Let \( f: (\mathbb{Z}, \mathbb{Z}) \rightarrow (\mathbb{R}, \mathbb{R}) \). 

\[ f \circ \pi \text{ is a lift of } f \]

\[ (x, y) \quad \quad (x, y) \quad \quad (x', y') \]

\[ (x, y) \rightarrow (x, y) \rightarrow (x', y') \]
**Homotopy Lifting Lemma**

Let \( F : (\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_0) \to (S^1, \mathbb{L}_0) \).

\[ \exists F : (\mathbb{L}_0, \mathbb{L}_1, \mathbb{L}_0) \to (\mathbb{R}, \mathbb{L}_0) \]

with \( F = \pi \circ F \).

Recall maps \( \tilde{f}_n : (\mathbb{L}_0, \mathbb{L}_1, \mathbb{R}) \to (\mathbb{R}, \mathbb{Z}) \)

and \( f_n : (\mathbb{L}_0, \mathbb{L}_1, \mathbb{R}) \to (S^1, \mathbb{L}_0) \)

with \( \tilde{f}_n(\xi) = nt \)

and \( f_n = \pi \circ \tilde{f}_n \).

**Thm**

Given \( f : (\mathbb{L}_0, \mathbb{L}_1, \mathbb{R}) \to (S^1, \mathbb{L}_0) \)

there exists a unique \( nt \in \mathbb{Z} \) s.t.

\[ f \sim_p f_n. \]
We have already seen that there exists an \( n \in \mathbb{N} \) such that \( f \sim_p f_n \).

**Quick Review** By the L.H. I a lift

\[ \tilde{f} : ([0,1], \{0\}) \to (\mathbb{R}, \{0\}) \]

As \( f(\iota) = \pi \circ \tilde{f}(\iota) = 0 \Leftrightarrow \tilde{f}(\iota) \in \mathbb{N} \)

and \( \tilde{f}(\iota) = n \) for some \( n \in \mathbb{N} \).

Then we claim \( f \sim_p f_n \).

Define \( \tilde{G} : [0,1] \times [0,1] \to \mathbb{R} \) as a straight line homotopy by

\[ \tilde{G}(s, \iota) = (1-\iota)\tilde{f}(s) + \iota \tilde{f}_n(s) \]

and let \( G(s, \iota) = \pi \circ \tilde{G}(s, \iota) \).
Note \[ \tilde{g}'(x) = (1-t) \tilde{g}'(x) + t \tilde{f}'(x) = 0 \text{ mod } \mathcal{F} \]

\[ \mathcal{G}(903 \times 100) \subset \mathcal{G}(103 \times 100) \subset \mathbb{C} \]

\[ \mathcal{G}(10,13 \times 100) \cong [\mathbb{C}] \]

\[ \Rightarrow f \sim_p f_1. \]
Now we show that if \( f \sim_p f_m \) then \( n = k \).

Let \( G : \{0,1\} \times \{0,1\} \to S^1 \) be the homotopy realizing \( f \sim_p f_m \).

Then \( G([0,1] \times \{0,1\}) = \{0\} \).

In particular \( G(0,0) = \{0\} \).

By the H.L.L. there exists a lift

\[ \tilde{G} : ([0,1] \times \{0,1\}, \{0,0\}) \to (\mathbb{R}, \{0\}) \]

\[ \tilde{G}_0(s) = \tilde{G}(s,0) \]

Note that \( \tilde{G}_1 \) is a lift of \( f_m \), \( G_0 \equiv \tilde{G}_1 \).

By the uniqueness of lifts \( \tilde{G}_1 = \tilde{f}_m \) \( \tilde{G}_1(0) = 0 \).

Since \( \tilde{f}_m \) is also a lift of \( f_m \).
\[ H \subseteq \{(f, x) \} \]

\[ \pi \circ \tilde{g}(t) = \tilde{g}(t) = \tilde{x} \]

\[ C : \Box \rightarrow \bigcirc \]

\[ \Rightarrow \quad \tilde{g}(t) \circ \tilde{x} = \tilde{g}(t) \circ \tilde{x} = \\text{discrete} \]

\[ \Rightarrow \quad \tilde{g}(t) : \tilde{x} \circ \tilde{x} = 0 \quad \Rightarrow \quad \tilde{g}(t) = 0 \]
Given \( f : (\mathbb{R}, \mathbb{R}) \rightarrow (\mathbb{R}, \mathbb{Z}) \),

there exists a unique \( n \in \mathbb{Z} \) such that \( f \sim_p f_n \).
We need to prove the 2 lifting lemmas. First we need:

**Lemma** Every \( x \in S' \) has a nbd \( U \) such that if \( CIR \) is a connected component of \( \pi^{-1}(U) \) then \( \pi/\alpha \) is a homeomorphism to \( U \).

\( U \) is evenly covered.
• If \( t^n, t' \in \mathbb{R} \) and \( \pi(t) = \pi(t') \), then
  \[ t = t' + nh, \quad n \in \mathbb{Z} \setminus \{0\} \]
  \[ \Rightarrow |t-t'| > 1. \]

• The closure of any bounded set in \( \mathbb{IR} \) is compact.

• If \( K \subset \mathbb{IR} \) is compact and \( \text{diam } K < 1 \) then \( \pi|_K \) is a homeomorphism onto its image.

• If \( A \subset \mathbb{IR} \) then \( \pi|_A \) is a homeomorphism onto its image.

• If \( U \) is an open interval in \( \mathbb{IR} \) of width \( < 1 \) then \( \pi|_U \) is a homeomorphism onto its image.
Let $\mathbf{U} = \pi(\mathbf{U})$. Note every $x \in \mathbf{S}'$ is contained in such a $\mathbf{U}$.

What is $\pi^{-1}(\mathbf{U})$? $\pi^{-1}(\mathbf{U}) = \{\mathbf{U}_n\}$

Let $\mathbf{U}_n$ be the translate of $\mathbf{U}$ by $n$.

Then $\pi(\mathbf{U}_n) = \mathbf{U}$ &

$\pi \mid \mathbf{U}_n$ will be a homeomorphism to $\mathbf{U}$. 
LIFTING LEMMA

Let \( f : (\mathbb{R}, \mathbb{R}) \to (\mathbb{R}, \mathbb{R}) \).

\[ \exists \tilde{f} : (\mathbb{R}, \mathbb{R}) \to (\mathbb{R}, \mathbb{R}) \]

\[ \text{with} \quad f = \pi \circ \tilde{f} \]

**Proof**

We need to find a partition

\[ t_0 = 0 < t_1 < \ldots < t_n = 1 \]

of \( [0,1] \) such that for each interval \( [t_i, t_{i+1}] \) there is an evenly covered nbd. \( U_i \subset s' \) with \( f([t_i, t_{i+1}]) \subset U_i \).
Now assume $\tilde{f}$ is defined on $[0, t_i]$. As $f(t_i) \subset U_i$, we have $\tilde{f}(t_i) \in \pi^{-1}(U_i)$. If $i = 0$, we note $\tilde{f}(0) = 0$. Let $\tilde{U}_i$ be the component of $\pi^{-1}(U_i)$ that contains $\tilde{f}(t_i)$. $\pi_i^{-1}$ is the inverse of $\pi_i | U_i$. Extend $\tilde{f}$ to $[t_i, t_{i+1}]$ by $\tilde{f} |_{[t_i, t_{i+1}]} = \pi_i^{-1} \circ f$. 

