Homework 4

The goal of this project is to understand two parallel constructions. In the first we take a topological space X and a family of open, nested subspaces $X_1 \subset X_2 \subset \cdots$ whose union is all of X. In the second construction we take a group G and a family of nested subgroups $G_1 \subset G_2 \subset \cdots$ whose union is all of G. The connection between the two is the fundamental group. That is we fix some basepoint $x_0 \in X_1 \subset X_i \subset X$ then G will be $\pi_1(X, x_0)$ and the G_i will be $\pi_1(X_i, x_0)$.

In practice rather than start with an X and a G we will start with the X_i and the G_i and use them to construct the X and G. More explicitly we start with spaces X_i and continuous, injective, open maps $\phi_i \colon X_i \to X_{i+1}$ and use to construct X. For the groups will have the G_i and injective, homomorphisms $\psi_i \colon G_i \to G_{i+1}$ and from this will construct G. When we put the constructions together the G_i will be the fundamental group of X_i with $\psi_i = (\phi_i)_*$. Our goal will be to show that the fundamental group of the limiting space X is the limiting group G.

We observe that even when the maps ϕ_i are injective the induced maps on π_1 need not be. That will be an extra assumption in our construction. Strictly speaking it is not necessary. However, if we do not insist on this the group theory construction will need to be more complicated.

Topology

We now want to mimic the algebraic construction with topology.

Let $X_1, X_2, \ldots$ be topological spaces and

$$\phi_i \colon X_i \to X_{i+1}$$

injective, open maps. In particular $\phi_i(X_i)$ is open in X_{i+1}. We then define maps

$$\phi_{i,j} \colon X_i \to X_j$$

by

$$\phi_{i,j} = \phi_{j-1} \circ \cdots \circ \phi_i.$$

Let $\mathscr{X}$ be the disjoint union of the X_i. Just as we did with groups we can define a relation on $\mathscr{X}$ by setting $x_i \sim x_j$ with $x_i \in X_i$ and $x_j \in X_j$ if there exists a k with $\phi_{i,k}(x_i) = \phi_{j,k}(x_j)$. The proof that this is an equivalence relation is exactly the same as the proof for groups. We then let $X = \mathscr{X}/\sim$ be the quotient space and we give X the quotient topology with quotient maps $q \colon \mathscr{X} \to X$. Recall that for the quotient topology $U \subset X$ is open if and only if $q^{-1}(U)$ is open.

1. Show that $q(X_i)$ is open in X and that q restricted to $X_i \subset \mathscr{X}$ is homeomorphism to its image $q(X_i)$. You should use that the maps ϕ_i are open.

   Solution: We have that $q^{-1}(q(X_i))$ is a union of the following subsets of $\mathscr{X}$. It contains all of the X_j if $j \leq i$ and $\phi_{i,j}(X_i)$ if $j > i$. All of these sets are open since the individual X_j are open since the maps $\phi_{i,j}$ is open so

$\phi_{i,j}(X_i)$. Therefore $q^{-1}(q(X_i))$ is a union of opens sets and is open.

In fact the above argument implies that for any open $U \subset X_i$ we have that $q(U)$ is open. In fact, as every open set in $\mathscr{X}$ is a union of open sets in the individual X_i this implies that q is an open map.

Since the $\phi_{i,j}$ are injective if $x, x' \in X_i$ with $x \sim x'$ then $x = x'$. In particular q restricted to X_i is injective. Furthermore $q|_{X_i}$ is surjective to $q(X_i)$ by definition and therefore $q|_{X_i}$ is a continuous bijection to $q(X_i)$. We need to show that $(q|_{X_i})^{-1}$ is continuous. As q is an open map this follows from Lemma 3.4 in the notes.

Fix a point $x_0 \in X_0$ and let $G_i = \pi_1(X_i, \phi_{0,i}(x_0))$ and $\psi_i = (\phi_i)_*$. As before define $\psi_{i,j} = \psi_{j-1} \circ \cdots \circ \psi_i$. The G_i and $\psi_{i,j}$ are exactly as in our algebraic construction and determine a group $\vec{G}$.

2. Show that $\psi_{i,j} = (\phi_{i,j})_*$.

   Solution: This follows from the composition rule for the induced map. A formal proof follows by induction. Namely we have $\psi_{i,i+1} = (\phi_i)_* = (\phi_{i,i+1})_*$ by definition. Now assume that $\psi_{i,j-1} = (\phi_{i,j-1})_*$. Then

   $$\phi_{i,j} = \phi_{j-1} \circ (\phi_{j-2} \circ \cdots \phi_i) = \phi_{j,j-1} \circ \phi_{i,j-1}.$$

   Then by Lemma 4.9 we have

   $$\begin{aligned} \psi_{i,j} &= \psi_{j-1} \circ \cdots \circ \psi_i \\ &= \psi_{j-1} \circ \psi_{i,j-1} \\ &= (\phi_{j-1})_* \circ (\phi_{i,j-1})_* \\ &= (\phi_{i,j})_* \end{aligned}$$

   completing the induction step.

3. Show that $\pi_1(X, q(x_0)) \cong \vec{G}$. To simplify you can assume that the ψ_i are injective. You should use Problem 7 from the algebra worksheet where $G = \pi_1(X, q(x_0))$. Then you need to find injective homomorphisms $\psi^i \colon G_i \to \pi_1(X, q(x_0))$ with $\psi^i = \psi^j \circ \psi_{i,j}$ and then check that for every $[f] \in \pi_1(X, q(x_0))$ there is an i and $g_i \in G_i$ with $\psi^i(g_i) = [f]$. For this last statement you should use compactness to show that the image of f lies in $q(X_i)$ for some i.

   Solution: Let $\psi^i = (q|_{X_i})_*$. Then for $[f] \in \pi_1(X_i, \phi_{0,i}(x_0))$ we have

   $$\psi^i([f]) = [(q|_{X_i}) \circ f] = [q \circ f].$$

   On the other hand we have

   $$\psi^j \circ \psi_{i,j}([f]) = \psi^j \circ (\phi_{i,j})_*([f]) = \psi^j([\phi_{i,j} \circ f]) = [q \circ \phi_{i,j} \circ f] = [q \circ f].$$

Therefore $\psi^i = \psi^j \circ \psi_{i,j}$.

To apply problem 7 from the algebra worksheet we need to show that every $[f] \in \pi_1(X, q(x_0))$ is in the image of ψ^i for some i. For this let $U_i = f^{-1}(q(X_i))$. Since, by (1), $q(X_i)$ is open we have that the U_i are an open cover of the compact set $[0,1]$ and hence have a finite subcover $U_{i_1}, \dots, U_{i_k}$. We can assume that $i_k > i_j$ for $j = 1, \dots, k-1$. In particular $U_{i_k} \supset U_{i_j}$ so the image of f is contained in U_{i_k}. By (1) we also know that $q|_{X_{i_k}}$ is a homeomorphism to $q(X_{i_k})$ so we can take the inverse. Then $[(q|_{X_{i_k}})^{-1} \circ f] \in \pi_1(X_{i_k}, \phi_{0,i_k}(x_0))$ and

$$\psi^{i_k}([(q|_{X_{i_k}})^{-1} \circ f]) = [q \circ (q|_{X_{i_k}})^{-1} \circ f] = [f]$$

so $[f]$ is in the image of ψ^{i_k} and by problem 7 in the algebra worksheet $\pi_1(X, q(x_0))$ is isomorphic to $\vec{G}$.

Now we will use this to construct a space X with $\pi_1(X, x_0) \cong \mathbb{Q}$. We know how to do this on the group theory level from the algebra worksheet. We want the G_i to be isomorphic to $\mathbb{Z}$ and the maps ψ_i to be multiplication by $i+1$. Our goal will be to construct X_i and and maps $\phi_i \colon X_i \to X_{i+1}$ such that $\pi_1(X_i) = \mathbb{Z}$ and $(\phi_i)_*$ is multiplication by $i+1$.

4. Let
$$B^2 = \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\right\}$$
be the open disk and let
$$W = S^1 \times B^2.$$
Then W is a *solid torus*. Let $\pi \colon W \to S^1$ be projection to the first coordinate. Show that π_* is an isomorphism so that $\pi_1(W, w_0)$ is isomorphic to $\mathbb{Z}$.

Solution: We know from the last homework that $\pi_1(W)$ is the product of $\pi_1(S^1)$ and $\pi_1(B^2)$. Since the later group is trivial this shows that $\pi_1(W)$ is isomorphic to $\mathbb{Z}$. However, what we showed was even stronger. Namely, if $\sigma \colon W \to B^2$ is the projection to the second coordinate then the map

$$\Phi \colon \pi_1(W) \to \pi_1(S^1) \times \pi_1(B^2)$$

given by $\Phi([f]) = (\pi_*([f]), \sigma_*([f]))$ is an isomorphism. Again using that the second group is trivial we have that π_* is an isomorphism.

5. Viewing S^1 as the subset of points in the complex plane of norm 1 define maps $f_n \colon S^1 \to S^1$ by $f_n(z) = z^n$. Show that $(f_n)_* \colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is multiplication by n.

Solution: Recall the following maps: We have $g_n \colon [0,1] \to S^1$ given by $g_n(t) = (\cos 2\pi nt, \sin 2\pi nt)$. In Theorem 4.5 (where we called these maps

f_n) we show that $\pi_1(S^1) = \cup[g_n]$ and that the maps $[g_n] \mapsto n$ is an isomorphism form $\pi_1(S^1)$ to $\mathbb{Z}$. We also observe that $(f_n)_*([g_k]) = [f_n \circ g_k] = [g_{n\cdot k}]$ and therefore $(f_n)_*$ is multiplication by n.

6. Show that for any integer $n \in \mathbb{Z}$ there is a injective, continuous map

$$\phi_n \colon W \to W$$

such that the homomorphism $(\phi_n)_*$ is multiplication by n.

Here is how to do this. The maps f_n aren't injective unless $n = \pm 1$ and are not even homotopic to injective maps. However using the extra dimensions we can find injective, continuous maps $\phi_n \colon W \to W$ with $\pi \circ \phi_n = f_n \circ \pi$. Then the ϕ_n have the necessary property. As a starting point define functions

$$\eta_n \colon S^1 \to W$$

by $\eta_n(z) = (z^n, z/n)$. This function is injective and satisfies $\pi \circ \eta_n = f_n \circ \pi$. (Why?) You can think of η_n as a map with domain $S^1 \times \{0\} \subset W$ and you need to extend this to an injective function on W.

Solution: Define ϕ_n by

$$\phi_n(z,w) = (z^n, z/2 + c_n \cdot w)$$

where $c_n = \left|e^{2\pi i/n} - 1\right|/10$. This map is continuous. We need to show that it is injective. Assume that $\phi_n(z_0,w_0) = \phi_n(z_1,w_1)$. We first show that $z_0 = z_n$. For this we notice that we have $z_0^n = z_1^n$ so $z_1 = e^{2\pi ik/n} z_0$ for some $k \in \{0,\dots,n-1\}$ and if $|z_0 - z_1| < \left|e^{2\pi i/n} - 1\right|$ then $k = 0$ (and $z_0 = z_1$).

For this we have

$$z_0/2 + c_n \cdot w_0 = z_1/2 + c_n \cdot w_1 \implies |z_0 - z_1| = 2c_n|w_0 - w_1|.$$

Noting that $|w_0 - w_1| \le 2$ we get

$$|z_0 - z_1| \le 4c_n < \frac{2}{5}\left|e^{2\pi i/n} - 1\right| < \left|e^{2\pi i/n} - 1\right|.$$

Therefore $z_0 = z_1$. However, if $z_0 = z_1$ then the equalitiy

$$z_0/2 + c_n \cdot w_0 = z_1/2 + c_n \cdot w_1$$

implies that $w_0 = w_1$. Therefore ϕ_n is injective.

7. Use the previous problem and problem 8 from the algebra worksheet to construct a space X with $\pi_1(X, q(x_0)) \cong \mathbb{Q}$.

Solution: We let each $X_i = W$. Then by Problem 6 we can find injective maps $\phi_i \colon X_i \to X_{i+1}$ such that $\psi_i = (\phi_i)_*$ is multiplication by $i+1$.

We then let $X = \mathscr{X} / \sim$ be the quotient space as above. If we let $G_i = \pi_1(X_i, \phi_{0,i}(x_0)) \cong \mathbb{Z}$ (for some choice of $x_0 \in X_0$) then the G_i and homomorphisms $\psi + i$ determine a group $\vec{G}$. By (8) of the algebra worksheet we have that $\vec{G} \cong \mathbb{Q}$. By problem 3 we have that $\pi_1(X, q(x_0)) \cong \vec{G} \cong \mathbb{Q}$.