Homework 4

The goal of this project is to understand two parallel constructions. In the first we take a topological space X and a family of open, nested subspaces $X_1 \subset X_2 \subset \cdots$ whose union is all of X. In the second construction we take a group G and a family of nested subgroups $G_1 \subset G_2 \subset \cdots$ whose union is all of G. The connection between the two is the fundamental group. That is we fix some basepoint $x_0 \in X_1 \subset X_i \subset X$ then G will be $\pi_1(X, x_0)$ and the G_i will be $\pi_1(X_i, x_0)$.

In practice rather than start with an X and a G we will start with the X_i and the G_i and use them to construct the X and G. More explicitly we start with spaces X_i and continuous, injective, open maps $\phi_i \colon X_i \to X_{i+1}$ and use to construct X. For the groups will have the G_i and injective, homomorphisms $\psi_i \colon G_i \to G_{i+1}$ and from this will construct G. When we put the constructions together the G_i will be the fundamental group of X_i with $\psi_i = (\phi_i)_*$. Our goal will be to show that the fundamental group of the limiting space X is the limiting group G.

We observe that even when the maps ϕ_i are injective the induced maps on π_1 need not be. That will be an extra assumption in our construction. Strictly speaking it is not necessary. However, if we do not insist on this the group theory construction will need to be more complicated.

We begin with the group theory.

Groups

Let $G_0, G_1, \ldots$ be a family of groups and let

$$\psi_i \colon G_i \to G_{i+1}$$

be homomorphisms. We can then define for each $i < j$ a homomorphism

$$\psi_{i,j} \colon G_i \to G_j$$

with

$$\psi_{i,j} = \psi_{j-1} \circ \cdots \psi_{i+1} \circ \psi_i.$$

Note that if follows that if $i < j < k$ then

$$\psi_{i,k} = \psi_{j,k} \circ \psi_{i,j}.$$

Let $\mathscr{G}$ be the disjoint union of the G_i. Define a relation on $\mathscr{G}$ as follows: If $g_i \in G_i$ and $g_j \in G_j$ then $g_i \sim g_j$ if there exists a k with $\psi_{i,k}(g_i) = \psi_{j,k}(g_j)$.

1. Show that $\sim$ is an equivalence relation.

For $g \in \mathscr{G}$ denote the equivalence class by $[g]$.

Define an operation on equivalence classes as follows: If $g_i \in G_i$ and $g_j \in G_j$ choose a k larger than i and j and set

$$[g_i] \cdot [g_j] = [\psi_{i,k}(g_i) \cdot \psi_{j,k}(g_j)].$$

2. Show that this is a well defined operation on the set of equivalence classes.

Let $\vec{G}$ be the set of equivalence classes.

3. Show that $\vec{G}$ with the given operation is a group.

4. Define maps
$$\psi_{i,\infty} \colon G_i \to \vec{G}$$
by $\psi_{i,\infty}(g) = [g]$. Show that the $\psi_{i,\infty}$ are homomorphisms and that $\psi_{i,\infty} = \psi_{j,\infty} \circ \psi_{i,j}$.

5. If all of the $\psi_{i,j}$ are the trivial homomorphism, show that $\vec{G}$ is the trivial group.

6. If all of the $\psi_{i,j}$ are isomorphisms, show that the G_i are isomorphic to $\vec{G}$.

7. For this problem you can assume that the ψ_i are injective homomorphisms. Assume that G is another group and
$$\psi^i \colon G_i \to G$$
are injective homomorphisms with $\psi^i = \psi^j \circ \psi_{i,j}$ when $i < j$. Also assume that for every $g \in G$ there is an i and a $g_i \in G_i$ with $\psi^i(g_i) = g$. Show that there exists an isomorphism
$$\psi \colon \vec{G} \to G$$
with $\psi^i = \psi \circ \psi_{i,\infty}$.

8. Assume that all of $G_i \cong \mathbb{Z}$. Find homomorphisms
$$\psi_{i,j} \colon G_i \to G_j$$
such that $\vec{G} \cong \mathbb{Q}$.