Paths:
\[ f : [0,1] \rightarrow X \]

\[ \quad \quad \quad \rightarrow \quad \quad \quad \rightarrow \]

\[ \begin{array}{c}
X \text{ topological space } \\
\text{will usually assume } X \text{ is path connected. }
\end{array} \]

Homotopy of paths.

Perturbation of paths:
\[ f : [0,1] \rightarrow X \rightarrow X \]

DEFN

2 paths \( f \) and \( g \) are homotopic if \( \exists F : [0,1] \times [0,1] \rightarrow X \)

such that \( f = f_0 \) and \( g = f_1 \)
Notation \[ f \approx g \] if \( f \) and \( g \) are homotopic, we write \( f \approx g \).

**Lemma** Homotopy is an equivalence relation:

1. \( f \approx g \Leftrightarrow g \approx f \)
2. \( f \approx f \)
3. \( f \approx g \) and \( g \approx h \) imply \( f \approx h \)

**Proof**

\[ f \approx g \Leftrightarrow f \sim X \]

\[ f : [0,1] \times [0,1] \to X \]

\[ f = f_0 \quad \text{and} \quad g = f_1 \]

Let \( G : [0,1] \times [0,1] \to X \) be defined by

\[ G(s, t) = \begin{cases} f(s, t) & \text{if } t = 0 \\ f(s, 1-t) & \text{if } t = 1 \end{cases} \]
For \( f \sim f \) we define the homotopy by \( F(s, \epsilon) = f(s) \). (constant homotopy).

Transitivity: \( f \sim g \) \& \( g \sim h \) so

we have

\[
F(s, \epsilon) \quad \& \quad G(s, \epsilon)
\]

with

\[
f(s) = F(s, 0) \quad \& \quad g(s) = F(s, 1) = G(s, 0)
\]

\& \( h(s) = G(s, 1) \).

Define

\[
H(s, \epsilon) = \begin{cases} 
F(s, 2\epsilon) & 0 \leq \epsilon \leq \frac{1}{2} \\
G(s, 2\epsilon - 1) & \frac{1}{2} < \epsilon \leq 1 
\end{cases}
\]

As \( H \) is continuous this defines \( f \sim h \)

a homotopy from \( f \) to \( h \).
Exercise

Here we're using the following:

Fact:

\[ f : A \to X \]
\[ g : B \to X \]

Continuous function with

\[ f \circ g \] on \( A \cap B \).

Then

\[ h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases} \]

is continuous on \( A \cup B \).
**Prop** If $X$ is path connected then $\forall f, g : [0, 1] \to X$ we have $f \circ g$.

**Proof** We first assume that $f$ and $g$ are constant maps:
$f([0, 1]) = x \in X \; \& \; g([0, 1]) = y \in X$

Since $X$ is path connected we have $\gamma : [0, 1] \to X$
with $\gamma(0) = x \; \& \; \gamma(1) = y$.

Define $\pi(s, t) = t$ & $F(s, t) = \gamma \circ \pi(s, t) = \gamma(t)$.

$\Rightarrow f \sim g$
Need to show that an arbitrary
\[ f : \mathbb{D} \rightarrow X \]
is homotopic to a point.

\[ \epsilon = 0 \]
\[ \epsilon = 1/4 \]
\[ \epsilon = 1/2 \]
\[ \epsilon = 1 \]

\[ F(\epsilon, t) = f(s(1-t)) \]
gives a homotopy of \( f \) to
a constant map.

Given \( f, g : \mathbb{D} \rightarrow X \)
\[ f \sim \text{const.} \quad g \sim \text{const.} \]
and maps are homotopic
\[ \Rightarrow f \approx g. \]
Examples

Let \( X = [0,1] \) &
\[
\begin{align*}
  f : [0,1] & \to [0,1] \\
  f(0) & = 0 \quad \text{and} \quad f(1) = 1.
\end{align*}
\]

Then \( f \) is id.

Define
\[
  F(s,t) = (1-t) f(s) + ts
\]

where
\[
  F \text{ is continuous and } f_0(s) = F(s,0) = f(s) \quad \text{and} \quad f_1(s) = F(s,1) = s.
\]

We need to check that
the image of \( F \) is \([0,1]\).

Note that for any \( a \leq b \in \mathbb{R} \)
\[
a \leq (1-t)a + tb \leq b \quad \text{if} \quad t \in [0,1],
\]

since
\[
(1-t)a + tb \geq (1-t)a + ta = a
\]

& \quad (1-t)a + tb \leq (1-t)b + b = b.
$\mathbb{R}^2$

$t \mathbf{x}_0 \rightarrow (1-\epsilon) \mathbf{x}_1$

$\epsilon \in [0,1]$
More generally if \( f, g : [0, 1] \to \mathbb{R} \) we have the homotopy

\[
F(s, t) = (1-t)f(s) + t\,g(s)
\]
We can combine: A reparameterization of a path is homotopic to the original path.

If \( \varphi: [0,1] \rightarrow [0,1] \) is a homeomorphism and \( f: [0,1] \rightarrow X \) is a path, then \( f \sim \varphi f \).

We have seen that if \( X \) is path connected then all paths are homotopic.

That is, the equivalence relation of homotopy has only one equivalence class.

This is not very interesting!
To make things more interesting we can replace $\mathbb{R}^2$ with any topological space.

We can also look at maps of pairs.

In general it is much harder to show that maps are not homotopic.

The circle $S^1$.  

$S^1 = \{ (x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1 \}$

Why homeo?

$S^1 = \{ z \in \mathbb{C} | |z| = 1 \} = x + iy$

$S^1 = \mathbb{R}/\mathbb{Z}$ where $\mathbb{R}^\mathbb{Z}$.

$S^1 = \{ (x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1 \}$

and

$S^1 = \{ (x, y) \in \mathbb{R}^2 | \max \{ |x|, |y| \} = 1 \}$
\[ S' = \mathbb{R}/\sim \text{ where } \sim \in \mathbb{Z} \text{ and } \forall n \in \mathbb{Z}, \]
General definition of homotopy

\[ f, g : X \rightarrow Y \]

are homotopic \((f \simeq g)\) if \( f \)

\[ F : X \times [0,1] \rightarrow Y \]

s.t. \( f = F_{0} \) and \( g = F_{1} \).

**EXERCISE**

This is (still) an equivalence relation.

How can we define maps from \( S' \) to \( S' \)?

\[ S' = \{ z \in \mathbb{C} | |z| = 1 \} \]

Define \( f_{n} : S' \rightarrow S' \)

by \( f_{n}(z) = z^{n} \).

Since \( |z^{n}| = |z|^{n} = 1 \), this is a map from \( S' \) to \( S' \).
Alternative construction.

Define \( f_n : \mathbb{R} \to \mathbb{R} \) by \( \tilde{f}_n(x) = nx \).

Note that \( \tilde{f}_n(x+m) = n(x+m) = nx + nm \) so in our equivalence relation on \( \mathbb{R} \) we have \( f_n(x) \sim \tilde{f}_n(x+m) \).

\( f_n \) takes equivalence classes to equivalence classes.

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{\sim} & \mathbb{R}/\mathbb{Z} \\
\pi & : & \pi \\
X & \xrightarrow{\sim} & \pi(X) \\
f_n & : & \pi(X) \\
\end{array}
\]

Choose some \( x \in \mathbb{R} \) so \( \pi(x) = x \).

Define \( f_n(x) : \pi \circ \tilde{f}_n(x) \).