_n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Cap product

For $k \geq \ell$ we define

$$C_x(X;R) \times C^\ell(X;R) \xrightarrow{\frown} C_{k-\ell}(X;R)$$

as follows. If σ is a singular singular simplex and φ is a co-chain then

$$\sigma \frown \varphi = \varphi(\sigma_{[v_0 \cdots v_\ell]})\sigma_{[v_\ell \cdots v_k]}.$$

We then extend to chains linearly.

We also have

$$\partial(\alpha \frown \varphi) = (-1)^{\ell}(\partial\alpha \frown \varphi - \alpha \frown \delta\varphi)$$

which gives that the cap product gives a well defined map

$$H_k(X;R) \times H^{\ell}(X;R) \to H_{k-\ell}(X;R).$$

Theorem 0.27 (Poincaré Duality) *Let M be a compact, orientable manifold. Given $[M] \in H_n(M)$ define $D\colon H^k(M;\mathbb{Z}) \to H_{n-k}(M;\mathbb{Z})$ by $D([\varphi]) = [M] \frown [\varphi]$. Then D is an isomorphism.*

For any compact manifold, connected n-dimensional manifold let $[M]$ be the generator of $H_n(M;\mathbb{Z}_2)$ and define $D\colon H^k(M;\mathbb{Z}_2) \to H_{n-k}(M;\mathbb{Z}_2)$ by $D([\varphi]) = [M] \frown [\varphi]$. Then D is an isomorphism.

If $[\varphi] \in H^k(X;R)$ then

$$[\psi] \mapsto [\varphi \smile \psi]$$

is a homomorphism from $H^{\ell}(X;R)$ to $H^{k+\ell}(X;R)$. This defines a homomorphism from $H^k(X;R)$ to $\mathrm{Hom}(H^{\ell}(X;R), H^{k+\ell}(X;R))$. If M is a closed, orientable n-manifold, $k+\ell = n$, and $R = \mathbb{Z}$ then this defines a map from $H^k(M;\mathbb{Z})$ to $\mathrm{Hom}(H^{\ell}(M;\mathbb{Z}),\mathbb{Z})$ since $H^n(M;\mathbb{Z}) \cong \mathbb{Z}$. For any closed manifold M with $R = \mathbb{Z}_2$ we have $H^n(M;\mathbb{Z}_2) \cong \mathbb{Z}_2$ so we get a homomorphism from $H^k(M;\mathbb{Z}_2)$ to $\mathrm{Hom}(H^{\ell}(M;\mathbb{Z}_2),\mathbb{Z}_2)$

Theorem 0.28 *Let M be a closed, orientable n-manifold with $k+\ell = n$. Then the homomorphism*

$$H^k(M;\mathbb{Z}) \to \mathrm{Hom}\left(H^{\ell}(M;\mathbb{Z}),\mathbb{Z}\right)$$

restricted to the free part of $H^k(M;\mathbb{Z})$ is an isomorphism.

For any closed manifold M the homomorphism

$$H^k(M;\mathbb{Z}_2) \to \mathrm{Hom}\left(H^{\ell}(M;\mathbb{Z}_2),\mathbb{Z}_2\right)$$

is an isomorphism.