Homework 4

The goal of this project is to understand two parallel constructions. In the first we take a topological space X and a family of open, nested subspaces $X_1 \subset X_2 \subset \cdots$ whose union is all of X. In the second construction we take a group G and a family of nested subgroups $G_1 \subset G_2 \subset \cdots$ whose union is all of G. The connection between the two is the fundamental group. That is we fix some basepoint $x_0 \in X_1 \subset X_i \subset X$ then G will be $\pi_1(X, x_0)$ and the G_i will be $\pi_1(X_i, x_0)$.

In practice rather than start with an X and a G we will start with the X_i and the G_i and use them to construct the X and G. More explicitly we start with spaces X_i and continuous, injective, open maps $\phi_i \colon X_i \to X_{i+1}$ and use to construct X. For the groups will have the G_i and injective, homomorphisms $\psi_i \colon G_i \to G_{i+1}$ and from this will construct G. When we put the constructions together the G_i will be the fundamental group of X_i with $\psi_i = (\phi_i)_*$. Our goal will be to show that the fundamental group of the limiting space X is the limiting group G.

We observe that even when the maps ϕ_i are injective the induced maps on π_1 need not be. That will be an extra assumption in our construction. Strictly speaking it is not necessary. However, if we do not insist on this the group theory construction will need to be more complicated.

Topology

We now want to mimic the algebraic construction with topology.

Let $X_1, X_2, \ldots$ be topological spaces and

$$\phi_i \colon X_i \to X_{i+1}$$

injective, open maps. In particular $\phi_i(X_i)$ is open in X_{i+1}. We then define maps

$$\phi_{i,j} \colon X_i \to X_j$$

by

$$\phi_{i,j} = \phi_{j-1} \circ \cdots \circ \phi_i.$$

Let $\mathscr{X}$ be the disjoint union of the X_i. Just as we did with groups we can define a relation on $\mathscr{X}$ by setting $x_i \sim x_j$ with $x_i \in X_i$ and $x_j \in X_j$ if there exists a k with $\phi_{i,k}(x_i) = \phi_{j,k}(x_j)$. The proof that this is an equivalence relation is exactly the same as the proof for groups. We then let $X = \mathscr{X}/\sim$ be the quotient space and we give X the quotient topology with quotient maps $q \colon \mathscr{X} \to X$. Recall that for the quotient topology $U \subset X$ is open if and only if $q^{-1}(U)$ is open.

1. Show that $q(X_i)$ is open in X and that q restricted to $X_i \subset \mathscr{X}$ is homeomorphism to its image $q(X_i)$. You should use that the maps ϕ_i are open.

Fix a point $x_0 \in X_0$ and let $G_i = \pi_1(X_i, \phi_{0,i}(x_0))$ and $\psi_i = (\phi_i)_*$. As before define $\psi_{i,j} = \psi_{j-1} \circ \cdots \circ \psi_i$. The G_i and $\psi_{i,j}$ are exactly as in our algebraic construction and determine a group $\vec{G}$.

2. Show that $\psi_{i,j} = (\phi_{i,j})_*$.

3. Show that $\pi_1(X, q(x_0)) \cong \vec{G}$. To simplify you can assume that the ψ_i are injective. You should use Problem 7 from the algebra worksheet where $G = \pi_1(X, q(x_0))$. Then you need to find injective homomorphisms $\psi^i \colon G_i \to \pi_1(X, q(x_0))$ with $\psi^i = \psi^j \circ \psi_{i,j}$ and then check that for every $[f] \in \pi_1(X, q(x_0))$ there is an i and $g_i \in G_i$ with $\psi^i(g_i) = [f]$. For this last statement you should use compactness to show that the image of f lies in $q(X_i)$ for some i.

Now we will use this to construct a space X with $\pi_1(X, x_0) \cong \mathbb{Q}$. We know how to do this on the group theory level from the algebra worksheet. We want the G_i to be isomorphic to $\mathbb{Z}$ and the maps ψ_i to be multiplication by $i+1$. Our goal will be to construct X_i and and maps $\phi_i \colon X_i \to X_{i+1}$ such that $\pi_1(X_i) = \mathbb{Z}$ and $(\phi_i)_*$ is multiplication by $i+1$.

4. Let
$$B^2 = \left\{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\right\}$$
be the open disk and let
$$W = S^1 \times B^2.$$
Then W is a *solid torus*. Let $\pi \colon W \to S^1$ be projection to the first coordinate. Show that π_* is an isomorphism so that $\pi_1(W, w_0)$ is isomorphic to $\mathbb{Z}$.

5. Viewing S^1 as the subset of points in the complex plane of norm 1 define maps $f_n \colon S^1 \to S^1$ by $f_n(z) = z^n$. Show that $(f_n)_* \colon \pi_1(S^1, 1) \to \pi_1(S^1, 1)$ is multiplication by n.

6. Show that for any integer $n \in \mathbb{Z}$ there is a injective, continuous map
$$\phi_n \colon W \to W$$
such that the homomorphism $(\phi_n)_*$ is multiplication by n.

   Here is how to do this. The maps f_n aren't injective unless $n = \pm 1$ and are not even homotopic to injective maps. However using the extra dimensions we can find injective, continuous maps $\phi_n \colon W \to W$ with $\pi \circ \phi_n = f_n \circ \pi$. Then the ϕ_n have the necessary property. As a starting point define functions
$$\eta_n \colon S^1 \to W$$
by $\eta_n(z) = (z^n, z/n)$. This function is injective and satisfies $\pi \circ \eta_n = f_n \circ \pi$. (Why?) You can think of η_n as a map with domain $S^1 \times \{0\} \subset W$ and you need to extend this to an injective function on W.

7. Use the previous problem and problem 8 from the algebra worksheet to construct a space X with $\pi_1(X, q(x_0)) \cong \mathbb{Q}$.