Given \( f : ([0,1], \{0,1\}) \to (X, \{x_0\}) \)

we let \([f]\) be the equivalence class of maps homotopic to \(f\) as pairs.

As a set \(\Pi_1(X, x_0)\) is equivalence classes \([f]\).

We’ve defined \(f \ast g\).

We need to show that

\([f] \ast [g] = [f \ast g] \) is well defined.

That is we need to show

if \( f_0 \sim_p f_1 \) \& \( g_0 \sim_p g_1 \)

then \([f_0 \ast g_0] = [f_1 \ast g_1]\).
Let $F, G$ be the two homotopies (between $f_0 \times f_1$ and $g_0 \times g_1$).

Define

$$H(s, t) = \begin{cases} 
F(2s, t) & 0 \leq s \leq \frac{1}{2} \\
G(2s-1, t) & \frac{1}{2} \leq s \leq 1
\end{cases}$$

$H$ is a homotopy between $f_0 \times g_0$ and $f_1 \times g_1$.

$$\Rightarrow [f_0 \times g_0] = [f_1 \times g_1].$$

**Closure**
Identity

Need to find a map

\[ e_{x_0} : ([0,1], \{0,1\}) \rightarrow (X, f(x_0)) \]

such that

\[ f \circ e_{x_0} = e_{x_0} \circ f \].

The constant map!

\[ e_{x_0} (\varepsilon) = x_0. \]

Again, \( f \circ e_{x_0} \neq f \) but only

\[ f \circ e_{x_0} \sim_p f. \]

Again, need to reparameterize:
\[ G(s, t) = \begin{cases} \frac{s}{1+t} & \text{if } s \leq \frac{1}{4} + \frac{\epsilon}{2} \\ \frac{1}{2} & s > \frac{1}{4} + \frac{\epsilon}{2} \end{cases} \]

\[ S = (1 - t) \frac{1}{2} + t = \frac{1}{2} + \frac{\epsilon}{2} \]

\[ h_0 = f \ast e_\chi \]

\[ H(s, t) = (f \ast e_\chi) \circ G(s, t) \implies h_1 = t \]

Also need to check that \( H(0, \epsilon) \)
\[ = 1 + (2, \epsilon) = \chi_0 \]
We'll "wind back" $f$. 

**Inverse**

Given $f : (0,13, x_0 , y_0) \rightarrow (X, \epsilon X)$

need to find $\bar{f}$ s.t.

$f \ast \bar{f} \sim \bar{f} \ast f \sim \epsilon_{X_0}$.

**Define:** $\bar{f}(\epsilon) = f(1-\epsilon)$
\[ f \ast g_x(x) = \begin{cases} 
 f(2x) & t \leq \frac{1}{2} \\
 \bar{f}(2x-1) & t > \frac{1}{2}
\end{cases} \]

\[ f_t(s) = \begin{cases} 
 f(2t) & s \leq \frac{1-\epsilon}{2} \\
 f(\frac{1-t}{2}) & \frac{1-\epsilon}{2} < s \leq \frac{\epsilon+1}{2} \\
 \bar{f}(2s-1) & \frac{\epsilon+1}{2} < s 
\end{cases} \]

**Pause on purple**

**Still need to prove Assoc!**

\[
(f \ast g) \ast h \preceq_p f \ast (g \ast h) \\
\frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4} \quad \frac{1}{4}
\]
The fundamental group of $S^1$ We have done all the work to calculate $\pi_1(S^1, \ast)$.

An element $[f]$ of $\pi_1(S^1, \ast)$ is an equivalence class of maps $f: ([0,1], \{0,1\}) \rightarrow (S^1, \ast)

We have seen that $[f] = [g]$ for a unique $n \in \mathbb{Z}$. This defines a map

$\phi : \pi_1(S^1, \ast) \rightarrow \mathbb{Z}$

by $\phi([f]) = n$

where $n$ is the unique integer such that $[f] = [g]$. Clearly $\phi([f_n]) = n$ so this map is surjective. By uniqueness it is also injective $\Rightarrow$ the map is a bijection.

We have also seen that $[f_1 \ast f_m] = [f_{n+m}]$ so

$\phi((f_1 \ast f_m)) = \phi([f_{n+m}]) = n+m = \phi(f_1) + \phi(f_m)$

Therefore $\phi$ is a homomorphism. A homomorphism that is a bijection is an isomorphism.

What is the lift of $f_n \ast f_m$?

\[ f_n \ast f_m \sim (12, 0) \]

\[ (\mathbb{C}, 12, 0) \xrightarrow{f_n \ast f_m} (S^1, \ast) \]

\[ \text{What is } \tilde{f}_n(2) = ? \]

\[ \tilde{f}_n(0) = 0 \]

\[ \text{Maybe } f_n \ast f_m = \tilde{f}_n \ast \tilde{f}_m \]

\[ f_n \ast f_m = \begin{cases} \tilde{f}_n(2t) & t \leq \frac{1}{2} \\ \tilde{f}_n(2t-1) + n & t > \frac{1}{2} \end{cases} \]
\[ f_n \ast f_m \text{ is continuous since } \]
\[ f_n(2(\chi_0)) = f(2(\chi_0-1) + n = n \]
\[ f_n \ast f_m \text{ is a lift since } \]
\[ f_n \ast f_m = \Xi \circ f_n \ast f_m \]
\[ f_n \ast f_m (x) = 0 . \]

What is \[ f_n \ast f_m (2) = f_n(01+n \]
\[ = n+n \]
\[ \Rightarrow f_n \ast f_m \sim f_n \oplus f_m \]