Notes on the Fundamental Group and Covering Spaces
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1 Introduction

A central question in topology is to distinguish when two topological spaces are not homeomorphic. This is typically a difficult question.

- A space with one point is not homeomorphic to a space with two points. More generally two spaces that are homeomorphic have the same cardinality.
- Another way to distinguish two spaces is to count the number of connected components - if two spaces are homeomorphic they will have the same number of connected components. If X is a connected space with infinitely many points (the real line $\mathbb{R}$ for example) and Y is two disjoint copies of the X then X and Y will have the same cardinality but will not be homeomorphic.
- The real line $\mathbb{R}$ and the circle

  $$S^1 = \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\right\}$$

  have the same cardinality and are both connected. However, they are not homeomorphic. One way to see this is that $\mathbb{R}$ is not compact while S^1 is compact.
- The closed interval $[0,1]$ and S^1 also have the same cardinality and are connected but are not homeomorphic. However, they are both compact so we need a new method to show this. Instead we note that when we remove the point $1/2$ from $[0,1]$ the space becomes disconnected but when we remove any point from S^1 it remains connected.
- A similar method can be used to show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic. When we remove any point from $\mathbb{R}$ the space becomes disconnected but when we do this to $\mathbb{R}^2$ the space remains connected. However, this method fails to distinguish between $\mathbb{R}^2$ and $\mathbb{R}^3$ or, more generally, if $n \neq m$ are both ≥ 2 it will not distinguish between $\mathbb{R}^n$ and $\mathbb{R}^m$.

Problem 1 *Show that* $[0,1]$, $(0,1]$ *and* $(0,1)$ *are mutually not homeomorphic.*

That $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ if and only if $n = m$ is called *invariance of domain*. This theorem of Brower is one of the first achievements of algebraic topology. We will not prove this in general in this class but will show that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ if $n \neq 2$.

A topological space X is *path connected* if for every two points $x_0, x_1 \in X$ there exists a continuous map

$$\gamma\colon [0,1] \to X$$

with $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Here is another, equivalent characterization of being path connected. Let

$$S^0 = \left\{x \in \mathbb{R} | x^2 = 1\right\}$$

and

$$D^1 = \left\{x \in \mathbb{R} | x^2 \leq 1\right\}.$$

Then X is path connected if every map

$$f\colon S^0 \to X$$

has a continuous extension to D^1.

Here is one way to generalize the concept of being path connected. Let

$$S^n = \left\{x \in \mathbb{R}^{n+1} \,\middle|\, |x|^2 = 1\right\}$$

and

$$D^n = \left\{ x \in \mathbb{R}^n \mid |x|^2 \leq 1 \right\}.$$

We say that a topological space X has *Property* SD_n if every map

$$f\colon S^n \to X$$

has a continuous extension to D^{n+1}.

This is not a standard definition.

Question 2 *$\mathbb{R}^n$ has property SD_m for all n and m and this is not difficult to prove. However, $\mathbb{R}^{n+1} \setminus \{0\}$ does not have SD_n. You should try to prove this! We will eventually show this when $n = 2$. This fact is how you show that $\mathbb{R}^n$ and $\mathbb{R}^m$ are homeomorphic if and only if $n = m$.*

2 Homotopy

If X and Y are topology spaces and

$$f, g\colon X \to Y$$

are two maps then f and g are homotopic if there exists a continuous map

$$F\colon X \times [0,1] \to Y$$

with $f(x) = F(x,0)$ and $g(x) = F(x,1)$. The map F is a *homotopy* between f and g. We write $f \sim g$ if f and g are homotopic.

Lemma 2.1 *Let X be a topological space. Then any two continuous maps $f, g\colon X \to \mathbb{R}^n$ are homotopic.*

Proof: Define

$$F(x,t)\colon X \times [0,1] \to \mathbb{R}^n$$

by

$$F(x,t) = (1-t)f(x) + tg(x).$$

□

The homotopy F is a *straight line homotopy*. Essentially all homotopies we construct will be based on this.

Lemma 2.2 *Let X be a singleton and Y a topological space. The $f, g\colon X \to Y$ are homotopic if and only if $f(X)$ and $g(X)$ are in the same path component of Y.*

Proof: We have $X = \{x\}$. If $f(x)$ and $g(x)$ are in the same path component of Y then there exists

$$\gamma\colon [0,1] \to Y$$

with $\gamma(0) = f(x)$ and $\gamma(1) = g(x)$. This is essentially a homotopy. That is define

$$F\colon X \times [0,1] \to Y$$

by $F(x,t) = \gamma(t)$.

On the other hand if $F\colon X \times [0,1] \times Y$ is a homotopy between f and g then $\gamma(t) = F(x,t)$ is a path between $f(x)$ and $g(x)$ so $f(X)$ and $g(X)$ are in the same path component of Y. □

This very simple fact is the basis for showing that two maps aren't homotopic in a large variety of situations.

Lemma 2.3 *Homotopy is an equivalence relation.*

Proof: Transitivity is the trickiest of the 3 properties to prove. Assume that $f \sim g$ and $g \sim h$ and let F be a homotopy between f and g and G a homotopy between g and h. Then

$$H\colon X \times [0,1] \to Y$$

defined by

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq 1/2 \\ G(x,2t-1) & 1/2 < t \leq 1 \end{cases}$$

is a homotopy between f and h. □

The following problem is needed to show that H is continuous.

Problem 3 *Let X be a topological space and $A, B \subset X$ closed subspaces whose union is X. Let $f\colon A \to Y$ and $g\colon B \to Y$ be continuous such that $f = g$ on $A \cap B$. Then there exists a unique continuous map $h\colon X \to Y$ with $h(x) = f(x)$ on A and $h(x) = g(x)$ on B.*

Problem 4 *Show that $f \sim f$ and if $f \sim g$ then $g \sim f$.*

Problem 5 *Show that a topological space X has property* SD_n *if and only if every continuous map $f\colon S^n \to X$ is homotopic to a constant map. Show that $\mathbb{R}^m$ has property* SD_n *for all n and m.*

If $X = [0,1]$ there is more refined concept of homotopy called path homotopy. Here is the definition. If $f, g\colon [0,1] \to Y$ are continuous paths with $f(0) = g(0)$ and $f(1) = g(1)$ then they are *path homotopic* if there exists a continuous map

$$F\colon [0,1] \times [0,1] \to Y$$

with $f(t) = F(t,0)$, $g(t) = F(t,1)$, $F(0,s) = f(0) = g(0)$, and $F(1,s) = f(1) = g(1)$.

Here is some useful notation: If X and Y are topological spaces and $A \subset X$ and $B \subset Y$ are subspaces then we write $f\colon (X,A) \to (Y,B)$ when f is a map from X to Y with $f(A) \subset B$. This is a *map of pairs*. We will often use this when $X = [0,1]$, $A = \{0,1\}$ and $B = \{y_0\}$ for some basepoint in $y_0 \in Y$. That is a map

$$f\colon ([0,1], \{0,1\}) \to (Y, \{y_0\})$$

is a map of the interval into Y that takes both endpoints to the basepoint y_0. This is a *loop* in Y.

Lemma 2.4 *Path homotopy is an equivalence relation.*

Proof: The proof is the same as for regular homotopy. One just needs to check that the homotopies constructed fix the endpoints. □

Lemma 2.5 *Two paths $f, g\colon [0,1] \to \mathbb{R}^n$ are path homotopic if and only if $f(0) = g(0)$ and $f(1) = g(1)$.*

Proof: As before define $F\colon [0,1] \times [0,1] \to \mathbb{R}^n$ by

$$F(s,t) = (1-t)f(s) + tg(s).$$

Then if $f(0) = g(0)$ we have

$$F(0,t) = (1-t)f(0) + tg(0) = (1-t)f(0) + tf(0) = f(0)$$

and similarly $F(1,t) = f(1)$ if $f(1) = g(1)$. Therefore F is path homotopy between f and g. □

3 $[0,1]$, $\mathbb{R}$ and S^1

We have seen that all maps to $\mathbb{R}^n$ are homotopic and all paths are homotopic exactly when there endpoints agree. The simplest example where there is more than one homotopy class is maps from S^1 to itself. Rather than study this directly we will first study path homotopies to S^1. We'll need some setup.

Define

$$p\colon \mathbb{R} \to S^1$$

by

$$p(t) = (\cos 2\pi t, \sin 2\pi t)$$

and define maps

$$\tilde{f}_n\colon [0,1] \to \mathbb{R} \qquad \text{and} \qquad f_n\colon [0,1] \to S^1$$

by

$$\tilde{f}_n(t) = nt \qquad \text{and} \qquad f_n = p \circ \tilde{f}_n.$$

Theorem 3.1 *Every*

$$f\colon ([0,1],\{0,1\}) \to \left(S^1,(1,0)\right)$$

is path homotopic to a unique f_n.

3.1 Lifts

The proof of Theorem 3.1 requires several steps. Essentially we want to use the map p to convert the problem to one about maps from the interval to $\mathbb{R}$. A key tool we use is the notion of a *lift* of a map. We'll only define it here for the special case of lifts of maps to S^1 to $\mathbb{R}$ but we'll later see that this is just an example of a more general concept.

If

$$f\colon X \to S^1$$

is a continuous maps then a *lift* of f to $\mathbb{R}$ is a continuous map

$$\tilde{f}\colon X \to \mathbb{R}$$

with $f = p \circ \tilde{f}$. Lifts, when they exist, are not unique. To see this let $\tilde{f}$ be a lift of f and for each $n \in \mathbb{Z}$ define $\tilde{f}_n$ by $\tilde{f}_n(x) = \tilde{f}(x) + n$. Since $p(t+n) = p(t)$ we have $p \circ \tilde{f} = p \circ \tilde{f}_n$ so $\tilde{f}_n$ is also a lift of f.

To get a unique lift we need to fix basepoints. Take a map of pairs

$$f\colon (X,x_0) \to \left(S^1,(1,0)\right).$$

Then we can get a unique lift by fixing where we send the basepoint x_0. Since $p^{-1}(0,0) = \mathbb{Z}$ we need to choose an integer and zero is a natural choice. We will look for a lift

$$\tilde{f}\colon (X,x_0) \to (\mathbb{R},0).$$

We will eventually give a very general criteria for lifts existing and being unique. We start with this:

Proposition 3.2 (Path lifting lemma for S^1) *Let*

$$f\colon ([0,1],0) \to \left(S^1,(0,0)\right)$$

be a continuous map of pairs. Then there exists a unique lift

$$\tilde{f}\colon ([0,1],0) \to (\mathbb{R},0).$$

That is $\tilde{f}$ is the unique continuous map with $f = p \circ \tilde{f}$ and $\tilde{f}(0) = 0$.

To prove this we first need some properties of the map p.

A subset $A \subset \mathbb{R}$ is $\mathbb{Z}$*-invariant* if for all $n \in \mathbb{Z}$ when $t \in A$ we have $t+n \in A$. We observe the following.

- If $B \subset S^1$ then $p^{-1}(B) \subset \mathbb{R}$ is $\mathbb{Z}$-invariant.
- The intersection of a collection (of any cardinality) of $\mathbb{Z}$-invariant sets is $\mathbb{Z}$ invariant. If $A \subset \mathbb{R}$ is non-empty then the intersection $\tilde{A}$ of all $\mathbb{Z}$-invariant sets containing A is the smallest $\mathbb{Z}$-invariant set containing A. That is $\tilde{A}$ is a subset of any other $\mathbb{Z}$-invariant set containing A.
- If $A \subset \mathbb{R}$ and $B = p(A)$ then $p^{-1}(B)$ is the smallest $\mathbb{Z}$-invariant subset that contains A. In particular, if A is $\mathbb{Z}$-invariant then $A = p^{-1}(B)$.

Problem 6 *Prove the above bullets.*

Lemma 3.3 *The map p is open. That is the p-image of an open set is open.*

Proof: Recall that the topology on $S^1 \subset \mathbb{R}^2$ is the subspace topology so a set in S^1 is open if it is the intersection of an open set in $\mathbb{R}^2$ with S^1. We claim that for every $t \in \mathbb{R}$ and every $\varepsilon > 0$ the p-image of the interval $(t-\varepsilon, t+\varepsilon)$ is open. This can be done explicitly and we omit the details.

Now assume that $V \subset \mathbb{R}$ is open. To show that $p(V)$ is open we need to show that every $(x,y) \in p(V)$ has an open neighborhood that is contained in $p(V)$. Choose $t \in V$ with $p(t) = (x,y)$. Since t is open there exists and $\varepsilon > 0$ such that $(t-\varepsilon, t+\varepsilon) \subset V$. The p-image of this interval is contained in $p(V)$ and by the previous paragraph it is open in S^1. Therefore (x,y) has an open neighborhood in S^1 that is contained in $p(V)$ so $p(V)$ is open. □

Here is a general fact that we'll find very useful.

Lemma 3.4 *Let*

$$f\colon X \to Y$$

be an open map. Assume that $V \subset X$ is open and f restricted to V is injective. Then f is a homeomorphism from V to $U = f(V)$.

Proof: Since f restricted to V is injective there is a well defined inverse $f^{-1}\colon U \to V$. Let $A \subset V$ be open (in the subspace topology on V). Then A is the intersection of an open set in X with V. As the intersection of two open subsets is open we have that A is open in X. Therefore $(f^{-1})^{-1}(A) = f(A)$ is open in Y and as $f(A) = f(A) \cap U$, we have that $f(A)$ is open in U. This implies that f^{-1} is continuous and that f is homeomorphism from V to U. □

A neighborhood $U \subset S^1$ is *evenly covered* (with respect to $p\colon \mathbb{R} \to S^1$) if for every component V of $p^{-1}(U)$ the restriction $p|_V$ is a homeomorphism from V to U.

Proposition 3.5 *Every point in S^1 has an evenly covered neighborhood.*

Proof: Given $x, y \in \mathbb{R}$ with $0 < y - x < 1$ define

$$A = \bigcup_{n \in \mathbb{Z}} (x+n, y+n).$$

This is a disjoint collection of open intervals and on each interval p is injective (since $p(t) = p(s)$ if and only if $s = t+m$ for some $m \in \mathbb{Z}$ but if $t, s \in (x+n, y+n)$ then $|t-s| < 1$). The set A is also the smallest $\mathbb{Z}$-invariant set that contains any of the individual intervals $(x+n, y+n)$ so $U = p(A) = p((x+n, y+n))$ for all n. Lemmas 3.3 and 3.4 then imply that the restriction of p to each interval $(x+n, y+n)$ is a homeomorphism to U so U is evenly covered. The map p is surjective and clearly for any $t \in \mathbb{R}$ we can construct a set A that contains t. This implies that every point in S^1 has an evenly covered neighborhood. □

Proposition 3.2, Existence of lifts: We can now prove the existence of a lift. Recall that we have a map

$$f\colon [0,1] \to S^1.$$

We claim that there exists a partition $s_0 = 0 < s_1 < \cdots < s_n = 1$ such that for each interval $J_i = [s_{i-1}, s_1]$ we have $p(J_i) \subset U_i$ where U_i is an evenly covered neighborhood. We'll prove the existence of the lift assuming the claim.

We inductively prove that the lift exists on the interval $[0, s_i]$. If $i = 0$ then the interval is just the point 0 and we define $\tilde{f}(0) = 0$. Now assume that $\tilde{f}$ is defined on $[0, s_{i-1}]$ and we'll show that it can be extended to $[0, s_i]$. Let V_i be the component of $p^{-1}(U_i)$ that contains $\tilde{f}(s_{i-1})$ and let $q_i\colon U_i \to V_i$ be the inverse of $p|_{V_i}$ given by the evenly covered property. Then define $\tilde{f} = q_i \circ f$ on $[s_{i-1}, s_i]$. Note that both $\tilde{f}(s_{i-1})$ and $q_i \circ f(s_{i-1})$ lie in $V_i \cap p^{-1}(f(s_{i-1}))$. As there is a unique such point in the intersection we have $\tilde{f}(s_{i-1}) = q_i \circ f(s_{i-1})$ and therefore the extension of $\tilde{f}$ to $[s_{i-1}, s_i]$ is continuous on $[0, s_i]$. By induction the lift exists on all of $[0,1]$.

Now we prove the existence of the partition. By the continuity of f and Proposition 3.5 every $t \in [0,1]$ is contained in the interior (relative to $[0,1]$) of a closed interval I_t such that $f(I_t)$ is contained in an evenly covered neighborhood. Then the interior of I_t cover $[0,1]$ and by compactness there is a finite subcover. We then choose s_i to be the endpoints in the intervals in the finite subcover. We choose the indices of the s_i so that $s_i < s_{i+1}$. Note that intervals $[0,t)$ and $(t,1]$ are open in $[0,1]$ and intervals of this type must be in the finite subcover. In particular, $s_0 = 0$ and the for the final point $s_n = 1$. By construction each of the $[s_i, s_{i+1}]$ will lie in some I_t and therefore $f([s_i, s_{i+1}])$ lies in an evenly covered neighborhood and we have constructed the desired partition. □

An open, closed and non-empty subset of a connected set is the entire set. This is a very useful fact! We will use this in the next proof and quite often later.

We state the uniqueness part of Proposition 3.2 as a separate lemma:

Lemma 3.6 *Let*

$$\tilde{f}_0, \tilde{f}_1\colon ([0,1], 0) \to (\mathbb{R}, 0)$$

be two lifts of f and let

$$A = \left\{ t \in [0,1] | \tilde{f}_0(t) = \tilde{f}_1(t) \right\}.$$

We will show that A is open, closed, and non-empty. This implies that $A = [0,1]$.

- Since $\tilde{f}_0(0) = 0 = \tilde{f}_1(0)$, we have $0 \in A$ and $A \neq \emptyset$.
- See Problem 7.
- The main work is to show that A is open. Assume that $t \in A$. Then, as above, t is in the interior of an interval I with $p(I)$ contained in an evenly covered neighborhood $U \subset S^1$. Since I is connected $\tilde{f}_0(I)$ and $\tilde{f}_1(I)$ will be connected. Since $p \circ \tilde{f}_i(I) = f(I)$ both of these sets must lie in $p^{-1}(U)$. As they are connected they must each lie in a single component of $p^{-1}(U)$ and since $\tilde{f}_0(t) = \tilde{f}_1(t)$ this must be same component V for both. Let $q\colon U \to V$ be the inverse of $p|_V$. Then

  $$\left(p \circ \tilde{f}_i\right)|_I = p|_V \circ \tilde{f}_i|_I = f|_i$$

  and composing on the left with q gives

  $$q \circ p|_V \circ \tilde{f}_i|_I = \tilde{f}_i|_I = q \circ f|_I.$$

  Therefore $\tilde{f}_0|_I = \tilde{f}_1|_I$ so $I \subset A$ and A is open.

□

Problem 7 *Let*

$$f_0, f_1 : X \to Y$$

be continuous maps. If Y is Hausdorff the set

$$A = \{x \in X | f_0(x) = f_1(x)\}$$

is closed.

The uniqueness statement easily generalizes:

Corollary 3.7 *Let X be path connected and*

$$f\colon (X, x_0) \to \big(S^1, (1,0)\big)$$

continuous. If

$$\tilde{f}_0, \tilde{f}_1\colon (X, x_0) \to (\mathbb{R}, 0)$$

are both lifts of f then $\tilde{f}_0 = \tilde{f}_1$.

Proof: Given $x \in X$ let γ: $[0,1] \to X$ be a path from x_0 to x. Then

$$f \circ \gamma\colon ([0,1], 0) \to \big(S^1, (1,0)\big)$$

is a continuous map of pairs and $\tilde{f}_0 \circ \gamma$ and $\tilde{f}_1 \circ \gamma$ are both lifts of $f \circ \gamma$ and by Lemma 3.6 they are equal. Therefore

$$\tilde{f}_0(x) = \tilde{f}_0 \circ \gamma(1) = \tilde{f}_1 \circ \gamma(1) = \tilde{f}_1(x)$$

so $\tilde{f}_0 = \tilde{f}_1$. □

Problem 8 *Let*

$$f\colon (\mathbb{R}, 0) \to \big(S^1, (0,0)\big)$$

be a continuous map of pairs. Show that there exists a unique lift

$$\tilde{f}\colon (\mathbb{R}, 0) \to (\mathbb{R}, 0).$$

A topological space X is a *topological tree* if X is locally path connected and for any distinct x and y in X there is an injective path $\alpha\colon [0,1] \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$ and for any other such injective path β there is a homeomorphism $\phi\colon [0,1] \to [0,1]$ with $\alpha = \beta \circ \phi$. A space is locally path connected path if for every $x \in X$ and every neighborhood U of x there is a path connected neighborhood U_0 of x with $U_0 \subset U$.

Problem 9 *Let X be a topological tree with basepoint $x_0 \in X$. If*

$$f\colon (X, x_0) \to \big(S^1, (1,0)\big)$$

is a continuous map of pairs there exists a unique lift

$$\tilde{f}\colon (X, x_0) \to (\mathbb{R}, 0).$$

Lemma 3.8 *Given continuous map*

$$f\colon ([0,1], 0, 1) \to \big(S^1, (1,0)\big)$$

is homotopic to a f_n.

Proof: Let

$$\tilde{f}\colon ([0,1],0) \to (\mathbb{R},0)$$

be the lift of f. Since $p \circ \tilde{f}(1) = f(1) = (1,0)$ we have $\tilde{f}(1) \in p^{-1}(1,0) = \mathbb{Z}$. That is $\tilde{f}(1) = n$ for some $n \in \mathbb{Z}$. Let

$$\tilde{F}\colon [0,1] \times [0,1] \times \mathbb{R}$$

be the path homotopy from $\tilde{f}$ to $\tilde{f}_n$ given by Lemma 2.5. Then $F = p \circ \tilde{F}$ is a path homotopy from f to f_n. □

Proposition 3.9 (Homotopy path lifting lemma for S^1) *Let*

$$F\colon \big([0,1]^2,(0,0)\big) \to \big(S^1,(0,0)\big)$$

be a continuous map of pairs. Then there exists a unique lift

$$\tilde{F}\colon \big([0,1]^2,(0,0)\big) \to (\mathbb{R},0).$$

Proof: The proof is very similar to the path lifting lemma. First we start with partitions $s_0 = 0 < s_1 < \cdots < s_n = 1$ and $t_0 = 0 < t_1 < \cdots < t_m = 1$. such that the F-image of each of the rectangles $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is contained in an evenly covered neighborhood of S^1. As before we will assume the existence of these partitions and finish the proof of the lemma.

Together the two partitions partition the square into a collection of rectangles $R_1, \dots, R_{nm}$ where $R_{i+(j-1)n} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$. Let Q_k be the union $R_1 \cup \cdots \cup R_k$ with $Q_0 = \{(0,0)\}$. We inductively define $\tilde{F}$ on Q_k noting that $\tilde{F}(0,0) = 0$ which defined $\tilde{F}$ on Q_0. The key point is that $Q_k \cap R_{k+1}$ is connected.

Now assume that $\tilde{F}$ has been defined on Q_k and let U_{k+1} be an evenly covered neighborhood that contains $F(R_{k+1})$. Then $F(Q_k \cap R_{k+1})$ is contained in U_{k+1} so $\tilde{F}(Q_k \cap R_{k+1})$ is contained in $p^{-1}(U_k)$. As $Q_k \cap R_{k+1}$ is connected, $\tilde{F}(Q_k \cap R_{k+1})$ will be contained in a single component V_{k+1} of $p^{-1}(U_{k+1})$ and we let $q_{k+1}\colon U_{k+1} \to V_{k+1}$ be the inverse of $p|_{V_{k+1}}$. Define $\tilde{F} = q_{k+1} \circ F$ on R_{k+1}. This will agree with the definition of $\tilde{F}$ on $Q_k \cap R_{k+1}$ so will be a continuous extension of $\tilde{F}$ to Q_{k+1}. Then by induction $\tilde{F}$ is defined on all of $[0,1]^2$.

We now construct the partitions. The continuity of F and Proposition 3.5 imply that $(s,t) \in [0,1]^2$ is contained in the interior of a rectangle $R_{s,t}$ (with horizontal and vertical sides) such that $F(R_{s,t})$ is contained in an evenly covered neighborhood in S^1. As before the interiors of $R_{s,t}$ are an open cover of the compact set $[0,1]^2$ so there is a finite subcover. We then take the s_i to be the coordinates of the vertical sides and the t_j to be the coordinates of the horizontal sides. (You should draw a picture!)

The uniqueness follows from Corollary 3.7. □

The proof of Theorem 3.8 is completed with the next lemma.

Lemma 3.10 *The maps f_n and f_m are path homotopic if and only if $n = m$.*

Proof: Let F be a path homotopy between f_n and f_m and let $\tilde{F}$ be the lift given by Proposition 3.9. As $F(0,t) = (1,0)$ we have $\tilde{F}(0,t) \subset p^{-1}((1,0)) = \mathbb{Z}$. The restriction of $\tilde{F}$ to $\{0\} \times [0,1]$ is a homotopy of $\tilde{F}|_{(0,0)}$ to $\tilde{F}|_{(0,1)}$. Then by Lemma 2.2 $\tilde{F}|_{(0,0)}$ and $\tilde{F}|_{(0,1)}$ are in same path components of $\mathbb{Z}$ - that is they are equal. Therefore $0 = F(0,0) = F(0,1)$. Similarly $F(1,0) = F(1,1)$.

We also have that the restriction of $\tilde{F}$ to $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ are lifts of f_n and f_m and since $F(0,0) = F(0,1) = 0$ they are lifts that take zero to zero. The maps $\tilde{f}_n$ and $\tilde{f}_m$ are also lifts of f_n and f_m with $\tilde{f}_n(0) = 0$ and $\tilde{f}_m(0) = 0$ so by the uniqueness from Proposition 3.2 we have that $\tilde{f}_n(t) = \tilde{F}(t,0)$ and $\tilde{f}_m(t) = \tilde{F}(t,1)$. Therefore

$$n = \tilde{f}_n(1) = \tilde{F}(1,0) = \tilde{F}(1,1) = \tilde{f}_m(1) = m.$$

□

Theorem 3.11 *The identity map of S^1 to itself doesn't extend to a continuous map of D^2. Therefore the space S^1 doesn't have* SD_1.

Proof: The maps f_0 and f_1 have image S^1. As S^1 is a subspace of $\mathbb{R}^2$ we can view them both as maps to $\mathbb{R}^2$ and take

$$F\colon [0,1]^2 \to \mathbb{R}^2$$

be the straight line homotopy between them. That is

$$F(t,s) = (1-s)f_0(t) + sf_1(t).$$

Note that image of F is contained in $D^2 \subset \mathbb{R}^2$.

Now assume that

$$G\colon D^2 \to S^1$$

is a continuous extension of the identity map on $S^1 \subset D^2$. Then $G \circ F$ is a path homotopy from f_0 to f_1. This is a contradiction so the extension G can't exist. □

Corollary 3.12 $\mathbb{R}^2 \setminus \{(0,0)\}$ *doesn't have* SD_1.

Proof: Define

$$r\colon \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$$

by

$$r(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right).$$

Note that r is the identity on S^1. Let

$$f\colon S^1 \to S^1 \subset \mathbb{R}^2 \setminus \{(0,0)\}$$

be a continuous map. If f has a continuous extension

$$F\colon D^2 \to \mathbb{R}^2 \setminus \{(0,0)\}$$

then $r \circ F$ is a continuous extension of f to S^1. Therefore if $\mathbb{R}^2 \setminus \{(0,0)\}$ has SD_1 so does S^1 and by Theorem 3.11, $\mathbb{R}^2 \setminus \{(0,0)\}$ doesn't have SD_1. □

Theorem 3.13 (Brower Fixed Point Theorem) *If*

$$f\colon D^2 \to D^2$$

is continuous there exists and $(x,y) \in D^2$ with $f(x,y) = (x,y)$.

Proof: Let

$$\Delta = \left\{(x,x) \in D^2 \times D^2\right\}$$

be the diagonal in $D^2 \times D^2$ and define

$$\sigma\colon D^2 \times D^2 \setminus \Delta \to S^1$$

by setting $\sigma(x,y)$ to be the unique point in S^1 such that the line segment in $\mathbb{R}^2$ from y to $\sigma(x,y)$ contains x. (Draw a picture!) Note that if $x \in S^1 \subset D^2$ then $\sigma(x,y) = x$. The map σ is continuous. (Why?)

Now assume that $f\colon D^2 \times D^2$ doesn't have fixed points. Then the graph of f given by

$$F\colon D^2 \to D^2 \times D^2$$

with $F(x) = (x, f(x))$ has image disjoint from Δ. Therefore we can take the composition $\sigma \circ F$. This will be a continuous map of D^2 to S^1 that is the identity when restricted to S^1. This contradicts Theorem 3.11 and therefore f must have a fixed point. □

3.2 Concatenation of paths

Let

$$f,g\colon [0,1]\to X$$

be paths with $f(1)=g(0)$ and we define the *concatenation* $f*g$ of f and g by

$$f*g(t)=\begin{cases} f(2t) & 0\le t\le 1/2 \\ g(2t-1) & 1/2<t\le 1. \end{cases}$$

Lemma 3.14 *Let*

$$f_0,f_1,g_0,g_1\colon [0,1]\to X$$

be paths with $f_0(1)=f_1(1)=g_0(0)=g_1(0)$. *If* $f_0\sim_p f_1$ *and* $g_0\sim_p g_1$ *then* $(f_0*g_0)\sim_p(f_1*g_1)$.

Proof: Let F and G be the homotopies between the f_i and g_i. Then define

$$H(s,t)=\begin{cases} F(2t,s) & 0\le t\le 1/2 \\ G(2t-1,s) & 1/2<t\le 1. \end{cases}$$

□

Lemma 3.15 *Let*

$$f,g,h\colon [0,1]\to X$$

be paths with $f(1)=g(0)$ *and* $g(1)=h(0)$. *Then* $(f*g)*h\sim_p f*(g*h)$.

Proof: Define $\sigma\colon [0,1]\to[0,1]$ by

$$\sigma(t)=\begin{cases} 2t & 0\le 0\le t\le 1/4 \\ t+1/4 & 1/4<t\le 1 \\ t/2+1/2 & 1/2<t\le 1. \end{cases}$$

The map σ is a homeomorphism of the interval $[0,1]$ to itself and we can take the straight line homotopy to the identity. That is let $\tau\colon [0,1]\times[0,1]\to[0,1]$ be given by

$$\tau(t,s)=(1-s)t+s\sigma(t).$$

Then the composition $(f*g)*h\circ\tau$ is a path homotopy from $(f*g)*h$ to $f*(g*h)$. □

Problem 10 *If* f_0 *and* f_1 *are homotopic (or path homotopic) then* $f_0\circ g$ *and* $f_1\circ g$ *are homotopic (or path homotopic). Similarly* $h\circ f_0$ *and* $h\circ f_1$ *are homotopic (or path homotopic).*

If $f\colon [0,1]\to X$ is a path define $\bar f\colon [0,1]\to X$ by

$$\bar f(t)=f(1-t).$$

For $x\in X$ define $\mathrm{id}_x\colon [0,1]\to X$ to be the constant map $\mathrm{id}_x(t)=x$.

Lemma 3.16 *If* $f\colon [0,1]\to X$ *is a path with* $f(0)=x_0$ *and* $f(1)=x_1$ *then*

$$f*\mathrm{id}_{x_1}\sim_p \mathrm{id}_{x_0}*f\sim_p f.$$

Proof: Define

$$F\colon [0,1]\times[0,1]\to X$$

by

$$F(t,s) = f * \mathrm{id}_{x_1}\left((1-s/2)t\right).$$

This defines a path homotopy between $f * \mathrm{id}_{x_1}$ and f.

We also define

$$G\colon [0,1]\times[0,1]\to X$$

by

$$G(t,s) = \mathrm{id}_{x_0} * f\left(s/2 + (1-s/2)t\right).$$

This is a path homotopy between $\mathrm{id}_{x_0} * f$ and f. □

Lemma 3.17 *If $f\colon [0,1]\to X$ is a path with $f(0)=x_0$ and $f(1)=x_1$ then*

$$f * \bar{f} \sim_p \mathrm{id}_{x_0} \qquad \textit{and} \qquad \bar{f} * f \sim_p \mathrm{id}_{x_1}.$$

Proof: Define

$$F\colon [0,1]\times[0,1]\to X$$

by

$$F(t,s) = \begin{cases} f(2(1-s)t) & 0\le t\le 1/2 \\ f(2(1-s)(1-t)) & 1/2 < t \le 1. \end{cases}$$

This gives the homotopy between $f * \bar{f}$ and id_{x_0}. Reversing the roles of f and $\bar{f}$ gives the homotopy between $\bar{f} * f$ and id_{x_1}. □

4 The fundamental group

Let X be a topological space and $x_0 \in X$ and basepoint. We can now define the *fundamental group*. Given a continuous map

$$f\colon ([0,1],\{0,1\}) \to (X,x_0)$$

we let $[f]$ be the equivalence class of paths that are path homotopic to f. Then $\pi_1(X,x_0)$ is the set of all equivalence classes. We want to give $\pi_1(X,x_0)$ the structure of a group. To do this we need to define a binary operation

$$\pi_1(X,x_0)\times\pi_1(X,x_0)\to\pi_1(X,x_0).$$

We need the following lemma:

Lemma 4.1 *Given $[f],[g]\in\pi_1(X,x_0)$ if $f_0\in[f]$ and $g_0\in[g]$ then $f_0 * g_0 \in [f*g]$.*

Proof: This follows from Lemma 3.14. Since $f_0\in[f]$ and $g_0\in[g]$ we have $f_0\sim_p f$ and $g_0\sim_p g$. Therefore by Lemma 3.14 we have $f_0 * g_0 \sim_p f*g$ so $f_0*g_0\in[f*g]$. □

We then define $[f]\cdot[g]=[f*g]$. Lemma 4.1 implies that this is well defined. We need to check that this operation satisfies the properties of a group.

Lemma 4.2 (Associativity) *If $[f],[g],[h]\in\pi_1(X,x_0)$ then*

$$([f]\cdot[g])\cdot[h] = [f]\cdot([g]\cdot[h]).$$

Proof: By definition

$$([f] \cdot [g]) \cdot [h] = [(f * g) * h]$$

and

$$[f] \cdot ([g] \cdot [h]) = [f * (g * h)].$$

By Lemma 3.15 we have $f * (g * h) \sim_p (f * g) * h$ so $f * (g * h) \in [(f * g) * h]$ and $[f * (g * h)] = [(f * g) * h]$. □

Lemma 4.3 (Identity) *For all* $[f] \in \pi_1(X, x_0)$ *we have*

$$[f] \cdot [\mathrm{id}_{x_0}] = [\mathrm{id}_{x_0}] \cdot [f] = [f].$$

Proof: By Lemma 3.16 we have

$$f * \mathrm{id}_{x_0} \sim_p f \sim_p \mathrm{id}_{x_0} * f$$

so

$$[f] \cdot [\mathrm{id}_{x_0}] = [\mathrm{id}_{x_0}] \cdot [f] = [f].$$

□

Lemma 4.4 (Inverses) *For* $[f] \in \pi_1(X, x_0)$ *we have*

$$[f] \cdot [\bar{f}] = [\bar{f}] \cdot [f] = [\mathrm{id}_{x_0}].$$

Proof: This is a direct consequence of Lemma 3.17. □

$\pi_1\left(S^1, (1,0)\right)$

Theorem 4.5

$$\pi_1\left(S^1, (1,0)\right) \cong \mathbb{Z}$$

Proof: By Theorem 3.1 as a set $\pi_1\left(S^1, (1,0)\right)$ is $\{[f_n] | n \in \mathbb{Z}\}$. We define a map $\sigma\colon \pi_1\left(S^1, (1,0)\right)$ by $\sigma([f_n]) = n$. We'll show that σ is a group isomorphism. As the map is a bijection we just need to show that it is a homomorphism. That is we need to show that

$$\sigma([f_n] \cdot [f_m]) = \sigma([f_n]) + \sigma([f_m]) = n + m.$$

This is equivalent to showing that $[f_n] \cdot [f_m] = [f_{n+m}]$. To see this we need to construct the lift of $f_n * f_m$. Note that we can't just concatenate the lifts $\tilde{f}_n$ and $\tilde{f}_m$ since the right endpoint of $\tilde{f}_n$ is not the same as the left endpoint of $\tilde{f}_m$. However, we can still describe the lift explicitly:

$$\widetilde{f_n * f_m}(t) = \begin{cases} 2nt & 0 \le t \le 1/2 \\ 2m(t - 1/2) + n & 1/2 < t \le 1. \end{cases}$$

One can check directly that $\widetilde{f_n * f_m}$ is a lift of $f_n * f_m$ and that $\widetilde{f_n * f_m}(0) = 0$. We also have that $\widetilde{f_n * f_m}(1) = n + m$. Then we can let $\tilde{F}$ be the straight line homotopy between $\tilde{f}_{n+m}$ and $\widetilde{f_n * f_m}$:

$$\tilde{F}(t, s) = (1 - s)\tilde{f}_{n+m} + s\widetilde{f_n * f_m}.$$

This is a path homotopy and the composition $F = p \circ \tilde{F}$ is a path homotopy from f_{n+m} to $f_n * f_m$. Therefore $[f_{n+m}] = [f_n * f_m] = [f_n] \cdot [f_m]$ as claimed. □

4.1 Basic properties of the fundamental group

Theorem 4.6 *For all $x_0 \in \mathbb{R}^n$ we have*

$$\pi_1(\mathbb{R}^n, x_0) = \{[\mathrm{id}_{x_0}]\}.$$

Proof: Given any $[f] \in \pi_1(\mathbb{R}^n, x_0)$ the straight line homotopy between id_{x_0} and f is a path homotopy so $[f] = [\mathrm{id}_{x_0}]$. □

This is a special case of a more general phenomena. A space X is *contractible* if the identity map is homotopic to a constant map to x_0 for some $x_0 \in X$. Equivalently there is a continuous map

$$F\colon X \times [0,1] \to X$$

with $F(x,0) = x$ and $F(x,1) = x_0$ for all $x \in X$. We then have:

Theorem 4.7 *If X is contractible then there exists an $x_0 \in X$ such that $\pi_1(X,x_0) = \{[\mathrm{id}_{x_0}]\}$.*

Proof: Let

$$F\colon X \times [0,1] \to X$$

be the map that contracts X. That is $F(x,0) = x$ and $F(x,1) = x_0$ for some $x_0 \in X$. Now let $[f] \in \pi_1(X,x_0)$. Then $G(t,s) = F(f(t),s)$ is a homotopy of f to id_{x_0} but it is not necessarily a path homotopy. Let $\gamma(t) = (t,x_0)$. We let $\phi\colon [0,1]^2 \to [0,1]^2$ with the following properties. First assume that the left vertical side is mapped to $(0,0)$, the right vertical side is mapped to $(1,0)$ and ϕ is the identity on the bottom side. Then on the top side we map the first quarter to the left side of the square, the second quarter to the top side and the remaining half to the right side. This map will be continuous on the boundary and can be continuously extended to the entire square. For example we can set $\phi(t,s) = (1-s)\phi(t,0) + s\phi(t,1)$. We then let $G = F \circ \phi$.

Note that $f(t) = G(t,0)$, $(\gamma * \mathrm{id}_{x_0}) * \bar{\gamma}(t) = G(t,1)$ and $G(0,s) = G(1,s) = x_0$ so G is path homotopy from f to $(\gamma * \mathrm{id}_{x_0}) * \bar{\gamma}$. By Lemmas 3.16 and 3.17 we have $(\gamma * \mathrm{id}_{x_0}) * \bar{\gamma} \sim_p \mathrm{id}_{x_0}$. Therefore $f \sim_p \mathrm{id}_{x_0}$ so $\pi_1(X,x_0) = \{[\mathrm{id}_{x_0}]\}$. □

Problem 11 *A space X has property* SD_1 *if and only if $\pi_1(X,x_0) = [\mathrm{id}_{x_0}]$.*

There are spaces that have trivial fundamental group but are not contractible (for example S^n for $n \geq 2$). However, we need some more methods to show this.

The next thing we consider is how the fundamental group depends on the choice of basepoint. Let

$$\alpha\colon [0,1] \to X$$

be a path with $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Note that for $[f] \in \pi_1(X,x_0)$ that path $\bar{\alpha} * f * \alpha$ represents an element in $\pi_1(X,x_1)$. Using this we define a map

$$\hat{\alpha}\colon \pi_1(X,x_0) \to \pi_1(X,x_1)$$

by setting $\hat{\alpha}([f]) = [\bar{\alpha} * f * \alpha]$.

Lemma 4.8 *Let $\alpha, \beta\colon [0,1] \to X$ be paths.*

1. *The map $\hat{\alpha}\colon \pi_1(X,\alpha(0)) \to \pi_1(X,\alpha(1))$ is a homomorphism.*
2. *If $\alpha(1) = \beta(0)$ then $\widehat{\alpha * \beta} = \hat{\beta} \circ \hat{\alpha}$.*
3. *If $\alpha(0) = \beta(0)$, $\alpha(1) = \beta(1)$, and $\alpha \sim_p \beta$ then $\hat{\alpha} = \hat{\beta}$.*

4. For all $x_0 \in X$ we have $\hat{\mathrm{id}}_{x_0} = \mathrm{id}$.

5. The map $\hat{\alpha}$ is an isomorphism.

Proof: We first show (1). We have

$$\begin{aligned}
\hat{\alpha}([f]\cdot[g]) &= \hat{\alpha}([f*g]) \\
&= [\bar{\alpha}*f*g*\alpha] \\
&= [\bar{\alpha}*f*\mathrm{id}_{x_0}*g*\alpha] \qquad \text{Lemma 3.16} \\
&= [\bar{\alpha}*f*\alpha*\bar{\alpha}*g*\alpha] \qquad \text{Lemma 3.17} \\
&= \hat{\alpha}([f])\cdot\hat{\alpha}([g]).
\end{aligned}$$

For (2) we have

$$\begin{aligned}
\widehat{\alpha*\beta}([f]) &= [\overline{\alpha*\beta}*f*\alpha*\beta] \\
&= [\bar{\beta}*\bar{\alpha}*f*\alpha*\beta] \\
&= \hat{\beta}\circ\hat{\alpha}([f]).
\end{aligned}$$

Next we have (3):

$$\begin{aligned}
\hat{\alpha}([f]) &= \bar{\alpha}*f*\alpha \\
&= \bar{\beta}*f*\beta \\
&= \hat{\beta}([f]).
\end{aligned}$$

For (4) we note that $\mathrm{id}_{x_0} = \bar{\mathrm{id}}_{x_0}$ and apply Lemma 4.4 to get

$$\begin{aligned}
\hat{\mathrm{id}}_{x_0}([f]) &= [\bar{\mathrm{id}}_{x_0}*f*\mathrm{id}_{x_0}] \\
&= [f].
\end{aligned}$$

For (5) we let $\beta = \bar{\alpha}$. By (2) $\widehat{\alpha*\beta} = \hat{\beta}\circ\hat{\alpha}$. By Lemma 3.17 we have $\alpha*\beta \sim_p \mathrm{id}_{x_0}$ where $\alpha(0) = \beta(1) = x_0$. Then by (3) and (4) $\hat{\beta}\circ\hat{\alpha} = \hat{\mathrm{id}}_{x_0} = \mathrm{id}$. Since id is a bijection this implies that $\hat{\alpha}$ is injective and $\hat{\beta}$ is surjective. Reversing the roles of α and β we get that $\hat{\beta}$ is injective and $\hat{\alpha}$ is surjective. In particular they are both bijections and, as by (1) they are homomorphisms, they are isomorphisms. □

Note that the isomorphism $\hat{\alpha}$ depends on the path α not just the endpoints $\alpha(0)$ and $\alpha(1)$.

Problem 12 *Let*

$$\alpha,\beta\colon [0,1]\to X$$

be paths with $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1)$. Then for all $[f] \in \pi_1(X,x_0)$ we have

$$\hat{\beta}^{-1}\circ\hat{\alpha}([f]) = [\beta*\bar{\alpha}]\cdot[f]\cdot[\beta*\bar{\alpha}]^{-1}.$$

Let

$$f\colon X\to Y$$

be a continuous map between topological spaces. As above we will see that the topological map f induces an algebraic map f_* between $\pi_1(X,x_0)$ and $\pi_1(Y,y_0)$ where $y_0 = f(x_0)$. In particular we define

$$f_*\colon \pi_1(X,x_0)\to\pi_1(Y,f(x_0))$$

by

$$f_*([g]) = [f\circ g].$$

This is well defined since if

$$g, g_0 \colon ([0,1], \{0,1\}) \to (X, x_0)$$

with $g_0 \in [g]$ then the compositions

$$f \circ g, f \circ g_0 \colon ([0,1], \{0,1\}) \to (Y, f(x_0))$$

are paths in $\pi_1(Y, f(x_0))$ and $f \circ g_0 \in [f \circ g]$.

Lemma 4.9 *If*

$$f \colon X \to Y$$

is continuous the map

$$f_* \colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

given by

$$f_*([g]) = [f \circ g]$$

is well defined and a homomorphism.

If

$$g \colon Y \to Z$$

is a continuous map then

$$(g \circ f)_* = g_* \circ f_*.$$

Proof: We have already seen that f_* is well defined. Let $[h_0], [h_1] \in \pi_1(X, x_0)$. Then

$$\begin{aligned} f_*([h_0] \cdot [h_1]) &= f_*([h_0 * h_1]) \\ &= [f \circ h_0 * h_1] \\ &= [(f \circ h_0) * (f \circ h_1)] \\ &= [f \circ h_0] \cdot [f \circ h_1] \\ &= f_*([h_0]) \cdot f_*([h_1]). \end{aligned}$$

For the composition rule we take $[h] \in \pi_1(X, x_0)$ and see that

$$\begin{aligned} (g \circ f)_*([h]) &= [(g \circ f) \circ h] \\ &= [g \circ (f \circ h)] \\ &= g_*([f \circ h]) \\ &= g_* \circ f_*([h]). \end{aligned}$$

□

Theorem 4.10 *Let*

$$f_0, f_1 \colon X \to Y$$

be homotopic with homotopy

$$F \colon X \times [0,1] \to Y.$$

For $x_0 \in X$ let

$$\alpha \colon [0,1] \to Y$$

be the path $\alpha(t) = F(x_0, 1-t)$. Then

$$(f_0)_* = \hat{\alpha} \circ (f_1)_*.$$

Proof: Let $[g] \in \pi_1(X, x_0)$ and define

$$G\colon [0,1] \times [0,1] \times X$$

by $G(t,s) = (g(t), s)$. Then $F \circ G$ is a homotopy from $f_0 \circ g$ to $f_1 \circ g$ but it is not a path homotopy. Of course not such path homotopy is possible (in general) as the two paths will have different endpoints. To define a path homotopy we first define a map

$$\Phi\colon [0,1] \times [0,1] \to [0,1] \times [0,1]$$

by

$$\Phi(t,s) = (1-s)(t,0) + s\Phi_1(t)$$

where

$$\Phi_1(t) = \begin{cases} (0,3t) & 0 \leq t \leq 1/3 \\ (3t-1,1) & 1/3 < t \leq 2/3 \\ (1,-3t+3) & 2/3 < t \leq 1. \end{cases}$$

You should draw a picture of the maps Φ_1 and Φ. The map Φ_1 takes the interval to the left, top, and right sides of the square $[0,1]^2$ and Φ is the straight line homotopy from the path traversing the bottom half of the square to Φ_1.

We now define

$$H\colon [0,1] \times [0,1] \to Y$$

by $H = F \circ G \circ \Phi$. Then H is a path homotopy as $H(0,s) = H(1,s) = f_0(x_0)$. As usual we define $h_s(t) = H(t,s)$ and we have $[h_0] = [h_1]$ in $\pi_1(Y, f(x_0))$.

To complete the proof we first note that $f_0 \circ g(t) = h_0(t)$ and therefore $(f_0)_*([g]) = [h_0]$. For h_1 we observe that $\bar{\alpha}(t) = H(t/3, 1)$, $f_1 \circ g(t) = H(t/3 + 1/3, 1)$ and $\alpha(t) = H(t/3 + 2/3, 1)$ so $h_1 \sim_p \bar{\alpha} * f_1 \circ g * \alpha = \hat{\alpha} \circ f_1([g])$. As $[h_0] = [h_1]$ this gives $(f_0)_* = \hat{\alpha} \circ (f_1)_*$ as claimed. □

A continuous map $f\colon X \to Y$ is *homotopy equivalence* if there exists a $g\colon Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity. Then g is a *homotopy inverse* for f and X and Y are homotopy equivalent.

Proposition 4.11 *If $f\colon X \to Y$ is a homotopy equivalence then for all $x_0 \in X$ the homomorphism*

$$f_*\colon \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof: Let $g\colon Y \to X$ be the homotopy inverse to f. The $g \circ f$ is homotopic to the identity map $\mathrm{id}_X\colon X \to X$ with homotopy $F\colon X \times [0,1] \to X$. Let $\alpha(t) = F(x_0, t)$. By Theorem 4.10 we have $(\mathrm{id}_X)_* = \hat{\alpha} \circ (g \circ f)_*$. As both $(\mathrm{id}_X)_*$ and $\hat{\alpha}$ are isomorphism we have that $(g \circ f)_*$ is an isomorphism. Since $(g \circ f)_* = g_* \circ f_*$ this implies that f_* is injective and g_* is surjective. Reversing roles of f and g we have that g_* is injective and f_* is surjective. Therefore both are bijections and isomorphisms. □

Given $A \subset X$ let $\iota\colon A \hookrightarrow X$ be the inclusion map. Then $r\colon X \to A$ is a *retract* if $r \circ \iota = \mathrm{id}_A$.

Proposition 4.12 *If $A \subset X$ with inclusion $\iota\colon A \hookrightarrow X$ and $r\colon X \to A$ is a retract then ι_* is injective and r_* is surjective.*

Proof: We have that $r_* \circ \iota_* = (r \circ \iota)_* = (\mathrm{id}_A)_*$ is an isomorphism. In particular it is a bijection which implies that ι_* is injective and r_* is surjective. □

Note that this immediately implies that there is no retraction $r\colon D^2 \to S^1$ as the fundamental group of D^2 is trivial while the fundamental group of S^1 is $\mathbb{Z}$ and there is no surjective map from the trivial group to $\mathbb{Z}$. In fact the statement that there is no retraction from D^2 to S^1 is equivalent to the statement that the identity map from S^1 to itself does not extend to a continuous map from D^2 to S^1. This was Theorem 3.11. The core idea of our previous proof of Theorem 3.11 is essentially the same as the proof using the fundamental group.

This also gives a new viewpoint of the proof of the Brower Fixed Point Theorem (Theorem 3.13). In particular we saw that if $f\colon D^2 \to D^2$ doesn't have a fixed point then there is a retraction from D^2 to S^1. But we just saw that such a retraction doesn't exist.

A *deformation retraction* of X to $A \subset X$ is a homotopy

$$F\colon X \times [0,1] \to X$$

such that $f_0 = \mathrm{id}_X$, f_1 is a retraction to A and $F(x,t) = x$ for all $x \in A$. Then A is a *deformation retract* of X.

Proposition 4.13 *If*

$$F\colon X \times [0,1] \to X$$

is a deformation retract to $A \subset X$ then the retraction $f_1\colon X \to A$ is a homotopy equivalence with homotopy inverse the inclusion map $\iota\colon A \hookrightarrow X$.

Proof: By the definition of a retraction $f_1 \circ \iota = \mathrm{id}_A$ (which is clearly homotopic to the id_A as every map is homotopic to itself). On the other hand $\iota \circ f_1 = f_1$ and F is a homotopy from f_1 to the identity. □

There is a retraction

$$r\colon \mathbb{R}^{n+1} \setminus \{0\} \to S^n$$

given by

$$r(x) = \frac{x}{|x|}.$$

This retraction can also be extended to a deformation retract

$$F\colon \mathbb{R}^{n+1} \setminus \{0\} \times [0,1] \to S^n$$

by setting

$$F(x,t) = (1-t)x + tr(x).$$

Problem 13 *Let X and Y be topological spaces and*

$$p_X\colon X \times Y \to X \qquad \text{and} \qquad p_Y\colon X \times Y \to Y$$

the projections. Define

$$\Phi\colon \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

by

$$\Phi([f]) = ((p_X)_*([f]), (p_Y)_*([f]))$$

and show that Φ is an isomorphism.

Problem 14 *Let X be topological space and $A, B \subset X$ open subspaces such that $X = A \cup B$ and $A \cap B$ is path connected. Let $x_0 \in A \cap B \subset X$ be a basepoint and show that if $\pi_1(A, x_0)$ and $\pi_1(B, x_0)$ are trivial then $\pi_1(X, x_0)$ is trivial. Use this to show that $\pi_1(S^n, x_0)$ is trivial if $n \geq 2$. I can give you some hints on how to do this but I'd first like you to spend some time thinking about it on your own.*

5 Covering spaces

Let

$$p\colon \tilde{X} \to X$$

be a continuous map between topological spaces. A neighborhood $U \subset X$ is *evenly covered* (with respect to p) if for every component V of $p^{-1}(U)$ the restriction $p|_V$ is homeomorphism from V to U. Then $p\colon \tilde{X} \to X$ is a *covering space* if every $x \in X$ has an evenly covered neighborhood.

The map $p\colon \mathbb{R} \to S^1$ is an example of a covering space. We will see that much of what we did there works much more generally.

Note that a covering space includes all three objects: the two spaces X and $\tilde{X}$ along with the map p. For example S^1 covers itself in more than one way. That is S^1 will play the role of both X and $\tilde{X}$ but the maps p will be different. It will be most convenient to define them if we consider S^1 as a subset of the complex plane $\mathbb{C}$. That is let

$$S^1 = \{z \in \mathbb{C} | |z| = 1\}.$$

Then for positive integers n define

$$p_n\colon S^1 \to S^1$$

by

$$p_n(z) = z^n.$$

Note that if $|z| = 1$ then $|p_n(z)| = |z^n| = |z|^n = 1^n = 1$ so p_n is indeed a map from S^1 to itself. We also observe that $(p_n)^{-1}(w)$ has n points for all $w \in S^1$. The evenly covered neighborhoods for these covers are the same as the evenly covered neighborhoods of our much discussed map $p\colon \mathbb{R} \to S^1$. Namely for $a, b \in \mathbb{R}$ with $0 < b - a < 2\pi$ define

$$U_{(a,b)} = \left\{z \in S^1 | z = e^{i\theta} \text{ for } \theta \in (a,b) \text{ with } b - a < 2\pi\right\}.$$

Then

$$p_n^{-1}\left(U_{(a,b)}\right) = \bigcup_{k=0}^{n-1} U_{((a+2k\pi)/n,(b+2k\pi)/n)}$$

and on each of then n components p restricts to a homeomorphism to $U_{(a,b)}$.

Problem 15 *Define*

$$p\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$$

by

$$p(z) = e^z.$$

Show that is a covering space.

Problem 16 *Let $p\colon \tilde{X} \to X$ be a covering space with X connected and assume that for some $x_0 \in X$ that $p^{-1}(x_0)$ contains n elements where n is a positive integer. Show that $p^{-1}(x)$ contains n elements for all $n \in X$.* (**Hint:** Let $U_k = \{x \in X | p^{-1}(x) \text{ contains } k \text{ points}\}$ and let $U_\infty = \{x \in X | p^{-1}(x) \text{ contains infinitely many points}\}$. Show that the U_k and U_∞ are open and use the assumption X is connected to show that at most one of them can be non-empty.)

Problem 17 *Let $p\colon \tilde{X} \to X$ be a covering space with X connected, compact and Hausdorff. Show that if $\tilde{X}$ is compact then $p^{-1}(x)$ contains finitely many points for all $x \in X$.*

Define an equivalence relation on $S^2 \subset \mathbb{R}^3$ by $x \sim -x$ and let $\mathbb{R}\mathrm{P}^2 = S^2/\sim$ be the quotient space. Then the quotient map

$$q\colon S^2 \to \mathbb{R}\mathrm{P}^2$$

is a covering space. In the quotient topology $V \subset \mathbb{R}\mathrm{P}^2$ is open if $q^{-1}(V)$ is open in S^2. To construct evenly covered neighborhoods we will use that the embedding of S^2 in $\mathbb{R}^3$. Note that if $x \in S^2 \subset \mathbb{R}^3$ than the distance between x and $-x$ is 2 so if A is subset of $\mathbb{R}^3$ of diameter < 2 (that is if $x, y \in A$ then $|x-y| < 2$) and $x \in A \cap S^2$ then $-x \notin A$. Now let $B_\varepsilon(x)$ be the open ball in $\mathbb{R}^3$ of radius ε centered at $x \in \mathbb{R}^3$. Then for $x \in S^2$ let

$$U_x = S^2 \cap B_{1/2}(x).$$

This is a connected open neighborhood of S^2 and q is injective on it. If we let $V_{[x]} = p(U_x)$ then $p^{-1}\left(V_{[x]}\right) = U_x \cup U_{-x}$. The two neighborhoods U_x and U_{-x} are disjoint and the restriction of q to each of them is a bijection to $V_{[x]}$. The map q is an open map so by Lemma 3.4 the restriction of q to U_x or U_{-x} is a homeomorphism to $V_{[x]}$. Therefore $V_{[x]}$ is an evenly covered neighborhood and $q\colon S^2 \to \mathbb{R}\mathrm{P}^2$ is a covering space.

Proposition 5.1 (Path lifting lemma) *Let*

$$p\colon \tilde{X} \to X$$

be a covering space. Let $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$ be basepoints with $p(\tilde{x}_0) = x_0$. Then for all

$$f\colon ([0,1],0) \to (X, x_0)$$

be a continuous map. Then there exists a unique continuous map

$$\tilde{f}\colon ([0,1],0) \to (\tilde{X}, \tilde{x}_0)$$

with $f = p \circ \tilde{f}$.

Proof: We claim that there exists a partition $s_0 = 0 < s_1 < \cdots < s_n = 1$ such that for each interval $J_i = [s_{i-1}, s_1]$ we have $p(J_i) \subset U_i$ where U_i is an evenly covered neighborhood. We'll prove the existence of the lift assuming the claim.

We inductively prove that the lift exists on the interval $[0, s_i]$. If $i = 0$ then the interval is just the point 0 and we define $\tilde{f}(0) = \tilde{x}_0$. Now assume that $\tilde{f}$ is defined on $[0, s_{i-1}]$ and we'll show that it can be extended to $[0, s_i]$. Let V_i be the component of $p^{-1}(U_i)$ that contains $\tilde{f}(s_{i-1})$ and let $q_i\colon U_i \to V_i$ be the inverse of $p|_{V_i}$ given by the evenly covered property. Then define $\tilde{f} = q_i \circ f$ on $[s_{i-1}, s_i]$. Note that both $\tilde{f}(s_{i-1})$ and $q_i \circ f(s_{i-1})$ lie in $V_i \cap p^{-1}(f(s_{i-1}))$. As there is a unique such point in the intersection we have $\tilde{f}(s_{i-1}) = q_i \circ f(s_{i-1})$ and therefore the extension of $\tilde{f}$ to $[s_{i-1}, s_i]$ is continuous on $[0, s_i]$. By induction the lift exists on all of $[0,1]$.

Now we prove the existence of the partition. By the continuity of f every $t \in [0,1]$ is contained in the interior (relative to $[0,1]$) of a closed interval I_t such that $f(I_t)$ is contained in an evenly covered neighborhood. Then the interior of I_t cover $[0,1]$ and by compactness there is a finite subcover. We then choose s_i to be the endpoints in the intervals in the finite subcover. We choose the indices of the s_i so that $s_i < s_{i+1}$. Note that intervals $[0,t)$ and $(t,1]$ are open in $[0,1]$ and intervals of this type must be in the finite subcover. In particular, $s_0 = 0$ and the for the final point $s_n = 1$. By construction each of the $[s_i, s_{i+1}]$ will lie in some I_t and therefore $f([s_i, s_{i+1}])$ lies in an evenly covered neighborhood and we have constructed the desired partition.

We prove uniqueness in Lemma 5.2 below. □

The careful reader will notice that the above proof is a copy and paste of the proof of Proposition 3.2 with minor changes and we have removed that part of the proof were we showed that the map $p\colon \mathbb{R} \to S^1$ is a covering space as that is now part of our assumptions.

While Problem 7 required that the image space was Hausdorff this assumption it is not necessary for the the uniqueness of lifts.

Lemma 5.2 *Let $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ be a covering space and assume that Y is connected and $f\colon (Y,y_0) \to (X,x_0)$ is continuous. If $\tilde{f}_0, \tilde{f}_1\colon (Y,y_0) \to (\tilde{X},\tilde{x}_0)$ with $f_i = p \circ \tilde{f}_i$ then $\tilde{f}_0 = \tilde{f}_1$.*

Proof: The strategy is the same as before. We let

$$A = \{y \in Y | \tilde{f}_0(y) = \tilde{f}_1(y)\}$$

and we'll show that A is open, closed and non-empty. We first not that A is non-empty since $y_0 \in A$.

Next we show that A is open. Assume that $y \in A$. Let U be an evenly covered neighborhood of $f(y)$ and let V be the component of $p^{-1}(U)$ that contains $\tilde{f}_0(y) = \tilde{f}_1(y)$. Let $W = \tilde{f}_0^{-1}(V) \cap \tilde{f}_1^{-1}(V)$. As W is the intersection of two open sets it is also open. On W we have that $\tilde{f}_i = (p|_V)^{-1} \circ f$ so $W \subset A$ and A is open.

A similar argument shows that the complement of A is open (and hence A is closed). Assume that $y \in Y \setminus A$ and, as before, let U be an evenly covered neighborhood. Let V_0 and V_1 be the components of $p^{-1}(U)$ that contain $\tilde{f}_0(y)$ and $\tilde{f}_1(y)$, respectively. Then $V_0 \cap V_1 = \emptyset$ since $\tilde{f}_0(y) \neq \tilde{f}_1(y)$. We then let $W = \tilde{f}_0^{-1}(V_0) \cap \tilde{f}_1^{-1}(V_1)$. Again, W is open since it is the intersection of two open sets. On W we have $\tilde{f}_i = (p|_{V_i})^{-1} \circ f$ so $\tilde{f}_0(W) \subset V_0$ and $\tilde{f}_1(W) \subset V_1$ and $\tilde{f}_0(W) \cap \tilde{f}_1(W) = \emptyset$. This implies that $W \subset Y \setminus A$ so $Y \setminus A$ is open and A is closed.

We have shown that A is open, closed, and non-empty. Therefore, as Y is connected, $A = Y$ and $\tilde{f}_0 = \tilde{f}_1$ on all of Y. $\square$

Similarly the Homotopy Lifting Lemma also generalizes to arbitrary covering spaces.

Proposition 5.3 (Homotopy Lifting Lemma) *Let*

$$p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$$

be a covering space and let

$$F\colon \big([0,1]^2,(0,0)\big) \to (X,x_0)$$

be a continuous map. Then there exists a unique continuous map

$$\tilde{F}\colon \big([0,1]^2,(0,0)\big) \to \big(\tilde{X},\tilde{x}_0\big)$$

with $F = p \circ \tilde{F}$.

Again the proof is the same as the proof for the covering space $p\colon \mathbb{R} \to S^1$. We will not repeat it.

Lemma 5.4 *Let*

$$p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$$

be a covering space and let

$$F\colon ([0,1]\times[0,1],(0,0)) \to (X,x_0)$$

be a path homotopy. Then the lift

$$\tilde{F}\colon ([0,1]\times[0,1],(0,0)) \to \big(\tilde{X}_0,\tilde{x}_0\big)$$

is path homotopy.

Proof: We first observe that for any $x \in X$ the pre-image $p^{-1}(x)$ is a discrete subset of $\tilde{X}$. To see this take an evenly covered neighborhood of U of x. Then each $\tilde{x}$ there is a component V of $p^{-1}(U)$ that contains $\tilde{x}$. Since $p|_V$ is a homeomorphism, and hence a bijection, to U we have $V \cap p^{-1}(x) = \{\tilde{x}\}$ and therefore $\tilde{x}$ is open in the subspace topology on $p^{-1}(x)$.

Since F is a path homotopy $F(0,t) = F(0,0) = x_0$ for all $t \in [0,1]$. Similarly if $F(1,0) = x_1$ then $F(1,t) = x_1$ for all $t \in [0,1]$. Therefore $\tilde{F}(\{0\}\times[0,1]) \subset p^{-1}(x_0)$ and as $p^{-1}(x_0)$ is discrete this implies that $\tilde{F}$ is constant on $\{0\}\times[0,1]$. The same argument gives that $\tilde{F}$ is constant on $\{1\}\times[0,1]$ and hence $\tilde{F}$ is a path homotopy. $\square$

Proposition 5.5 *If* $p\colon (\tilde{X},\tilde{x}_0)\to(X,x_0)$ *is a covering map then* $p_*\colon \pi_1(\tilde{X},\tilde{x}_0)\to\pi_1(X,x_0)$ *is injective.*

Proof: For $[\tilde{f}]\in\pi_1(\tilde{X},\tilde{x}_0)$ assume that $p_*([\tilde{f}])=[\mathrm{id}_{\tilde{x}_0}]$. Then there exists a path homotopy

$$F\colon [0,1]\times[0,1]\to X$$

from $\mathrm{id}_{\tilde{x}_0}$ to $p\circ\tilde{f}$. Let

$$\tilde{F}\colon [0,1]\times[0,1]\to\tilde{X}$$

be the unique lift given by Proposition 5.3. By Lemma 5.4 this will be a path homotopy from $\tilde{f}_0(t)=\tilde{F}(t,0)$ and $\tilde{f}_1(t)=\tilde{F}(t,1)$. Then $\tilde{f}_0$ is a lift of id_{x_0} with $\tilde{f}_0(0)=\tilde{x}_0$ and $\tilde{f}_1$ is a lift of $p\circ\tilde{f}$ with $\tilde{f}_1(0)=\tilde{x}_0$ (since $\tilde{F}(0,s)$ is constant). Since $\mathrm{id}_{\tilde{x}_0}$ and $\tilde{f}$ are also lifts id_{x_0} and $p\circ\tilde{f}$ with the same initial point by the unique of lifts (Proposition 5.1) we have $f_0=\mathrm{id}_{\tilde{x}_0}$ and $f_1=\tilde{f}$. Therefore $[\mathrm{id}_{\tilde{x}_0}]=[\tilde{f}]$ and p_* is injective. □

Lemma 5.6 *Let* $p\colon \left(\tilde{X},\tilde{x}_0\right)\to(X,x_0)$ *be a covering map and* $f\colon (Y,y_0)\to(X,x_0)$ *a continuous map with* Y *path connected and locally path connected. For* $[g]\in\pi_1(Y,y_0)$ *let* $\tilde{g}\colon ([0,1],0)\to(\tilde{X},\tilde{x}_0)$ *be the unique lift. Then* $f_*([g])\in p_*\left(\pi_1(\tilde{X},\tilde{x}_0)\right)$ *if and only if* $\tilde{g}(1)=\tilde{x}_0$.

Proof: If $\tilde{g}(1)=\tilde{x}_0$ then $[\tilde{g}]\in\pi_1\left(\tilde{X},\tilde{x}_0\right)$ and $p_*([\tilde{g}])=[p\circ\tilde{g}]=[g]$. Therefore $f_*([g])\in\pi_1(X,x_0)$.

Now assume $f_*([g])\in\pi_1(X,x_0)$. Now let $[h]$ be the unique element of $\pi_1\left(\tilde{X},\tilde{x}_0\right)$ with $p_*([h])=f_*([g])$. Note that by construction h is the lift of $p\circ h$ with $h(0)=\tilde{x}_0$.

As $p_*([h])=f_*([g])$ we have that $f\circ g$ and $p\circ h$ are path homotopic via a path homotopy F. That is F is a path homotopy with $f\circ g(t)=F(t,0)$ and $h\circ p(t)=F(t,1)$. By Proposition 5.3, there exists a unique lift $\tilde{F}\colon [0,1]^2\to\tilde{X}$ with $\tilde{F}(0,0)=\tilde{x}_0$. As F is path homotopy by Lemma 5.4 $\tilde{F}$ is a path homotopy. In particular, $\tilde{F}(0,1)=\tilde{F}(0,0)=\tilde{x}_0$. As both h and $\tilde{F}(t,1)$ are lifts of $h\circ p$ that take zero to $\tilde{x}_0$ by uniqueness of lifts $h(t)=\tilde{F}(t,1)$. Therefore $\tilde{F}(1,1)=h(1)=\tilde{x}_0$. Again using that $\tilde{F}$ is a path homotopy we have that $\tilde{F}(1,0)=\tilde{F}(1,1)=\tilde{x}_0$. Therefore both $\tilde{g}$ and $\tilde{F}(t,0)$ are lifts of g that take zero to $\tilde{x}_0$ so $\tilde{g}(1)=\tilde{F}(1,t)=\tilde{x}_0$. □ .

Theorem 5.7 *Let* $p\colon (\tilde{X},\tilde{x}_0)\to(X,x_0)$ *be a covering map and* $f\colon (Y,y_0)\to(X,x_0)$ *a continuous map with* Y *path connected and locally path connected. Then there exists a lift* $\tilde{f}\colon (Y,y_0)\to(\tilde{X},\tilde{x}_0)$ *if and only if* $f_*(\pi_1(Y,y_0))\subset p_*\left(\pi_1(\tilde{X},\tilde{x}_0)\right)$. *If the lift exists it is unique.*

Proof: We first show that if there exist a lift $f_*(\pi_1(Y,y_0))\subset p_*\left(\pi_1(\tilde{X},\tilde{x}_0)\right)$. This follows from the fact that $f_*=(p\circ\tilde{f})_*=p_*\circ\tilde{f}_*$ so the image of f_* lies in the image of p_* as whenever a map is written as a composition the image must lie in the image of the left map.

Now we assume $f_*(\pi_1(Y,y_0))\subset p_*\left(\pi_1(\tilde{X},\tilde{x}_0)\right)$. We are now define the lift $\tilde{f}$. Given $y\in Y$ choose a path $\alpha\colon [0,1]$ from y_0 to y and let $\tilde{\alpha}$ be the lift of $f\circ\alpha$ with $\tilde{\alpha}(0)=\tilde{x}_0$. We then set $\tilde{f}(y)=\tilde{\alpha}(1)$. To show that $\tilde{f}$ is well defined we need to show that the definition is independent of the choice of path α. Let $\beta\colon [0,1]\to Y$ be another path from y_0 to y with $\tilde{\beta}$ the lift of $f\circ\beta$. The path $\beta*\bar{\alpha}$ represents an element of $\pi_1(Y,y_0)$ so if $\widetilde{\beta*\bar{\alpha}}$ is the lift of $\beta*\bar{\alpha}$ that takes 0 to $\tilde{x}_0$ by Lemma 5.6 we have $\widetilde{\beta*\bar{\alpha}}(1)=\tilde{x}_0$. Then $\widetilde{\beta*\bar{\alpha}}(t/2)$ and $\widetilde{\beta*\bar{\alpha}}((1-t)/2)$ are lifts of $f\circ\beta$ and $f\circ\alpha$ that take 0 to $\tilde{x}_0$ so by the uniqueness of lifts $\widetilde{\beta*\bar{\alpha}}(t/2)=\tilde{\beta}(t)$ and $\widetilde{\beta*\bar{\alpha}}((1-t)/2)=\tilde{\alpha}(t)$. Therefore $\tilde{\beta}(1)=\widetilde{\beta*\bar{\alpha}}(1/2)=\tilde{\alpha}(1)$ so $\tilde{f}$ is well defined.

Finally we show that $\tilde{f}$ is continuous. Let $V\subset\tilde{X}$ be open. We need to show that every $y\in\tilde{f}^{-1}(V)$ has an open neighborhood U_y with $U_y\subset\tilde{f}^{-1}(V)$. If $x=f(y)$ let U_x be an evenly covered neighborhood of x such that a component V_x of $p^{-1}(U_x)$ is contained in V. Let $q\colon U_x\to V_x$ be the inverse of $p|_{V_x}$. Then $f^{-1}(U_x)$ is open in Y and since Y is locally path connected there is a path connected neighborhood U_y of y that is contained in $f^{-1}(U_x)$. We'll show that $U_y\subset\tilde{f}^{-1}(V)$.

Let α be a path from y_0 to y. For $y_1\in U_y$ let β be a path from y to y_1 that is contained in U_y. Then $\alpha*\beta$ is a path from y_0 to y_1. If $\tilde{\alpha}$ is the lift of $f\circ\alpha$ with $\tilde{\alpha}(0)=\tilde{x}_0$ then $\tilde{\alpha}*q\circ f\circ\beta$ is a lift of $f\circ\alpha*\beta$ and therefore $\tilde{f}(y_1)=\tilde{\alpha}*q\circ f\circ\beta(1)=q(f(y))\in V_x\subset V$. Therefore $U_y\subset\tilde{f}^{-1}(V)$ proving the continuity of $\tilde{f}$. □

Problem 18 *Let* $p\colon (\tilde{X},\tilde{x}_0) \to (X,x_0)$ *be a covering space. Given* $f,g\colon ([0,1],\{0,1\}) \to (X,x_0)$ *with lifts* $\tilde{f},\tilde{g}\colon ([0,1],0) \to (\tilde{X},\tilde{x}_0)$ *show that* $\tilde{f}(1) = \tilde{g}(1)$ *if and only if* $[f]\cdot[g]^{-1} \in p_*(\pi_1(\tilde{X},\tilde{x}_0))$.

Problem 19 *Let* $p\colon \tilde{X} \to X$ *be a covering space and* X *path connected and locally path connected. Let* $U \subset X$ *be open and path connected and assume that for* $x_0 \in U \subset X$ *we have that* $\iota_*(\pi_1(U,x_0))$ *is a trivial subgroup of* $\pi_1(X,x_0)$ *where* $\iota\colon U \hookrightarrow X$ *is the inclusion map. Show that* U *is an evenly covered neighborhood.*

Proposition 5.8 *For* $i = 0,1$ *let*

$$p_i\colon \left(\tilde{X}_i,\tilde{x}_i\right) \to (X,x_0)$$

be covering spaces. If $(p_0)_*(\pi_1(\tilde{X}_0,\tilde{x}_0)) = (p_1)_*(\pi_1(\tilde{X}_1,\tilde{x}_1))$ *there exists a unique homeomorphism* $\phi\colon (\tilde{X}_0,\tilde{x}_0) \to (\tilde{X}_1,\tilde{x}_1)$ *with* $p_1 \circ \phi = p_0$.

Proof: We can apply Theorem 5.7 where p_0 plays the role of p and p_1 plays the role of f. Then there exists a lift $\tilde{p}_1\colon (\tilde{X}_1,\tilde{x}_1) \to (\tilde{X}_0,\tilde{x}_0)$ with $p_1 = p_0 \circ \tilde{p}_1$. Reversing the roles of p_0 and p_1 we get a lift $\tilde{p}_0\colon (\tilde{X}_0,\tilde{x}_0) \to (\tilde{X}_1,\tilde{x}_1)$ with $p_0 = p_1 \circ \tilde{p}_0$. We then have $p_0 = p_0 \circ (\tilde{p}_1 \circ \tilde{p}_0)$. Therefore $\tilde{p}_1 \circ \tilde{p}_0$ is a lift of p_0 to the cover $p_0\colon (\tilde{X}_0,\tilde{x}_0) \to (X,x_0)$.

To be more explicit: We can let $\tilde{X}_0$ be both the cover and the space Y in Theorem 5.7. In this case the existence of the lift is obvious - it is just the identity map $\mathrm{id}_{\tilde{X}_0}\colon \tilde{X}_0 \to \tilde{X}_0$. What is important is the uniqueness property which tells us that $\mathrm{id}_{\tilde{X}_0} = \tilde{p}_1 \circ \tilde{p}_0$. Similarly we have $\mathrm{id}_{\tilde{X}_1} = \tilde{p}_0 \circ \tilde{p}_1$. Therefore $\tilde{p}_0$ and $\tilde{p}_1$ are inverses of each other and are homeomorphisms so $\phi = \tilde{p}_0$ is the desired homeomorphism. □

5.1 Group actions

If X is a topological space let $\mathbf{Homeo}(X)$ be the set of homeomorphisms $\phi\colon X \to X$. This set is a group where the binary operation is composition. A *group action* on X is a choice of subgroup $G \subset \mathbf{Homeo}(X)$.

Example: For $s \in \mathbb{R}$ define $\phi_s \in \mathbf{Homeo}(\mathbb{R})$ by $\phi_s(t) = s+t$ and let $G = \{\phi_s\}$. Note that $\phi_0 = \mathrm{id}_{\mathbb{R}}$ and $\phi_{s_0} \circ \phi_{s_1} = \phi_{s_0+s_1}$ so G is a subgroup of $\mathbf{Homeo}(\mathbb{R})$ and the map $s \mapsto \phi_s$ is an isomorphism from $\mathbb{R}$ to G. We can also take $H = \{\phi_n\}_{n\in\mathbb{Z}}$. Then H is also a group and the map $n \mapsto \phi_n$ is an isomorphism from $\mathbb{Z}$ to H. The G is an $\mathbb{R}$-*action* and H is an $\mathbb{Z}$-*action*.

If G is a group action on X then the G-*orbit* of a point $x \in X$ is the set

$$[x]_G = \{g(x) \in X | g \in G\}.$$

Lemma 5.9 *Let* G *be a group action on* X. *Then* $y \in [x]_G$ *if and only if* $[y]_G = [x]_G$.

Proof: Note that $y \in [y]_G$ since the identity homeomorphism is contained in G. Therefore if $[y]_G = [x]_G$ we have $y \in [x]_G$.

If $y \in [x]_G$ then there exists a $g \in G$ with $g(x) = y$. If $z \in [y]_G$ then there exists an $h \in G$ with $h(y) = z$ so $h \circ g(x) = h(y) = z$. Therefore $z \in [x]_G$ and we have $[y]_G \subset [x]_G$. On the other hand if $z \in [x]_G$ then there exists $h \in G$ with $h(x) = z$ so $h \circ g^{-1}(y) = h(x) = z$. Therefore $z \in [y]_G$ and $[x]_G \subset [y]_G$. As $[y]_G \subset [x]_G$ and $[x]_G \subset [y]_G$ we have $[x]_G = [y]_G$. □

If G is a group action on X then X/G is the set of G-orbits with the quotient topology.

Problem 20 *Let* G *and* H *be as in the example. Show that* $\mathbb{R}/G$ *is a point and* $\mathbb{R}/H$ *is homeomorphic to* S^1.

The quotient topology

We briefly review the quotient topology. If X is a topological space, Y is a set and $q\colon X \to Y$ is a surjective map then the *quotient topology* on Y is the collection of sets $U \subset Y$ such that $q^{-1}(Y)$ is open in X. This collection of sets is a topology on Y and q is a continuous map in this topology. Furthermore any other topology on Y where q is continuous is a sub-topology of the quotient topology.

If $q\colon X \to Y$ is a quotient map and $f\colon X \to Z$ is continuous a general question is when there is another continuous map $h\colon Y \to Z$ with $f = h \circ q$. We have:

Theorem 5.10 *Let $q\colon X \to Y$ be a quotient map and $f\colon X \to Z$ continuous. Then there exists a continuous map $h\colon Y \to Z$ with $f = h \circ q$ if and only if f is constant on $q^{-1}(y)$ for all $y \in Y$.*

A short aside on relations and equivalence relations: If X is a set then a relation is a set of order pairs of elements of X or, equivalently, a subset of $X \times X$.

Problem 21 *Every relation is contained in a unique smallest equivalence relation.*

Because of this we can define an equivalence relation by choosing a relation and taking the corresponding equivalence relation.

Example: Let X be an interval $[0,1]$ and let $\sim$ be the equivalence relation given by $0 \sim 1$. We also let $Y = X/\sim$ be the quotient space with quotient map $q\colon X \to S^1$. Then we have a map $f\colon X \to S^1$ given by $f(t) = (\cos 2\pi t, \sin 2\pi t)$. The maps f and q satisfy the conditions of Theorem 5.10. For this we note that only one equivalence class has more than one point, the one containing 0 and 1. Since $f(0) = f(1)$ we have that f is constant on all equivalence classes so there exists a continuous map $g\colon Y \to S^1$ with $g \circ q = f$.

We claim that g is a homeomorphism. For this we observe that f and q are injective on the open interval $(0,1)$ and therefore so is g. Finally we notice that the equivalence class $\{0,1\}$ is mapped to $(1,0)$ by g and this point is not in the image of $(0,1)$ under f and it follows that f is injective. For surjectivity we note that f is surjective and this implies that g is surjective.

We have shown that g is a continuous bijection. We are left to show that the inverse is continuous. To see this let $y \in Y$. If, as an equivalence class, y is a singleton then $y = q(t)$ for some $t \in (0,1)$ and an open interval $I \subset (0,1)$ that contains y is saturated with respect to q so $q(I)$ is an open neighborhood of $y$in Y. We also have that $f(I) = g(q(I))$ is open in S^1 and y has an open neighborhood in Y that maps to an open set in S^1. If $y = \{0,1\} \in Y$ then $[0,1/2) \cup (1/2,1]$ is a saturated open set in $[0,1]$ (for the subspace topology on $[0,1]$) so its q-image is open Y. As its f-image is also open in S^1 we have an open neighborhood $y \in Y$ that maps to an open set in S^1 under g. This shows that g is an open map so g^{-1} is continuous and g is a homeomorphism.

Example: Now let X be the union of two intervals e_a and e_b both homeomorphic to $[0,1]$ and let $\sim$ be the equivalence relation obtained by identifying all four endpoints and denote the corresponding quotient space R_2 with quotient map $q\colon X \to R_2$.

We also let $\tilde{X}$ be a disjoint union of four intervals e_a^0, e_a^1, e_b^0 and e_b^1 and define an equivalence relation by identifying the endpoints 0_a^0, 1_a^1, 0_b^0, and 1_b^0 in one equivalence class and 0_a^1, 1_a^0, 0_b^1, and 1_b^1 in another equivalence class. We let $\tilde{q}\colon \tilde{X} \to \tilde{R}$ be the quotient map and quotient space.

We have a map $\tilde{f}\colon \tilde{X} \to X$ that is a homeomorphism from the individual a edges in $\tilde{X}$ to e_a and the two b edges in $\tilde{X}$ to e_b in X. We can now apply Theorem 5.10 where $\tilde{X}$ plays the role of X, $\tilde{q}$ is the quotient map q, the composition $q \circ \tilde{f}$ is the map f, $\tilde{R}$ is Y and R_2 is Z. There are only two equivalence class in $\tilde{R}$ that aren't singletons and their pre-image in $\tilde{X}$ is endpoints of the intervals. These endpoints are all mapped to a single point in R_2 under the map $q \circ \tilde{f}$ so we have a continuous map $f\colon \tilde{R} \to R_2$ by Theorem 5.10.

Example: Let $\mathrm{SL}_2(\mathbb{R})$ be 2-by-2 matrices with determinant one. This is a group with the operation of matrix multiplication. As a subspace of $\mathbb{R}^4$ it is also a topological space. Given $A \in \mathrm{SL}_2(\mathbb{R})$ define $\phi_A\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})$ by $\phi_A(B) = ABA^{-1}$. Note that

$$\det \phi_A(B) = \det\left(ABA^{-1}\right) = \det A \cdot \det B \cdot \det A^{-1} = 1$$

so if $B \in \mathrm{SL}_2(\mathbb{R})$ then $\phi_A(B) \in \mathrm{SL}_2(\mathbb{R})$. Therefore ϕ_A is a map from $\mathrm{SL}_2(\mathbb{R})$ to itself as claimed.

It is also a continuous map. To see this we can identify $\mathbb{R}^4$ with the space $\mathscr{M}_2$ of 2-by-2 matrices. Then ϕ_A is also a map for $\mathscr{M}_2$ to itself. As a map of $\mathbb{R}^4$ to itself it is a polynomial in each coordinate and therefore continuous. This implies that the restriction of ϕ_A to $\mathrm{SL}_2(\mathbb{R})$ is also continuous.

We also note that $\phi_A \circ \phi_B(C) = A(BCB^{-1})A^{-1} = (AB)C(AB)^{-1} = \phi_{AB}(C)$. In particular, $\phi_{A^{-1}} \circ \phi_A = \phi_{\mathrm{id}} = \mathrm{id}_{\mathrm{SL}_2(\mathbb{R})}$ where id is the identity matrix and $\mathrm{id}_{\mathrm{SL}_2(\mathbb{R})}$ is the identity map on $\mathrm{SL}_2(\mathbb{R})$. This implies ϕ_A and $\phi_{A^{-1}}$ are inverses of each other so ϕ_A is a homeomorphism of $\mathrm{SL}_2(\mathbb{R})$ to itself. The equation $\phi_A \circ \phi_B(C) = \phi_{AB}$ also implies that the map $A \mapsto \phi_A$ is a group homomorphism from $\mathrm{SL}_2(\mathbb{R})$ to $\mathbf{Homeo}(\mathrm{SL}_2(\mathbb{R}))$. Therefore $G = \{\phi_A\}$ is a group action on $\mathrm{SL}_2(\mathbb{R})$.

Problem 22 *Let* $\mathrm{tr}\colon \mathrm{SL}_2(\mathbb{R}) \to \mathbb{R}$ *be the trace map and let* $q\colon \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_2(\mathbb{R})/G$ *be the quotient map. Show that there exists a continuous map* $f\colon \mathrm{SL}_2(\mathbb{R})/G \to \mathbb{R}$ *with* $\mathrm{tr} = f \circ q$ *where* tr *is the trace map. For* $A, B \in \mathrm{SL}_2(\mathbb{R})$ *with* $\mathrm{tr}A \neq \pm 2$ *show that* $A \in [B]_G$ *if and only if* $\mathrm{tr}A = \mathrm{tr}B$*. More generally show that every* G*-orbit is of the form*

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}_G \quad or \quad \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}_G \quad or \quad \begin{bmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{bmatrix}_G \quad or \quad \begin{bmatrix} \pm 1 & \mp 1 \\ 0 & \pm 1 \end{bmatrix}_G.$$

Problem 23 *Describe the topology of* $\mathrm{SL}_2(\mathbb{R})/G$ *by giving an explicit description of open sets. First classify which orbits are closed subsets of* $\mathrm{SL}_2(\mathbb{R})$*.*

Deck actions

A group action G on X is a *deck action* if every $x \in X$ has a neighborhood U such that for all $g \in G \smallsetminus \mathrm{id}$ we have $U \cap g(U) = \emptyset$.

Theorem 5.11 *If* G *is a deck action on* X *then* $q\colon X \to X/G$ *is a covering map.*

Proof: For $x \in X$ let U be the open neighborhood given by the deck action property and let $[U]_G$ be the image of U in the quotient. We claim that $[U]_G$ is evenly covered. We first claim that $g(U) \cap h(U) = \emptyset$ if $g \neq h$ in G. This follows from the fact that

$$g(U) \cap h(U) = g\left(U \cap g^{-1}h(U)\right) = g(\emptyset) = \emptyset$$

since $g^{-1}h \neq \mathrm{id}$.

Now let

$$\tilde{U} = \bigcup_{g \in G} g(U)$$

be the disjoint union of the $g(U)$. As a union of open sets $\tilde{U}$ is open. We claim that q is a bijection to $[U]_G$ on each $g(U)$. By definition $[U]_G$ is the q-image of U so every point in $[U]_G$ is of the form $[x]_G$ for some $x \in U$. Then for each $g \in G$ we have $g(x) \in g(U)$ and $g(x) \in [x]_G$ so $q(g(x)) = [x]_G$ and q is a surjection from $g(U)$ to $[U]_G$. Now assume that $y \in g(U)$ with $q(y) = [x]_G$. Then there exists an $h \in G$ with $h(x) = y$ which in turn implies that $y \in g(U) \cap h(U)$ so the intersection is non-empty. Therefore $g = h$ so $y = h(x) = g(x)$. This implies that q is injective on $g(U)$ and hence is a bijection on $g(U)$.

We also show that q is an open map. Let $V \subset X$ be open. Given $x \in V$ we need to show that there exists an open neighborhood of $[x]_G$ in X/G that is contained in $q(V)$. As before let U be an open neighborhood of x given by the deck action property. Then $U_0 = U \cap V$ is also neighborhood of x and also satisfies the deck action property since if $U_0 \cap g(U_0) \neq \emptyset$ then $U \cap g(U) \neq \emptyset$. As before we also set

$$\tilde{U_0} = \bigcup_{g \in G} g(U_0).$$

This set is open and if $[U_0]_G = q(U_0)$ then $\tilde{U_0} = q^{-1}([U_0]_G)$. Since $\tilde{U_0}$ is open and in the quotient topology as set is open if and only if its pre-image is open we have that $[U_0]_G$ is open. Furthermore $[U_0]_G \subset q(V)$ and $[x]_G \in [U_0]_G$ so we have obtained the desired neighborhood of $[x]_G$ and $q(V)$ is open. Therefore q is an open map.

Recapping we have:

- The components of $\tilde{U} = q^{-1}([U]_G)$ are $g(U)$.
- Each $g(U)$ is open and q is a bijection from $g(U)$ to $[U]_G$.
- The map q is open.

Then by Lemma 3.4 the restriction $q|_{g(U)}$ is a homeomorphism to $[U]_G$ so $[U]_G$ is an evenly covered neighborhood and $q\colon X \to X/G$ is a covering space. □

Problem 24 *Let G be a deck action on a space X and let $H \subset G$ be a subgroup. Let $q\colon X \to X/G$ and $q_0\colon X \to X/H$ be the quotient maps (which are covering maps by the proposition). Show that there is a covering map $p\colon X/H \to X/G$ with $q = p \circ q_0$.*

A space X is *simply connected* if X is path connected and $\pi_1(X,x_0)$ is trivial for some (and hence any) $x_0 \in X$.

Theorem 5.12 *Let G deck action on a simply connected and Hausdorff space $\tilde{X}$ and let $X = \tilde{X}/G$. Then $\pi_1(X,x_0)$ is isomorphic to G for all $x_0 \in X$.*

Proof: By Theorem 5.11 we have that $q\colon \tilde{X} \to X$ is a covering map. Fix $\tilde{x}_0 \in q^{-1}(x_0)$. Then the map $\phi\colon G \to q^{-1}(x_0)$ given by $\phi(g) = g(x_0)$ is a bijection.

Given $\tilde{x}_1 \in q^{-1}(x_0)$ fix a path $\gamma\colon [0,1] \to \tilde{X}$ with $\gamma(0) = \tilde{x}_0$ and $\gamma(1) = \tilde{x}_1$. Then $q \circ \gamma$ represents an element of $\pi_1(X,x_0)$. We claim that $[q \circ \gamma]$ only depends on $\tilde{x}_1$ and not the choice of path γ. Let $\alpha\colon [0,1] \to \tilde{X}$ be another path with $\gamma(0) = \tilde{x}_0$ and $\gamma(1) = \tilde{x}_1$. Since $\pi_1(\tilde{X},\tilde{x}_0)$ is trivial the paths γ and α are path homotopic so $q \circ \gamma$ and $q \circ \alpha$ are path homotopic and $[q \circ \gamma] = [q \circ \alpha]$. Therefore the map $\psi\colon q^{-1}(\tilde{x}_0) \to \pi_1(X,x_0)$ given by $\psi(\tilde{x}_1) = [q \circ \gamma]$ is well defined.

We'll show that the composition $\psi \circ \phi$ is an isomorphism from G to $\pi_1(X,x_0)$. First we show that it is a homomorphism. For $g_0, g_1 \in G$ let $\gamma_i\colon [0,1]$ be paths with $\gamma_i(0) = \tilde{x}_0$ and $\gamma_i(1) = g_i(\tilde{x}_0)$. Note that the composition $g_0 \circ \gamma_1$ is a path with $g_0 \circ \gamma_1(0) = g_0(\tilde{x}_1)$ so the concatenation $\gamma_0 * (g_0 \circ \gamma_1)$ is defined. Furthermore $\gamma_0 * (g_0 \circ \gamma_1)(0) = \tilde{x}_0$ and $\gamma_0 * (g_0 \circ \gamma_1)(1) = g_0(\gamma_1(1)) = g_0 \circ g_1(\tilde{x}_0)$ so $\gamma_0 * (g_0 \circ \gamma_1)$ is a path from $\tilde{x}_0$ to $g_0 \circ g_1(\tilde{x}_0)$. Therefore

$$\begin{aligned}
\psi \circ \phi(g_0 \circ g_1) &= [q \circ \gamma_0 * (g_0 \circ \gamma_1)] \\
&= [(q \circ \gamma_0) * (q \circ (g_0 \circ \gamma_1)] \\
&= [q \circ \gamma_0] \cdot [q \circ \gamma_1] \\
&= \psi \circ \phi(g_0) \cdot \psi \circ \phi(g_1)
\end{aligned}$$

implying that $\psi \circ \phi$ is a homomorphism.

Now we show that $\psi \circ \phi$ is injective. If $\psi \circ \phi(g) = [q \circ \gamma] = [\mathrm{id}_{x_0}]$ then γ is a lift of $q \circ \gamma$ that takes 0 to $\tilde{x}_0$ and therefore by Lemma 5.6 we have $\gamma(1) = \tilde{x}_0$. This implies that $\phi(g) = \tilde{x}_0$ and $g = \mathrm{id}_G$.

Finally we show that $\psi \circ \phi$ is surjective. Let $[f] \in \pi_1(X,x_0)$ and $\tilde{f}\colon ([0,1],0) \to (\tilde{X},\tilde{x}_0)$ the lift. Then $\tilde{f}(1) = \tilde{x}_1 \in q^{-1}(x_0) = [\tilde{x}_0]_G$ and there exists a $g \in G$ such that $g(\tilde{x}_0) = \tilde{x}_1$. Therefore $\phi(g) = \tilde{x}_1$

and as $\tilde{f}$ is a path from $\tilde{x}_0$ to $\tilde{x}_1$ we have $\psi(\tilde{x}_1) = [q \circ \tilde{f}] = [f]$. This combines to give $\psi \circ \phi(g) = [f]$ so $\psi \circ \phi$ is surjective. □

Let $C(X,Y)$ is the set of continuous maps between topological spaces X and Y. This set has a natural topology, the *compact-open topology.* Let K be a compact subset of X and U an open subset of Y. Then we let

$$V(K,U) = \{f \in C(X,Y) \,|\, f(K) \subset U\}.$$

The compact-open topology is smallest topology on $C(X,Y)$ that contains $V(K,U)$ for all pairs of compact sets $K \subset X$ and open sets $U \subset Y$.

Problem 25 *If Y is a metric space with metric d_Y show that $f_i \to f$ in the compact-open topology in $C(X,Y)$ if and only if for every compact set $K \subset X$ and $\varepsilon > 0$ there exists an $n = n(K,\varepsilon)$ with $d_Y(f(x), f_i(x)) \leq \varepsilon$ if $i \geq n$ and $x \in K$.*

As homeomorphisms are continuous maps $\mathbf{Homeo}(X)$ is a subset of $C(X,X)$ and as such inherits the subspace topology when we take the compact-open topology on $C(X,X)$.

Problem 26 *Above we saw the map $A \mapsto \phi_A$ is a map from $\mathrm{SL}_2(\mathbb{R})$ to $\mathbf{Homeo}(\mathrm{SL}_2(\mathbb{R}))$. Show that this map is continuous.*

Problem 27 *If $G \subset \mathbf{Homeo}(X)$ is a deck action on X show that G is discrete in $\mathbf{Homeo}(X)$. Show that the converse fails. That is find a space X and a discrete subgroup G of $\mathbf{Homeo}(X)$ that is not a deck action. (There are many ways to do this. One way is to let $X = S^1$ and find a subgroup of $\mathbf{Homeo}(S^1)$ that is isomorphic to $\mathbb{Z}$. In fact one can find such a subgroup where every orbit is dense.)*

6 Free groups

Let X be a figure eight and let $x_0 \in X$ be the point where the two circles meet. We would like to give an algebraic description of $\pi_1(X, x_0)$. This is a very important group.

Let $\mathscr{A}$ be a set with a fixed point free involution $x \mapsto \bar{x}$. That is $\bar{\bar{x}} = x$ and $x \neq \bar{x}$. Let $\tilde{\mathscr{W}} = \tilde{\mathscr{W}}_{\mathscr{A}}$ be the collection of finite sequences $x_1 x_2 \cdots x_n$ where the x_i are in $\mathscr{A}$. Then $\mathscr{W}$ is the collection of *words* in the *alphabet* $\mathscr{A}$. A word $x_1 x_2 \cdots x_n$ is *reduced* if $x_i \neq \bar{x}_{i+1}$ for $i = 1, \dots, n-1$ and let $\mathscr{W} = \mathscr{W}_{\mathscr{A}}$ be the set of reduced words. Note that both $\tilde{\mathscr{W}}$ and $\mathscr{W}$ include the empty word - the word with no letters.

We will define a group structure on $\mathscr{W}$. We need to define the binary operation $\mathscr{W} \times \mathscr{W} \to \mathscr{W}$. There is an obvious operation - concatenation. Given $w_0 = x_1 x_2 \cdots x_n$ and $w_1 = y_1 y_1 \cdots y_m$ define

$$w_0 * w_1 = x_1 x_2 \cdots x_n y_1 \cdots y_m.$$

If we consider this as an operation on all words then it is clearly a binary operation from $\tilde{\mathscr{W}} \times \tilde{\mathscr{W}}$ to $\tilde{\mathscr{W}}$. It is also easy to see that the empty word is an identity and that the associative property holds. However, there will not be inverses. On the other hand if we restrict to reduced words the concatenation $w_0 * w_1$ may not be reduced. There is a natural reduction process. For example if $w_0 = x_1 \cdots x_n$ and $w_1 = \bar{x}_n \cdots \bar{x}_1$ then we will reduce $w_0 * w_1$ to the empty word. In general if $w_0 = x_1 x_2 \cdots x_n$ and $w_1 = y_1 y_1 \cdots y_m$ are reduced words let k be the largest non-negative integers such hat $x_{n-i+1} = \bar{y}_i$ for $i \leq k$ and define

$$w_0 \cdot w_1 = x_1 x_2 \cdots x_{n-k} y_{1+k} \cdots y_m.$$

Lemma 6.1 *Let $w_0, w_1 \in \mathscr{W}$ then $w_0 \cdot w_1 \in \mathscr{W}$.*

Proof: Let $w_0 \cdot w_1 = z_1 z_2 \cdots z_{n+m-2k}$. If $i \le n-k-1$ then $z_i = x_i$ and $z_{i+1} = x_{i+1}$ so, since $w_0 \in \mathscr{W}$, we have $z_I = x_i \neq \bar{x}_{i+1} = \bar{z}_{i+1}$. If $i \ge n-k+1$ then $z_i = y_{i+k-n}$ and $z_i = y_{i+k-n+1}$ so again $z_i \neq \bar{z}_{i+1}$. The final case is $i = n-k$. In this case by assumption $z_{n-k} = x_{n-k} \neq \bar{y}_{1+k} = z_{n-k+1}$ by assumption. □

For reasons that will be soon clear we let $\mathrm{id} \in \mathscr{W}$ be the empty word.

Lemma 6.2 *If $w \in \mathscr{W}$ with $w = x_1 x_2 \cdots x_n$ then define $\bar{w} = \bar{x}_n \bar{x}_{n-1} \cdots \bar{x}_1$. Then*

$$w \cdot \mathrm{id} = \mathrm{id} \cdot w = w \qquad \textit{and} \qquad w \cdot \bar{w} = \bar{w} \cdot w = \mathrm{id}.$$

Proof: This follows directly from the definitions. □

This is almost everything we need to to give show that $(\mathscr{W}, \cdot)$ has the structure of a group. The last thing we need to check is associativity. This is surprisingly difficult!

6.1 Graphs

We define a *directed graph*. A (directed) graph is a union of *vertices* and *(directed) edges*. Let $\mathscr{V}$ be a set and $\mathscr{E}$ a collection of pairs of elements in $\mathscr{V}$. We use $\mathscr{V}$ and $\mathscr{E}$ to construct a topological space. To be a bit pedantic we let $\tilde{\mathscr{V}}$ be the topological space obtained by giving $\mathscr{V}$ the discrete topology. We also let $\tilde{\mathscr{E}}$ be a disjoint collection of intervals $[0,1]$ and assume that $e \mapsto \tilde{e}$ is a bijection from $\mathscr{E}$ to $\tilde{\mathscr{E}}$. In general we won't distinguish between $\mathscr{V}$ (vertices as a set) and $\tilde{\mathscr{V}}$ (vertices as a topological space) or $\mathscr{E}$ (pairs of vertices) and $\tilde{\mathscr{E}}$ (the associated copies of the unit interval), but we will do so now when making definitions. If $e \in \mathscr{E}$ then e^- is the left vertex and e^+ is the right vertex while $\tilde{e}^-$ be the left endpoint of $\tilde{e}$ and $\tilde{e}^+$ is the right endpoint of $\tilde{e}$. We define an equivalence relation on $\mathscr{V} \sqcup \tilde{\mathscr{E}}$ by setting $v \sim \tilde{e}^{\pm}$ if $v = e^{\pm}$. This defines a relation which extends to a unique smallest equivalence relation. (See problem 21 below.) We then define the graph to be the quotient space

$$\mathscr{G} = \tilde{\mathscr{V}} \sqcup \tilde{\mathscr{E}} / \sim.$$

The graph is determined by the vertices and edges so we write $\mathscr{G} = (\mathscr{V}, \mathscr{E})$. We will always assume that $\mathscr{G}$ is connected. In particular every vertex will be the endpoint of at least on edge.

If $e = v_0 v_1$ is an ordered pair of vertices in $\mathscr{V}$ we let $\bar{e} = v_1 v_0$. We let $\bar{\mathscr{E}}$ be the set of edges of $\mathscr{E}$ with e replaced by $\bar{e}$. Then the graph $\bar{\mathscr{G}} = (\mathscr{V}, \bar{\mathscr{E}})$ is homeomorphic to $\mathscr{G}$ so the pair $(\mathscr{V}, \mathscr{E})$ contains more information than just the topology of the graph. This is why $\mathscr{G}$ is a *directed* graph. The extra structure makes it easier to define and to work with.

Certain maps between edges of graphs determine a continuous map between the graphs. For $i = 0, 1$ let $\mathscr{G}_i = (\mathscr{V}_i, \mathscr{E}_i)$ be graphs and let $\phi_{\mathscr{E}} \colon \mathscr{E}_0 \to \mathscr{E}_1 \sqcup \bar{\mathscr{E}}_1$ be a map. The map $\phi_{\mathscr{E}}$ is *allowable* for all edges $e_0, e_1 \in \mathscr{E}_0$ if $e_0^- \sim e_1^-$ then $\phi_{\mathscr{E}}(e_0)^- \sim \phi_{\mathscr{E}}(e_1)^-$ with a similar statement if $-$ is replaced with $+$.

Lemma 6.3 *If $\phi_{\mathscr{E}}$ is allowable then there is a unique map $\phi_{\mathscr{V}} \colon \mathscr{V}_0 \to \mathscr{V}_1$ such that $\phi_{\mathscr{E}}(e)^{\pm} = \phi_{\mathscr{V}}(e^{\pm})$. Let $\tilde{\phi} \colon \tilde{\mathscr{V}}_0 \sqcup \tilde{\mathscr{E}}_0 \to \tilde{\mathscr{V}}_1 \sqcup \tilde{\mathscr{V}}_1$ be the continuous maps such that for each $v \in \mathscr{V}$ we have $\tilde{\phi}(\tilde{v}) = \widetilde{\phi_{\mathscr{V}}(v)}$ and for each $e \in \mathscr{E}_0$ we have that $\tilde{\phi}$ is the natural homeomorphism between $\tilde{e}$ and $\widetilde{\phi_{\mathscr{E}}(e)}$. Then there is a unique continuous map $\phi \colon \mathscr{G}_0 \to \mathscr{G}_1$ with $\phi \circ q_0 = q_1 \circ \phi$ where $q_i \colon \tilde{\mathscr{V}}_i \sqcup \tilde{\mathscr{E}}_i \to \tilde{\mathscr{V}}_i \sqcup \tilde{\mathscr{E}}_i / \sim$ are the quotient maps.*

Proof: For every vertex $v \in \mathscr{V}_0$ there is an edge $e \in \mathscr{E}_0$ with $v = e^{\pm}$. Define $\phi_{\mathscr{V}}(v) = \phi_{\mathscr{E}}(e)^{\pm}$. As $\phi_{\mathscr{E}}$ is allowable this map is well defined.

To see that ϕ is continuous we apply Theorem 5.10 where $X = \tilde{\mathscr{V}}_0 \sqcup \tilde{\mathscr{E}}_0$, Y is the quotient graph $\mathscr{G}_0$, Z is the quotient graph $\mathscr{G}_1$ and $q_1 \circ \tilde{\phi} \colon \tilde{\mathscr{V}}_0 \sqcup \tilde{\mathscr{E}}_0 \to \mathscr{G}_1$ is the map f. If $x \in \mathscr{G}_0$ is not a vertex then $q_0^{-1}(x)$ is a singleton so clearly $q_1 \circ \tilde{\phi}$ restricted to $q_0^{-1}(x)$ is constant. If $x \in \mathscr{G}_0$ is a vertex then $q_0^{-1}(x)$ is a single vertex and a collection of endpoints of edges. Here the allowability condition implies that $q_1 \circ \tilde{\phi}$ is constant on $q_0^{-1}(x)$. The existence and uniqueness of ϕ then follows from Theorem 5.10. □

One simple example is when $\mathscr{V} = \{v\}$ is a single vertex and $\mathscr{E} = \{e = vv\}$ is a single edge. Then $\mathscr{G}$ is homeomorphic to the circle. As there is only one edge the set $\mathscr{E} \sqcup \bar{\mathscr{E}}$ has two elements and there are two possible maps $\phi_{\mathscr{E}}$ from $\mathscr{E}$ to $\mathscr{E} \sqcup \bar{\mathscr{E}}$. Namely either $\phi_{\mathscr{E}}(e) = e$ or $\phi_{\mathscr{E}}(e) = \bar{e}$. As there is only one vertex both of these maps are allowable. In the first case the induced map is the identity. In the second case it is a homeomorphism but is not the identity.

Given $\phi_{\mathscr{E}} \colon \mathscr{E} \to \mathscr{E} \sqcup \bar{\mathscr{E}}$ define $\bar{\phi}_{\mathscr{E}} \colon \mathscr{E}_0 \sqcup \bar{\mathscr{E}}_0 \to \mathscr{E}_1 \sqcup \bar{\mathscr{E}}_1$ by $\bar{\phi}_{\mathscr{E}}(e) = \phi_{\mathscr{E}}(e)$ if $e \in \mathscr{E}_0$ and $\bar{\phi}_{\mathscr{E}}(\bar{e}) = \overline{\phi_{\mathscr{E}}(e)}$ if $\bar{e} \in \bar{\mathscr{E}}_0$.

Let $\mathscr{G}_2 = (\mathscr{V}_2, \mathscr{E}_2)$ be another graph. Note that if $(\phi_{\mathscr{E}})_i \colon \mathscr{E}_i \to \mathscr{E}_{i+1} \sqcup \bar{\mathscr{E}}_{i+1}$ are allowable maps the restriction of $(\bar{\phi}_{\mathscr{E}})_1 \circ (\bar{\phi}_{\mathscr{E}})_0$ to $\mathscr{E}_0$ is allowable. If $(\bar{\phi}_{\mathscr{E}})_0$ is a bijection then the inverse, restricted to $\mathscr{E}_1$, is allowable. The composition $(\bar{\phi}_{\mathscr{E}})_0^{-1} \circ (\bar{\phi}_{\mathscr{E}})_0$ is the identity and therefore the induced continuous map from on $\mathscr{G}_0$ to itself is the identity. This implies that if $\bar{\phi}_{\mathscr{E}}$ is a bijection then the induced map on graphs is a homeomorphism.

Problem 28 *Let X be at topological space and $x_0 \in X$ a basepoint. Assume that for every compact set $K \subset X$ there is a simply connected neighborhood $U \subset X$ with $x_0 \in X$ and $K \subset U$. Show that X is simply connected.*

Problem 29 *Let X be a topological space and $x_0 \in X$ a basepoint. Assume that for every compact set K there is a path connected open set U such that $K \subset U$, $x_0 \in U$ and if $\iota_U \colon U \to X$ is the inclusion map then $(\iota_U)_*$ is the trivial homomorphism. Show that X is simply connected.*

6.2 The figure eight

Let X be the figure eight. We can realize X as a graph with one vertex v and two edges e_a and e_b. Note that both edges are just the pair vv but we allow this. That $\mathscr{V}_X = \{v\}$ and $\mathscr{E}_X = \{e_a = vv, e_b = vv\}$.

We will construct a graph that is simply connected and that covers X.

Let $\mathscr{V}_{\tilde{X}} = \mathscr{W}$, the set of reduced words and let $\mathscr{E}_{\tilde{X}}$ be pairs of reduced words of the form $(w, w*x)$ with $x \in \{a, b, \bar{a}, \bar{b}\}$ a single letter. If $w = \emptyset$ is the empty word then there are four possible edges $w(w*x)$ where x varies of all four letters of the alphabet $\mathscr{A}$. If w is not empty and the last letter is y then there are three possible edges $w(w*x)$ where x varies over $\{a, b, \bar{a}, \bar{b}\} \smallsetminus \{y\}$. Let $\tilde{X} = (\mathscr{V}_{\tilde{X}}, \mathscr{E}_{\tilde{X}})$.

Let $\mathscr{W}_n$ be the set of reduced words of length $\leq n$ and $\mathscr{V}_n$ the corresponding set of vertices. Let $\mathscr{E}_n \subset \mathscr{E}$ be the subset of edges where both vertices have lengths $\leq n$. Then $\tilde{X}_n = (\mathscr{V}_n, \mathscr{E}_n)$ is a graph. Note that there is an inclusion $\iota_n \colon \tilde{X}_n \to \tilde{X}_{n+1}$.

Lemma 6.4 *The graph $\tilde{X}_{n+1}$ deformation retracts to $\tilde{X}_n$.*

Rather than prove this directly will prove something more general that implies this lemma.

If $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ is graph the *valence* of a vertex $v \in \mathscr{V}$ is the number of half edges that have v as an endpoint. That is the the number of edges e with $v = e^-$ plus the number of edges with $e^+ = v$. Now let $\mathscr{V}' \subset \mathscr{V}$ be the collection of vertices whose valence is ≥ 2. We let $\mathscr{E}' \subset \mathscr{E}$ be the subset of edges neither of whose endpoints are a vertex of valence one. Then $\mathscr{G}' = (\mathscr{V}', \mathscr{E}')$ is a subgraph of $\mathscr{G}$.

Lemma 6.5 *The graph $\mathscr{G}$ deformation retracts to $\mathscr{G}'$ (unless $\mathscr{G} = (\{v_0, v_1\}, \{v_0 v_1\})$ in which case $\mathscr{G}'$ is the empty graph).*

Proof: Let $e \in \mathscr{E}$ be an edge such that the valence of e^+ is one. Then define

$$\tilde{F}_e \colon \tilde{e} \times [0,1] \to \tilde{e}$$

by $\tilde{F}_e(s,t) = (1-t)s$. If the valence of e^- is one then we define $\tilde{F}_e(s,t) = (1-t)s+t$. If neither the valence of e^+ or e^- is one then define $\tilde{F}_e(s,t) = s$. If $v \in \mathscr{V}$ has valence one and $e \in \mathscr{E}$ is the edge with $v = e^\pm$ then we define

$$\tilde{F}_v \colon \tilde{v} \times [0,1] \to \tilde{e}$$

by $\tilde{F}_v(\tilde{v},t) = t$ if $v = e^-$ and $\tilde{F}_v(\tilde{v},t) = 1-t$. If the valence of v is ≥ 2 then $\tilde{F}_v(\tilde{v}) = \tilde{v}$.

Now define

$$\tilde{F} \colon (\tilde{\mathscr{V}} \sqcup \tilde{\mathscr{E}}) \times [0,1] \to \tilde{\mathscr{V}} \sqcup \tilde{\mathscr{E}}$$

by $\tilde{F} = \tilde{F}_e$ when $\tilde{e} \in \tilde{\mathscr{E}}$ and $\tilde{F} = \tilde{F}_v$ when $\tilde{v} \in \tilde{V}$. If q is the quotient map $q \colon \tilde{\mathscr{V}} \sqcup \tilde{\mathscr{E}} \to \mathscr{G}$ then we apply Theorem 5.10 to $q \circ \tilde{F}$ to get a continuous map

$$F \colon \mathscr{G} \times [0,1] \to \mathscr{G}$$

with $F \circ q = q \circ \tilde{F}$. Then F is a deformation retract from $\mathscr{G}$ to $\mathscr{G}'$. □

Proof of Lemma 6.4: We apply the previous lemma where $\mathscr{G} = \tilde{X}_{n+1}$ and $\mathscr{G}' = \tilde{X}_n$. □

Corollary 6.6 *$\tilde{X}_n$ is simply connected.*

Proof: We induct on n. The base case is when $n = 0$ in which case $\tilde{X}_0$ is a point and therefore simply connected.

Now assume that $\tilde{X}_n$ is simply connected. Then $\tilde{X}_{n+1}$ deformation retracts to X_n so is also simply connected. □

Theorem 6.7 *$\tilde{X}$ is simply connected.*

Proof: The union of the interiors of the $\tilde{X}_n$ cover $\tilde{X}$ so given a compact set $K \subset \tilde{X}$ there are finitely many $\tilde{X}_{n_1}, \dots, \tilde{X}_{n_k}$ whose union contains K. Note that $\tilde{X}_n \subset \tilde{X}_m$ if $n \leq m$ so if n is the max of $\{n_1, \dots, n_k\}$ we have $K \subset \tilde{X}_n$. By Corollary 6.6, $\tilde{X}_n$ is simply connected and by Problem 28 $\tilde{X}$ is simply connected. □

We now define a map from $\tilde{X}$ to X and show that it is a covering map. We first define the map on edges $\phi_{\tilde{E}} \colon \mathscr{E}_{\tilde{X}} \to \mathscr{E}_X \sqcup \bar{\mathscr{E}}_X$. There are four types of edges. If e of the form $w(w*a)$ then $\phi_{\mathscr{E}}(e) = e_a$ while if e is of the form $w(w*\bar{a})$ we define $\phi_{\mathscr{E}}(e) = \bar{e}_a$. We similarly define $\phi_{\mathscr{E}}$ on edges $w(w*b)$ and $w(w*\bar{b})$ with the image e_b or $\bar{e}_b$. As $\mathscr{V}_X$ is a single element the map $\phi_{\mathscr{E}}$ is automatically allowable and induces a map $p \colon \tilde{X} \to X$. We'll show that this is a covering map by giving a more general condition for maps between graphs to be covering maps.

Before stating the condition for a covering map we make a definition. For an edge e let $e^+_{1/3}$ be the third of the edge e that contains the endpoint e^+ and define $e^-_{1/3}$ to be the third of the edge that contains the endpoint e^-. For a vertex $v \in \mathscr{V}$ we then define

$$\mathscr{E}_{1/3}(v) = \left\{ e^\pm_{1/3} | e^\pm = v \right\}.$$

For $i = 0, 1$ let $\mathscr{G}_i = (\mathscr{V}_i, \mathscr{E}_i)$ be graphs and $\phi_{\mathscr{E}} \colon \mathscr{E}_0 \to \mathscr{E}_1 \sqcup \bar{\mathscr{E}}_1$ be an allowable map determining a map $\phi \colon \mathscr{G}_0 \to \mathscr{G}_1$. The map $\phi_{\mathscr{E}}$ induces a map on the one-third edges.

Lemma 6.8 *The induced map ϕ is a covering map if and only $\phi_{\mathscr{E}}$ is a bijection from $(\mathscr{E}_0)_{1/3}(v)$ to $(\mathscr{E}_1)_{1/3}(\phi_{\mathscr{V}}(v))$ for all $v \in \mathscr{V}$.*

Proof: For each $v \in \mathscr{V}_0$ let $\mathscr{N}_{1/3}(v)$ be the union of v and the $e^\pm_{1/3}$ where $e^\pm = v$. The condition of the lemma is equivalent to the restriction of ϕ to $\mathscr{N}_{1/3}(v)$ is a homeomorphism to $\mathscr{N}_{1/3}(\phi_{\mathscr{V}}(v))$. Then for each $v \in \mathscr{V}_1$, $\mathscr{N}_{1/3}(v)$ is an evenly covered neighborhood since $p^{-1}(\mathscr{N}_{1/3}(v))$ is the disjoint union of $\mathscr{N}_{1/3}(v')$ where $v' \in \phi^{-1}_{\mathscr{V}}(v)$.

For an edge e let $\overset{\circ}{e}$ its interior. Then $\phi^{-1}\left(\overset{\circ}{e}\right)$ is a union of interior of edges of $\mathscr{G}_0$. Therefore $\overset{\circ}{e}$ is also evenly covered.

Every point in $\mathscr{G}_1$ is contained in a set of one of these two types so this shows that p is a covering map. □

Applying this lemma p we have:

Corollary 6.9 *The map $p\colon \tilde{X} \to X$ is a covering map.*

Whenever we have a covering space $p\colon \tilde{X} \to X$ with $\tilde{X}$ simply connected we can realize X as a quotient of $\tilde{X}$ be a deck action.

Theorem 6.10 *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space with $\tilde{X}$ simply connected. For each $\tilde{x}_1 \in p^{-1}(x_0)$ there is a unique homeomorphism $\phi_{\tilde{x}_1}\colon (\tilde{X}, \tilde{x}_0) \to (\tilde{X}, \tilde{x}_1)$ with $p \circ \phi_{\tilde{x}_1} = p$ and the $\{\phi_{\tilde{x}_i}\}$ for a subgroup of* **Homeo**$(\tilde{X})$ *that form a deck action with quotient X.*

Proof: The existence of $\phi_{\tilde{x}_1}$ follows from Proposition 5.8. Note that $\mathrm{id}_{\tilde{X}}(\tilde{x}_0) = \tilde{x}_0$ and $p \circ \mathrm{id}_{\tilde{X}} = p$ so by the uniqueness statement of Proposition 5.8, $\mathrm{id}_{\tilde{X}} = \phi_{\tilde{x}_0}$. If $x_1, x_2 \in p^{-1}(x_0)$ then

$$p \circ \phi_{\tilde{x}_1} \circ \phi_{\tilde{x}_2} = p \circ \phi_{x_2} = p$$

so $\phi_{\tilde{x}_3} = \phi_{\tilde{x}_1} \circ \phi_{\tilde{x}_2}$ where $\tilde{x}_3 = \phi_{\tilde{x}_1} \circ \phi_{\tilde{x}_2}(\tilde{x}_0)$. This shows that the $\{\phi_{x_i}\}$ form a subgroup G of **Homeo**$(\tilde{X})$.

Given $\tilde{y} \in \tilde{X}$ let U be an evenly covered neighborhood of $y = p(\tilde{y})$. Let V be the component of $p^{-1}(V)$ that contains $\tilde{y}$. Assume that $\phi_{\tilde{x}_0} \in G$ is not the identity. Note that both $\tilde{y}$ and $\phi_{\tilde{x}_0}(y)$ are both in $p^{-1}(y)$ since $p(\phi_{\tilde{x}_0}(\tilde{y})) = p(\tilde{y}) = y$ so again applying Proposition 5.8 there is a unique homeomorphism ϕ of $\tilde{X}$ with $\phi(\tilde{y}) = \phi_{\tilde{x}_0}(\tilde{y})$ and $p \circ \phi = p$. However, $\phi_{\tilde{x}_0}$ has both these properties so $\phi = \phi_{\tilde{x}_0}$. We also note that if $\phi(\tilde{y}) = \tilde{y}$ then ϕ (and hence $\phi_{\tilde{x}_0}$) is the identity. By assumption $\phi_{\tilde{x}_0}$ is not the identity so $\phi_{\tilde{x}_0}(\tilde{y}) = \phi(\tilde{y}) \neq \tilde{y}$. Since $\phi_{\tilde{y}}$ is a homeomorphism with $p \circ \phi_{\tilde{y}} = p$ we have that $\phi_{\tilde{y}}(V)$ is a component of $p^{-1}(U)$. As $p^{-1}(y) \cap \phi_{\tilde{x}_0}(V) = \{\phi_{\tilde{x}_0}(\tilde{y})\}$ we have that V and $\phi_{\tilde{x}_0}(V)$ are disjoint so that G is a deck action.

Finally we show that the G-orbit $[\tilde{y}]_G$ is equal to $p^{-1}(y)$. We apply Proposition 5.8 again. If $\tilde{y}_0 \in p^{-1}(y)$ we have a homeomorphism ϕ with $\phi(\tilde{y}) = \tilde{y}_0$. Let $\tilde{x}_1 = \phi(\tilde{x}_0)$. Then ϕ is the unique homeomorphism such that $\phi(\tilde{x}_0) = \tilde{x}_1$ and $p \circ \phi = p$. This implies that $\phi = \tilde{\phi}_{\tilde{x}_0}$ so $\phi \in G$ and $p^{-1}(y) \subset [\tilde{y}]_G$. On the other hand if $\tilde{y}_0 \in [\tilde{y}]_G$ then there exists a $\phi_{\tilde{x}_1} \in G$ with $\phi_{\tilde{x}_1}(\tilde{y}) = \tilde{y}_0$. Since $p(\tilde{y}_0) = p(\phi_{\tilde{x}_1}(\tilde{y})) = p(\tilde{y}) = y$ we have $\tilde{y}_0 \in p^{-1}(y)$. Therefore $[\tilde{y}]_G \subset p^{-1}(y)$ and the two sets are equal. □

Problem 30 *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space with $\tilde{X}$ simply connected. Let H be a subgroup of $\pi_1(X, x_0)$ and show that there is a covering space $p_H\colon (X_H, x_H) \to (X, x_0)$ with*

$$(p_H)_*(\pi_1(X_H, x_H)) = H.$$

Problem 31 *Let $p\colon (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be a covering space and assume that $N = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$.*

1. *For each $\tilde{x}_1 \in p^{-1}(x_0)$ show that there is a unique homeomorphism $\tilde{\phi}_{\tilde{x}_1}\colon \tilde{X} \to \tilde{X}$ with $\phi_{\tilde{x}_1}(\tilde{x}_0) = \tilde{x}_1$. (**Hint:** Use Proposition 5.8.)*

2. *Let $G = \{\phi_{\tilde{x}_1}\}$ where the $\tilde{x}_1$ range over all points in $p^{-1}(x_0)$. Show that G is a subgroup of* **Homeo**$(\tilde{X})$.

3. *Let $\gamma_0, \gamma_1\colon [0,1] \to \tilde{X}$ be paths with $\gamma_i(0) = \tilde{x}_0$ and $\gamma_i(1) = \tilde{x}_1$. Show that $[p \circ \gamma_0] = [f] \cdot [p \circ \gamma_1]$ with $[f] \in N$.*

We saw in the homework that $(\mathscr{W},\cdot)$ is a group. By Theorem 6.10 for each vertex w in $\tilde{X}$ there is a homeomorphism

$$\psi_w \colon \tilde{X} \to \tilde{X}$$

with $\psi_w(\emptyset) = w$. (This is not to be confused with the bijection ϕ_w of the set of reduced words that we defined in the homework.)

Lemma 6.11 *If v and w are reduced words and x is a single letter than*

$$\psi_v(w\cdot x) = \psi_v(w)\cdot x$$

and

$$\psi_{w\cdot x} = \psi_w \circ \psi_x.$$

Proof: The vertices w and $w\cdot x$ are the vertices of a edge e that is mapped by p to an edge $e' \in \mathscr{E}_X \cup \bar{\mathscr{E}}_X$. In particular if $x \in \{a,b\}$ $e' = e_x$ while if $x \in \{\bar{a},\bar{b}\}$ then $e' = \bar{e}_x$. If we replace w with $\psi_v(w)$ then the same claim holds. Since $p = p\circ\psi_v$ we must have that p maps e and $\psi_v(e)$ to the same edge so the vertices of $\psi_v(e)$ must be $\psi_v(w)$ and $\psi_v(w)\cdot x$. Therefore $\psi_v(w\cdot x) = \psi_v(w)\cdot x$.

For the second part we use the first part to see that $\psi_w(\emptyset\cdot x) = \psi_w(\emptyset)\cdot x = w\cdot x$. Therefore $\psi_w\circ\psi_x(\emptyset) = \psi_w(x) = w\cdot x$. We also have that $p\circ\psi_w\circ\psi_x = p\circ\psi_x = p$ and since $\psi_{w\cdot x}$ is the unique homeomorphism of $\tilde{X}$ with $\psi_{w\cdot x}(\emptyset) = w\cdot x$ and $p\circ\psi_{w\cdot x} = p$ we have that $\psi_{w\cdot x} = \psi_w\circ\psi_x$. □

Theorem 6.12 *The map $w \mapsto \psi_w$ is an isomorphism from the free group $F_2 = (\mathscr{W},\cdot)$ to the deck group for the action of $\pi_1(X,v)$ on $\tilde{X}$.*

Proof: The map is a bijection so we only need to check that it is a homomorphism. Let w_0 and w_1 be reduced words. We need to show that $\psi_{w_0\cdot w_1} = \psi_{w_0}\circ\psi_{w_1}$. We'll induct on the length of w_1. If w_1 has length zero (i.e. $w_1 = \emptyset$) then $\psi_\emptyset$ is the identity so $\psi_{w_0}\circ\psi_\emptyset = \psi_{w_0} = \psi_{w_0\cdot\emptyset}$.

Now assume that the statement holds for words of length n and assume that $w_1 = w_1'\cdot x$ where w_1' has length n and x is a single letter. Then by Lemma 6.11 we have $\psi_{w_1} = \psi_{w_1'}\circ\psi_x$ so

$$\psi_{w_0}\circ\psi_{w_1} = \psi_{w_0}\circ(\psi_{w_1'}\circ\psi_x) = \psi_{w_0\cdot w_1'}\circ\psi_x = \psi_{(w_0\cdot w_1')\cdot x} = \psi_{w_0\cdot(w_1'\cdot x)} = \psi_{w_0\cdot w_1}.$$

□

6.3 Attaching spaces

Let X and Y be topological spaces, $B\subset Y$ a subspace and $f\colon B\to X$ a continuous map. Then $X\sqcup_f Y$ is the quotient space obtained from the disjoint union of X and Y with the equivalence relation generated by $y\sim f(y)$ for $y\in B\subset Y$. A graph is an example of this construction where X is discrete set, Y is a disjoint union of closed intervals and B is the endpoints of the intervals.

A more general example is an *n-dimensional CW complex*. This is defined inductively. A 0-dimensional CW complex is just a discrete set. We then inductively define an n-dimensional CW complex to be $X^n = X^{n-1}\sqcup_f Y$ where X^{n-1} is an $(n-1)$-dimensional CW complex, Y is a disjoint union of n-balls and B is the boundary of the balls (a disjoint union of $(n-1)$-dimensional spheres). This is a very important constructions as CW complexes are natural spaces to do algebraic topology and many naturally appearing spaces can be given the structure of a CW-complex.

An n-dimensional CW complex is successively built from lower dimensional CW complexes $X^0\subset X^1\subset\cdots\subset X^n$. Then X^k is the *k-skeleton* of X^n.

Theorem 6.13 *Let X^n be a connected CW complex. Then the inclusion $X^1\hookrightarrow X^n$ is surjective on π_1.*

Proof: Let $v \in X^0 \subset X$ be a point in the 0-skeleton and $[f] \in \pi_1(X^n, v)$. We want to that f is path homotopic to a path in X^1. The proof is inductive: We assume that the image of f lies in X^k and then show that there is path homotopy into X^{k-1} is $k \geq 2$.

Recall that $X^k = X^{k-1} \sqcup Y$ where Y is a disjoint union of k-balls. Let Y' be obtained from Y by removing the center of each ball. We first claim that there is path homotopy of f into $X^{k-1} \sqcup_f Y'$. Let Z be obtained from Y by taking the sub-balls of radius 1/2 for each ball in Y. The open set Z embeds in $X^k = X^{k-1} \sqcup_f Y$ and $X^{k-1} \sqcup_f Y'$ and Z are an open cover of X^k. We can then find a partition $t_0 = 0 < t_1 < \cdots < t_m = 1$ so that $f([t_i, t_{i+1}])$ is contained in either $X^{k-1} \sqcup_f Y'$ or in Z and $f(t_i) \in (X^{k-1} \sqcup_f Y') \cap Z$ for $i = 1, \ldots, m-1$. We then perform a path homotopy on each interval contained in Z to an interval that misses the center of the balls. This is a path homotopy of f into $X^{k-1} \sqcup_f Y'$.

Finally we observe that Y' deformation retracts to $B \subset Y' \subset Y$ and that this determines a deformation retraction of $X^{k-1} \sqcup_f Y'$ to X^{k-1}. Composing this deformation retract with f (which now lies in $X^{k-1} \sqcup_f Y'$) we get a path homotopy of f into X^{k-1}. □

We would like to calculate π_1 for a CW complex X^n. However, we will only do this when $n = 2$ and X^0 is a single point. We make some preliminary comments about free groups and their normal subgroups.

Let $\mathscr{S}$ be a set and let $\bar{\mathscr{S}} = \{\bar{x} | x \in \mathscr{S}\}$. There is then an involution of the set $\mathscr{S} \cup \bar{\mathscr{S}}$ that takes each $x \in \mathscr{S}$ to $\bar{x} \in \bar{\mathscr{S}}$ (so $\bar{\bar{x}} = x$). As before we let $\tilde{\mathscr{W}}$ and $\mathscr{W}$ be words and reduced words in $\mathscr{S} \cup \bar{\mathscr{S}}$ and for reduced words w_0, w_1 we define $w_0 \cdot w_1$ as before. Then $F_{\mathscr{S}} = (\mathscr{W}, \cdot)$ is the free group on the set $\mathscr{S}$. We can the make a graph $X_{\mathscr{S}}$ with one vertex v and an edge set $\mathscr{E}_{\mathscr{S}}$ indexed by $\mathscr{S}$. The proof that $\pi_1(X_{\mathscr{S}}, v)$ is isomorphic to $F_{\mathscr{S}}$ is the same as when we had $\mathscr{S} = \{a, b\}$.

Lemma 6.14 *Let G be a group and $S \subset G$ a subset. Then there is a unique subgroup $H \subset G$ such that H contains S and for any subgroup $H' \subset G$ that contains S we have $H \subset H'$. Similarly there is a normal subgroup N that contains S and for any other normal subgroup N' that contains S we have $N \subset N'$.*

Proof: Note that the intersection of subgroups is a subgroup and the intersection of normal subgroups is a normal subgroup. We then let H be the intersection of all subgroups that contain S and N the intersection of all normal subgroups that contain S. Note that G is a subgroup and a normal subgroup that contains S (possibly the only one!) so there is at least on subgroup in each intersection. □

We will apply this lemma when G is a free group. Namely, let $\mathscr{R}$ a collection of reduced words and we let

$$G = \langle\ \mathscr{S} \mid \mathscr{R}\ \rangle$$

be the quotient of the free group $\mathscr{F}_{\mathscr{S}}$ by the normal subgroup generated by $\mathscr{R}$. For example the group $\mathbb{Z}^2$ can be written as

$$\mathbb{Z}^2 = \left\langle\ a, b \mid ab\bar{a}\bar{b}\ \right\rangle.$$

In fact any group G can be written in this way. We then let $\mathscr{S} = \mathscr{G}$ and $\mathscr{R}$ the set of finite reduced words that represent trivial elements in G.

We also use the set $\mathscr{S}$ to construct a graph $X_{\mathscr{S}}$. We let the vertex set of the graph be single vertex $\{v\}$ and index the edge set $\mathscr{E}_{\mathscr{S}}$ by $\mathscr{S}$. That is for each α in we let e_α be an edge in $\mathscr{E}_\alpha$. As there is only one vertex v the endpoints of every edge in $\mathscr{E}_{\mathscr{S}}$ will be identified with v.

Theorem 6.15

$$\pi_1(X_{\mathscr{S}}, v) = F_{\mathscr{S}}$$

The proof of this theorem is essentially the same as the proof that $\pi_1\left(X_{\{a,b\}}, v\right) = F_2$.

Now let $\mathscr{R}$ be a collection of reduced words in $\mathscr{S} \cup \bar{\mathscr{S}}$ and let $\mathscr{D}_{\mathscr{R}}$ be a disjoint union of disks indexed by $\mathscr{R}$. That is for every reduced word $w \in \mathscr{R}$ there is a disk D_w in $\mathscr{D}_{\mathscr{R}}$. We then form a 2-dimensional CW complex $X_{\langle \mathscr{S} | \mathscr{R} \rangle} = X_{\mathscr{S}} \sqcup_f \mathscr{D}_{\mathscr{R}}$ as follows: Given a reduce word w define a map

$$f_w \colon S^1 \to X_{\mathscr{S}}$$

by dividing S^1 into $|w|$ intervals and then f_w will map ith interval to the edge corresponding to the ith letter of w. Then the map f restricted to ∂D_w is f_w.

Theorem 6.16

$$\pi_1 \left(X_{\langle \mathscr{S}, \mathscr{R} \rangle}, v \right) \cong \langle\ \mathscr{S} \mid \mathscr{R}\ \rangle$$

Proof (sketch): We will construct a simply connected covering space of $X_{\langle \mathscr{S} | \mathscr{R} \rangle}$.

Let N be the normal subgroup of $F_{\mathscr{S}}$ generated by $\mathscr{R}$. By Problem 24 there is a covering space $p_N \colon X_N \to X_{\langle \mathscr{S} | \mathscr{R} \rangle}$ such that $\operatorname{im} p_*$ is N.