Free products

If G_0 and G_1 are groups we want to define a new group $G_0 * G_1$, the *free product* of G_0 and G_1. We will do it in the special case when $G_0 = G_1 = \mathbb{Z}$.

Let $\mathscr{A} = \{a, b, \bar{a}, \bar{b}\}$. This is an *alphabet*. We let $\tilde{\mathscr{W}}$ be *words* in $\mathscr{A}$. That is elements of $\tilde{\mathscr{W}}$ are finite sequences $x_1 x_2 \cdots x_n$ with the $x_i \in \mathscr{A}$. We view $x \mapsto \bar{x}$ has an involution of $\mathscr{A}$. That is if $x = a$ or b then $\bar{x} = \bar{a}$ or $\bar{b}$ while if $x = \bar{a}$ or $\bar{b}$ then $\bar{x} = a$ or b. Then a word $x_1 x_2 \cdots x_n$ is *reduced* if $x_i \neq \bar{x}_{i+1}$ for $i = 1, \ldots, n-1$. We let $\mathscr{W}$ be the set of reduced words. Note that we consider the empty word, which we denote $\emptyset$, as an element of both $\tilde{\mathscr{W}}$ and $\mathscr{W}$.

We want to give $\mathscr{W}$ a group structure. We first define a binary operation $*$ on $\tilde{\mathscr{W}}$ by setting

$$w_0 * w_1 = x_1 x_2 \cdots x_n y_1 y_2 \cdots y_m$$

where $w_0 = x_1 x_2 \cdots x_n$ and $w_1 = y_1 y_2 \cdots y_m$. Note that the empty word is an identity for this operation and it is also associative but there are no inverses so $(\tilde{\mathscr{W}}, *)$ is not a group.

The first step in defining a group structure is to define an equivalence relation on $\tilde{\mathscr{W}}$ by setting

$$x_1 x_2 \cdots x \bar{x} \cdots x_n \sim x_1 x_2 \cdots x_n.$$

This defines a relation on $\tilde{\mathscr{W}}$ but as we proved in homework this determines a unique relation (which we also denote $\sim$).

Problem 1 *Show that each equivalence class contains a reduced word.*

Given $x \in \mathscr{A}$ define a map $\phi_x \colon \tilde{\mathscr{W}} \to \tilde{\mathscr{W}}$ by

$$\phi_x(x_1 x_2 \cdots x_n) = \begin{cases} x x_1 x_2 \cdots x_n & \text{if } \bar{x} \neq x_1 \\ x_2 x_3 \cdots x_n & \text{if } \bar{x} = x_1 \end{cases}$$

Problem 2 *Show that ϕ_x restricted to $\mathscr{W}$ is a bijection from $\mathscr{W}$ to itself.*

If $w = x_1 x_2 \cdots x_n$ is a word in $\mathscr{W}$ define a bijection $\phi_w \colon \mathscr{W} \to \mathscr{W}$ by

$$\phi_w = \phi_{x_1} \circ \phi_{x_2} \circ \cdots \phi_{x_n}.$$

For the special case when $w = \emptyset$ is the empty word we define $\phi_\emptyset$ to be the identity. We emphasize that when w is a single letter the bijection ϕ_w is defined by a concatenation/reduction operation but for general words the ϕ_w is a composition of these bijections. We will ultimately show that ϕ_w is a concatenation operation but this is something that we need to prove.

Problem 3 *Show that $\phi_{w_0} = \phi_{w_1}$ if $w_0 \sim w_1$.*

For a word w (as above) let $\bar{w} = \bar{x}_n \bar{x}_{n-1} \cdots \bar{x}_1$.

Problem 4 *If w is a reduced word show that $\phi_w(\emptyset) = w$ and $\phi_{\bar{w}}(w) = \emptyset$.*

Problem 5 *Use the previous problems to show that each equivalence class contains a unique reduced word.*

Problem 6 *If w_0 and w_1 are words show that $\phi_{w_0} \circ \phi_{w_1} = \phi_{w_0 * w_1}$.*

The set of bijections of a set is a group, with group law composition. We let $F_2 \subset \mathrm{Biject}(W)$ be the image of $\tilde{\mathscr{W}}$ under the map $w \mapsto \mathrm{Biject}(W)$.

Problem 7 *Show that* F_2 *is a subgroup of* $\text{Biject}(W)$ *and the map* $w \mapsto \phi_w$ *is a bijection from* W *to* F_2.

For reduced words $w_0 = x_1 x_2 \cdots x_n$ and $w_1 = y_1 y_2 \cdots y_m$ define

$$w_0 \cdot w_1 = x_0 \cdots x_{n-k} y_{k+1} \cdots y_m$$

where $x_{n-i} = \bar{y}_{i+1}$ for $i \leq k$ and $x_{n-k} \neq \bar{y}_{k+1}$.

Problem 8 *If* w_0 *and* w_1 *are reduced words show that* $\phi_{w_0} \circ \phi_{w_1}(\emptyset) = w_0 \cdot w_1$.

Problem 9 *Use what we have done above to show that* $(\mathscr{W}, \cdot)$ *is a group.*

We usually refer to $(\mathscr{W}, \cdot)$ as F_2, the *free group on 2 generators.*

We remark that $w_0 \cdot w_1$ could be taken as the definition of the binary operation on the set of reduced words $\mathscr{W}$. With this definition it is not hard to check that there is an identity and inverses. However, if we take this as the definition that it is difficult to check that the operation is associative.