BASIC LIFTING LEMMA

Let \( p: E \rightarrow B \) be a covering space, \( e_0 \in E \) a basepoint & \( b_0 = p(e_0) \in B \). Let \( f: ([0,1], e_0) \rightarrow (B, b_0) \)
be a continuous map. Then \( \exists! \) lift
\( \tilde{f}: ([0,1], e_0) \rightarrow (E, e_0) \).

Homotopy Lifting Lemma

Let \( p: (E, e_0) \rightarrow (B, b_0) \) be a covering space & \( F: [0,1] \times [0,1] \rightarrow B \) a continuous map with \( f(0,0) = b_0 \). Then \( \exists! \)
\( \tilde{F}: [0,1] \times [0,1] \rightarrow E \) with \( \tilde{F}(0,0) = e_0 \) and \( F = p \circ \tilde{F} \).

Cor

Let \( p: E \rightarrow B \) be a covering space & let
\( f, g: [0,1] \rightarrow B \)
be paths with \( f(0) = g(0) \) & \( f(1) = g(2) \), and \( f \sim p \circ g \). If \( \tilde{f}, \tilde{g}: [0,1] \rightarrow E \)
are lifts of \( f \) & \( g \) with \( \tilde{f}(0) = \tilde{g}(0) \)
then \( \tilde{f}(1) = \tilde{g}(2) \).

\( X \) is simply connected if \( X \) is path connected and \( \pi_1(X, x_0) = \{1\} \)
A LIFTING LEMMA

Assume that \( X \) is simply connected and locally path-connected. Let \( p : (E, e) \rightarrow (B, b_0) \) be a covering space. Fix a basepoint \( x_0 \in X \). Then any map \( f : (X, x_0) \rightarrow (B, b) \) has a unique lift \( \tilde{f} : (X, x_0) \rightarrow (E, e) \).

Proof

Given \( x \in X \) define \( \tilde{f}(x) \) by choosing a path \( x : [0, 1] \rightarrow X \) with \( x(0) = x_0 \) \& \( x(1) = x \). Then \( \tilde{f}(x) : [0, 1] \rightarrow B \) is a path to \( B \) with \( \tilde{f}(0) = b_0 \). By the lifting lemma, \( \tilde{f} \circ x \) has a lift \( \tilde{x} : [0, 1] \rightarrow E \) with \( \tilde{x}(0) = e_0 \) \& \( \tilde{x}(1) = \tilde{f}(x) \). We define \( \tilde{f}(x) = \tilde{x}(1) \). This is well defined since for any other path \( \tilde{B} \) with \( \tilde{f}(x) = \tilde{f}(x) \) \& \( \tilde{B}(1) = \tilde{f}(x) \) we have \( \tilde{x}(1) = \tilde{B}(1) \) by lemma.
**Question** Does path connected imply locally path connected?

To prove continuity we need to use that $X$ is locally path connected. That is $\forall x \in X$, and all nbds $U$ of $x$, there is a path connected nbd $V$ of $x$ with $V \subseteq U$.

$$f^{-1}(y) \text{ is a nbd of } x \in X$$

$VC f^{-1}(y)$ that is path connected
Let $U$ be an evenly covered nbhd. of $f(x)$. Then $f^{-1}(U)$ is a nbhd of $x$ in $X$ and there is a path connected nbhd $V$ of $x$ with $V \subset f^{-1}(U)$. Let $U_x$ be the component of $p_{\#}^{-1}(U)$ that contains $\widetilde{f}(x)$. Let $p_{\#}^{-1}$ be the inverse of the restriction of $p$ to $U_x$. We claim that $\tilde{f} = p_{\#}^{-1} \cdot f$ on $V$. Given $y \in V$ let $\beta: [0,1] \to V \subset X$ be a path with $\beta(0) = x$ and $\beta(1) = y$. Note that $p(\beta(0)) = f(x)$. So we can apply the lifting lemma to $f \beta$ to find a lift $\tilde{\beta}: [0,1] \to E$ with $\tilde{\beta}(0) = \tilde{f}(x)$.

We can also define a lift of $f \beta$ by taking $p_{\#}^{-1} \circ \tilde{\beta}$. As $\pi_{\#}^{-1} \circ \tilde{\beta}(0) = \tilde{f}(x)$ the uniqueness of lifts implies that $\tilde{\beta} = p_{\#}^{-1} \circ f \beta$.

To define $\tilde{f}(y)$ we need a path from $x_0$ to $y$. The concatenation $\alpha \star \beta$ is such a path, so $\tilde{f}(y) = \alpha \star \beta(1)$ where $\alpha \star \beta$ is the lift of $f(\alpha \star \beta)$. However, the concatenation $\alpha \star \beta$ is a (and hence the) lift of $f(\alpha \star \beta)$ so $\tilde{f}(y) = \alpha \star \beta(1) = \alpha \star \beta(1) = \beta(1) = p_{\#}^{-1} f(\beta(0)) = p_{\#}^{-1} f(y)$. 

Let $p: (E, e) \to (B, b)$ be a covering space. Then
$$p_*: \pi_1(E, e) \to \pi_1(B, b)$$
is the induced homomorphism.

We can apply the lifting lemma to $[f] \in \pi_1(B, b)$.

**Proposition** If $[f] \in \pi_1(B, b)$ is the lift of $f$ with $f(e) = e$, then $f(1) = e_0$.

**Proof** Choose $[g] \in \pi_1(E, e)$ such that $p_*([g]) = [f]$. Then $p \circ g \cong f$. By the lifting property, if $\tilde{p} \circ \tilde{g}$ is the lift of $p \circ g$, then $\tilde{p} \circ \tilde{g}(e) = \tilde{f}(1)$. But the (unique) lift of $p \circ g$ is $g$ so $g(1) = f(1) = e_0$.
**Proposition** \( p \) is injective.

**Proof** Assume that \( \pi_1(E, e_0) \) with \( p(e_0) = \text{id} \). Then there is a homotopy of pairs

\[ F : [0,1] \times [0,1] \to B \]

from \( p \) of to \( \text{id} \). By the homotopy lifting lemma there is a lift

\[ \tilde{F} : [0,1] \times [0,1] \to E \]

of \( F \) with \( \tilde{F}(0,0) = e_0 \). This is a homotopy of pairs from \( p \) to the \( \text{id} \), so \( \tilde{F}(1) = \text{id.} \)
**FINAL LIFTING LEMMA**

Assume $X$ is locally path connected, $p : (E, e_0) \rightarrow (B, b_0)$ and $f : (X, x_0) \rightarrow (B, b_0)$ with $f_* (\pi_1 (X, x_0)) \subseteq p_* (\pi_1 (E, e_0)) \subseteq \pi_1 (B, b_0)$.

Then there exists a lift $\tilde{f} : (X, x_0) \rightarrow (E, e_0)$. 

\[ X \rightarrowtail \xymatrix{ E \ar@{~[r]}_{f} & B } \]