**LIFTING LEMMAS**

Given $f: X \to B$, when can we find a lift $\tilde{f}: \tilde{X} \to E$?

Let $p: E \to B$ be a covering space, $e_0 \in E$ a basepoint & $b_0 = p(e_0) \in B$. Let $f: (\mathbb{R},(0,1)) \to (B,b_0)$ be a continuous map. Then $\exists$ lift $\tilde{f}: (\mathbb{R},(-1,1)) \to (E,e_0)$.

**PROOF**

The proof is almost exactly the same as when $B = S^1$ & $E = \mathbb{R}$.

1. Find $0 < t_0 < t_1 < \ldots < t_n = 1$ s.t. that $f([t_{i-1}, t_i)) \subset U_i$ where $U_i$ is evenly covered.
• Define $\tilde{f}(e) = e_0$

• Assume $\tilde{f}$ is defined on $[0, e_{i-1}]$ & let $\tilde{U}_i$ be the component of $p^{-1}(U_i)$ that contains $\tilde{f}(e_{i-1})$.

• Let $\tilde{p}_i$ be the inverse of $p$ restricted to $\tilde{U}_i$.

• Define $\tilde{f}$ on $[e_{i-1}, e_i]$ by $\tilde{f} = \tilde{p}_i \circ f$. 
Homotopy Lifting Lemma

Let \( p : (E,e_0) \to (B,b_0) \) be a covering space and \( f : [0,1] \times [0,1] \to B \) a continuous map with \( f(0,0) = b_0 \). Then, \( \exists! \) lift
\[
f : [0,1] \times [0,1] \to E \text{ with } f(0,0) = e_0.
\]

\[ f = p \circ \tilde{f} \]

PF

Again, the proof is the same.
Key corollary: if 2 paths have the same endpoints and are path homotopic then their lifts have the same endpoint.

**Cor** Let \( p : E \to B \) be a covering space & let

\[ f, g : [0, 1] \to B \]

be paths with \( f(0) = g(0) \), \( f(1) = g(1) \), and \( f \simeq p.g \). If \( \tilde{f}, \tilde{g} : I \to E \) are lifts of \( f \& g \) with \( \tilde{f}(0) = \tilde{g}(0) \) then \( \tilde{f}(1) = \tilde{g}(1) \).

**PF** Let \( F : [0,1] \times [0,1] \to B \)

be the homotopy between \( f \& g \).

Let \( \tilde{F} : [0,1] \times [0,1] \to E \) be the lift of \( F \) with \( \tilde{F}(0,0) = \tilde{F}(0,1) = \tilde{F}(1,0) \).

As \( F \) is a homotopy of pairs \( F \) is constant on \( [0,1] \times [0,1] \).

Also \( \tilde{F}(s) = \tilde{F}(s,t) \) is a lift of \( \tilde{F}(s) = \tilde{F}(s,t) \). Since \( \tilde{F}(s) = \tilde{F}(s,t) = \tilde{F}(s) = \tilde{F}(s) \) by the uniqueness of lifts \( \tilde{f}_0 = \tilde{f} \) & \( \tilde{f}_1 = \tilde{g} \).

Since \( F \) is constant on \( [1] \times [1] \) we have \( \tilde{f}_0(1) = \tilde{f}_1(1) \) \( \Rightarrow \tilde{f}(1) = \tilde{g}(1) \).
** DEFINITION **

If \( X \) is path connected \& \( \pi_1(X, \star) = \{e_3\} \) then \( X \) is simply connected.

** LEMMA **

Assume that \( X \) is simply connected. Then for any paths \( f, g : [0, 1] \to X \) with \( f(0) = g(0) \) \& \( f(1) = g(1) \) we have \( f \sim_p g \).

** PROOF **

\( f \ast \bar{g} \) represents an element of \( \pi_1(X, f(\alpha)) \). Since \( X \) is simply connected \( f \ast \bar{g} \sim_p \text{const} \).

\[ \Rightarrow f \ast \bar{g} \ast g \sim_p \text{const} \ast g \sim_p g \]

But \( \bar{g} \ast g \sim_p \text{const} \) \Rightarrow \( f \ast \bar{g} \ast g \sim_p f \ast \text{const} \sim_p f \).

Therefore \( f \sim_p g \). \hfill \square
Another Lifting Lemma

Assume that \( X \) is simply connected and that \( p: (E, e_0) \to (B, b_0) \) is a covering space. Fix a basepoint \( x_0 \in X \). Then any map

\[ f: (X, x_0) \to (B, b_0) \]

has a unique lift

\[ \tilde{f}: (X, x_0) \to (E, e_0). \]

**Proof:**

Given \( x \in X \), define \( \tilde{f}(x) \) by choosing a path

\[ \alpha: [0, 1] \to X \]

with \( \alpha(0) = x_0 \) and \( \alpha(1) = x \), and setting

\[ \tilde{f}(x) = \tilde{\alpha}(1) \]

where \( \tilde{\alpha} \) is the lift of \( \alpha \). This is well defined since for any other path \( \beta \) with \( \beta(0) = \alpha(0) \) and \( \beta(1) = \alpha(1) \), we have \( \tilde{\alpha}(1) = \tilde{\beta}(1) \) by Lemma.