MIDTERM  FRIDAY  2/27 IN CLASS
I’ll be out of town – leaving the dept at 2
on Wed.

Topics for midterm
- homotopies, path homotopies
- the fundamental group
- covering spaces
- lifting lemmas

Midterm
- true/false  example/counter-example questions
  • If f: f(g) & g(h) f(h).  
  • If [f], [g], [h] ∈ π₁(X, x₀) with
    • Give an example of a covering space
      p: (E, e₀) → (B, b₀) & a map
      f: (X, x₀) → (B, b₀) s.t f doesn’t
      have a lift.
    Using the same spaces find a map
g: (X, x₀) → (B, b₀)
      that does have a lift.

2 short proofs - I’ll give you 5 to choose from.
**RETRACTIONS**  \( X \) is a top. space, \( A \subset X \) subspace
\( r : X \to A \) is a retraction if
\( r \mid_A = \text{id} \). That is \( r(x) = x \) \( \forall x \in A \).

**THEOREM**  Let \( r : X \to A \) be a retraction and
\( i_A : A \to X \) the inclusion map. Then
1) \( r_* : \pi_1(X, x_0) \to \pi_1(A, r(x)) \) is surjective.
2) \( (i_A)_* : \pi_1(A, x_0) \to \pi_1(X, x_0) \) is injective.

**PROOF**  Let \( \text{id}_A : A \to A \) be the identity map.
Then \( \text{id}_A = r \circ i_A \) and therefore \( (\text{id}_A)_* = r_* \circ (i_A)_* \).

**GENERAL FACT**  If \( fg = h \) & \( h \) is injective
then \( g \) is injective (but not necessarily \( f \)).

If \( fg = h \) & \( h \) is surjective then
\( f \) is surjective (but not necessarily \( g \)).

The theorem follows directly from the general fact.

**COROLLARY**  Let \( X \) be a topological space & \( S' \subset X \) a subspace. Let \( x_0 \in S' \subset X \) be a basepoint.
If \( \pi_1(X, x_0) = \{1\} \) there is no retraction from \( X \) to \( S' \).

**PROOF**  Let \( r : X \to S' \) be a retraction. Then \( r_* \) is
surjective but \( \pi_1(X, x_0) = \{1\} \) while \( \pi_1(S', x_0) = \{1\} \). Contradiction.
Let $B^2 = \{(x,y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}$ and $x_0 \in B^2$ a basepoint.

The $\pi_1(B^2, x_0) \cong \mathbb{Z}$. Proof: Straight line homotopy.

If $(t,y) \in \pi_1 (B^2, x_0)$ define

$f(t) = (1-t)f(x) + t \cdot x_0$

Then $f \cong \alpha$, constant map $\alpha : B^2 \to x_0$.

**Corollary**

There is no retraction $r : B^2 \to S^1$.

**Proof**

Follows from previous corollary since $\pi_1(S^2) = \mathbb{Z}$.

**Brouwer Fixed Point Theorem in Dimension 2**

Every map $f : B^2 \to B^2$ has a fixed point.

$\exists x \in B^2$ s.t. $f(x) = x$.

**Proof**

We'll show that if $f$ doesn't have a fixed point then there is a retraction $r$ from $B^2$ to $S^2$.

The construction:

Why is this map continuous?

A formula:

$\forall x \in B^2$ choose $s, \in (0, 1]$

$s, \epsilon$

$1 - s \varepsilon f(x) + s \varepsilon \cdot x = 0$ & define

$r(x) = (1-s \varepsilon) f(x) + s \varepsilon \cdot x$.

We have that if $x \in S^1$,

$s = 1$ & $r(x) = x$. 