Computations with cup product

Let X be path connected. Define a 0-cocycle ε_1 on $X \times [0,1]$ as follows. For 0-chains we will identify points with the corresponding singular simplex so a 0-chain is of the form $\alpha = n_1 p_1 + \cdots + n_k p_k$. Then define

$$\varepsilon_1(\alpha) = \sum_{p_i \in X \times \{1\}} n_i.$$

Problem 1 *Show that $\delta\varepsilon_1$ is a generator of $H^1(X \times [0,1], X \times \{0,1\}; \mathbb{Z})$.*

Let $\psi = \delta\varepsilon_1$.

Given a singular k-simplex $\sigma \colon \Delta^k \to X$ then (a slight variation of) the prism operator defines a homomorphism $P \colon C_k(X) \to C_{k+1}(X \times [0,1])$. Recall how this works. Label the vertices of $\Delta^k \times [0,1]$ at level 0 as $v_0, \ldots, v_k$ and those at level 1 as $w_0, \ldots, w_k$. Then $j+1$ distinct vertices $\{u_0, \ldots, u_j\}$ in the collection $\{v_0, \ldots, v_k, w_0, \ldots, w_k\}$ span a j-simplex in $\Delta^k \times [0,1]$. A singular k-simplex σ determines a product map $\Delta^k \times [0,1] \to X \times [0,1]$ by the formula

$$(x,t) \mapsto (\sigma(x), t)$$

and we denote $\sigma_{[u_0 u_1 \cdots u_{k+1}]}$ to be the singular $k+1$ simplex in $X \times [0,1]$ obtained by restricting the product map to the simplex defined the vertices $\{u_0, \ldots, u_{k+1}\}$. We then define

$$P\sigma = \sum_{i=0}^{k} (-1)^{i+1} \sigma_{[v_0 \cdots v_i w_i \cdots w_k]}$$

and extend this to a homomorphism from k-chains in X to $k+1$-chains in $X \times [0,1]$. Let $\sigma_0 = \sigma_{[v_0, \ldots v_1]}$ and $\sigma_1 = \sigma_{[w_0, \ldots, w_k]}$. Then

$$\partial P\sigma = \sigma_0 - \sigma_1 + P(\partial\sigma).$$

This calculation is essentially the same as the calculation for the usual prism operator.

Problem 2 *Show that P defines a chain map from the singular chain complex $\{C_k(X)\}$ to the relative singular chain complex $\{C_{k+1}(X \times [0,1], X \times \{0,1\})\}$. Note that the indices differ by 1. For this to make sense for $C_0(X \times [0,1], X \times \{0,1\})$ we define $C_{-1}(X) = \{0\}$ and $P \colon C_{-1}(X) \to C_0(X \times [0,1], X \times \{0,1\})$ to be the zero map.*

The chain map P induces homomorphisms $P_* \colon H_k(X) \to H_{k+1}(X \times [0,1], X \times \{0,1\})$. We want to show that $k \geq 0$ that P_* is a homomorphism.

Note that $H_k(X \times \{0,1\})$ is isomorphic to $H_k(X \times \{0\}) \oplus H_k(X \times \{1\}) \cong H_k(X) \oplus H_k(X)$. We let $p_i \colon H_k(X \times \{0,1\}) \to H_k(X \times \{i\})$ be the projection homomorphism.

Problem 3 *Show that $P_* \colon H_k(X) \to H_{k+1}(X \times [0,1], X \times \{0,1\})$ is an isomorphism for $k \geq 0$. (***Hint:*** Use the long exact sequence on the pair $(X \times [0,1], X \times \{0,1\})$ to show that $p_0 \circ \partial$ induces an isomorphism. Then use that $P \circ p_0 \circ \partial$ is the identity on cycles to show that P_* is an isomorphism.)*

We stated this in class without proof:

Problem 4 *Let B be a chain complex of free abelian groups and $\iota \colon A \hookrightarrow B$ a sub-complex. That is A is a chain complex and ι is an injective chain map. If the the induced maps ι_* on homology are isomorphisms show that the induced maps ι^* on cohomology are isomorphisms. That is show that two cocycles φ and ψ on B are cohomologous in B if and only if their restrictions to A are cohomologous in A.*

The projection $X \times [0,1] \hookrightarrow X \times \{0\}$ induces a co-chain map $C^k(X;\mathbb{Z}) \to C^k(X \times [0,1];\mathbb{Z})$ and this map induces an isomorphism on cohomology. To (hopefully) reduce notational confusion we won't distinguish between a co-cycle on X and its image on $X \times [0,1]$.

Problem 5 *Show that the map from $H^k(X;\mathbb{Z}) \to H^{k+1}(X\times[0,1], X\times\{0,1\};\mathbb{Z})$ given by $[\varphi] \mapsto [\varphi]\cup[\psi]$ is an isomorphism. (***Hint:** *Use the previous problem to show that co-cycles in $C^{k+1}(X\times[0,1], X\times\{0,1\};\mathbb{Z})$ are determined by their values on chains of the form $P\sigma$.)*

The cup product on $X\times S^1$

Let p_X and p_{S^1} be the projections of $X\times S^1$ to X and S^1, respectively. These are retractions so the maps $(p_X)^*$ and $(p_{S^1})^*$ are injective. Therefore we will not distinguish between elements of $H^k(X;\mathbb{Z})$ and $H^k(S^1;\mathbb{Z})$ and their images in $H^k(X\times S^1;\mathbb{Z})$. In particular, let $[\psi]$ be a generator of $H^1(S^1;\mathbb{Z})\cong\mathbb{Z}$. Then we have a homomorphism $H^{k-1}(X;\mathbb{Z})\to H^k(X\times S^1;\mathbb{Z})$ given by

$$[\varphi]\mapsto[\varphi\cup\psi].$$

Combining this with the inclusion map of $H^k(X;\mathbb{Z})$ into $H^k(X\times S^1;\mathbb{Z})$ we have a homomorphism from the direct sum $H^{k-1}(X)\oplus H^k(X)$ to $H^k(X\times S^1;\mathbb{Z})$. We want to show that this map is an isomorphism. The basic strategy is the same with a few extra complications.

Let $I\subset S^1$ be a closed interval and $J\subset I\subset S^1$ a closed interval that is contained in the interior of I. Then by excision we have isomorphisms

$$H_k(X\times(S^1\smallsetminus J), X\times(I\smallsetminus J))\to H_k(X\times S^1, X\times I).$$

We also have that $(X\times(S^1\smallsetminus J), X\times(I\smallsetminus J))$ deformation retracts (as pairs) to $(X\times[0,1], X\times\{0,1\})$ giving isomorphisms from

$$H_k(X\times[0,1], X\times\{0,1\})\to H_k(X\times S^1, X\times(I\smallsetminus J)).$$

From our work above we also have an isomorphism

$$H_{k-1}(X)\to H_k(X\times[0,1], X\times\{0,1\}).$$

Composing we have an isomorphism

$$H_{k-1}(X)\to H_k(X\times S^1, X\times I).$$

From the long exact sequence of the pair $(X\times S^1, X\times I)$ we get the short exact sequence

$$0\longrightarrow H_k(X\times I)\longrightarrow H_k(X\times S^1)\longrightarrow H_k(X\times S^1, X\times I)\longrightarrow 0.$$

Swapping out the first term for $H_k(X)$ and the last term for $H_{k-1}(X)$ we get a short exact seqeunce

$$0\longrightarrow H_k(X)\longrightarrow H_k(X\times S^1)\longrightarrow H_{k-1}(X)\longrightarrow 0.$$

A version of the prism operator gives an isomorphism $S\colon C_{k-1}(X)\to C_k(X\times S^1)$ which will be a chain map and will give a splitting of the short exact sequence. In particular we have

$$H_k(X\times S^1)\cong H_k(X)\oplus H_{k-1}(X).$$

We need to understand this isomorphism as being induced from a chain map from $C_k(X)\oplus C_{k-1}(X)$ (which is a direct sum of chain complexes) and $C_k(X\times S^1)$.

From the first part we have a chain map

$$P\colon C_k(X)\to C_k(X\times[0,1], X\times\{0,1\})$$

that induces an isomorphism on homology. If we view $[0,1]$ as an subspace of S^1 whose interior if $S^1\smallsetminus I$ we have an inclusion of pairs

$$(X\times[0,1], X\times\{0,1\})\subset(X\times(S^1\smallsetminus J), X\times(I\smallsetminus J)).$$

This inclusion map is a homotopy equivalence of pairs so the induced chain map

$$C_k(X \times [0,1], X \times \{0,1\}) \to C_k(X \times (S^1 \setminus J), X \times (I \setminus J))$$

also induces an isomorphism on homology.

We also have an inclusion of pairs

$$(X \times (S^1 \setminus J), X \times (I \setminus J)) \hookrightarrow (X \times S^1, X \times I)$$

and the chain map induces an isomorphism on homology by excision.

Given a $p \in I \subset S^1$ we also have map

$$(X \times S^1, X \times I) \to (X \times S^1, X \times \{p\})$$

where the map is the identity on the first factor and the quotient map $S^1 \to S^1/I$ on the second factor.

Let

$$S_1 \colon C_{k-1}(X) \to C_k(X \times S^1, X \times \{p\})$$

be the composition of these chain maps. As the individual chain maps induce isomorphisms on homology so will their composition.

Finally we have a homomorphism

$$r \colon C_k(X \times S^1, X \times \{p\}) \to C_k(X)$$

with $r(\sigma) = 0$ if the image of σ is contained in $X \times \{p\}$ and $r(\sigma) = \sigma$ otherwise. This is **not** a chain map. However, the composition $S = r \circ S_0$ is:

Problem 6 *Show that S is a chain map.*

Let $\iota \colon X \times \{p\} \hookrightarrow X \times S^1$ be the inclusion map and define

$$\iota_\# \oplus S \colon C_k(X) \oplus C_{k-1}(X) \to C_k(X \times S^1)$$

by $\iota_\# \oplus S(\alpha, \beta) = \iota_\#(\alpha) + S\beta$.

Problem 7 *Show that $\iota_\# \oplus S$ induces an isomorphism from $H_k(X) \oplus H_{k-1}(X)$ to $H_k(X \times S^1)$.*

Now we can calculate the cup product. Recall that that we have projection maps p_X and p_{S^1} from $X \times S^1$ to X and S^1 that determine homomorphisms from $H^k(X;\mathbb{Z})$ and $H^k(S^1;\mathbb{Z})$ to $H^k(X \times S^1;\mathbb{Z})$. As mentioned above we will not distinguish between cohomology classes in $H^k(X;\mathbb{Z})$ and $H^k(S^1;\mathbb{Z})$ from their image in $H^k(X \times S^1;\mathbb{Z})$.

We have defined $[\psi]$ be the generator of $H^1(S;\mathbb{Z})$.

Problem 8 *The cohomology class $[\psi]$ is represented by a cocycle $\psi \in C^1(S^1;\mathbb{Z})$. Given an explicit description of ψ. In particular show that if a singular 1-simplex σ represents a cycle that generates $H_1(S^1)$ then $\psi(\sigma) = \pm 1$ while if σ is constant then $\psi(\sigma) = 0$. (**Hint:** Recall that we have a homomorphism $p \colon Z^1(S^1;\mathbb{Z}) \to C^1(S^1;\mathbb{Z})$. Use this to define ψ.)*

Problem 9 *Show that the homomorphism from $H^k(X;\mathbb{Z}) \oplus H^{k-1}(X;\mathbb{Z})$ to $H^k(X \times S^1;\mathbb{Z})$ given by*

$$([\alpha], [\varphi]) \mapsto [\alpha] + [\varphi] \cup [\psi]$$

is an isomorphism.