Groups

Let $G_0, G_1, \ldots$ be a family of groups and assume for each $i < j$ there is homomorphism

$$\phi_{i,j} : G_i \to G_j$$

such that if $i < j < k$ then

$$\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}.$$ 

Let $\mathcal{G}$ be the disjoint union of the $G_i$. Define a relation on $\mathcal{G}$ as follows: If $g_i \in G_i$ and $g_j \in G_j$ then $g_i \sim g_j$ if there exists a $k$ with $\phi_{i,k}(g_i) = \phi_{j,k}(g_j)$.

1. Show that $\sim$ is an equivalence relation.

For $g \in \mathcal{G}$ denote the equivalence class by $[g]$.

Define an operation on equivalence classes as follows: If $g_i \in G_i$ and $g_j \in G_j$ choose a $k$ larger than $i$ and $j$ and set

$$[g_i] \cdot [g_j] = [\phi_{i,k}(g_i) \cdot \phi_{j,k}(g_j)].$$

2. Show that this is a well defined operation on the set of equivalence classes.

Let $\mathcal{G}$ be the set of equivalence classes.

3. Show that $\mathcal{G}$ with the given operation is a group.

4. Define maps $\phi_{i,\infty} : G_i \to \mathcal{G}$ by $\phi_{i,\infty}(g) = [g]$. Show that the $\phi_{i,\infty}$ are homomorphisms and that $\phi_{i,\infty} = \phi_{j,\infty} \circ \phi_{i,j}$.

5. If all of the $\phi_{i,j}$ are the trivial homomorphism, show that $\mathcal{G}$ is the trivial group.

6. If all of the $\phi_{i,j}$ are isomorphisms, show that the $G_i$ are isomorphic to $\mathcal{G}$.

7. Assume that $G$ is another group and $\psi_i : G_i \to G$

are injective homomorphisms with $\psi_i = \psi_j \circ \phi_{i,j}$ when $i < j$. Show that there exists an injective homomorphism

$$\psi : \mathcal{G} \to G$$

with $\psi_i = \psi \circ \phi_{i,\infty}$. If for all $g \in G$ there exists an $i$ and a $g_i \in G_i$ such that $\psi_i(g_i) = g$ then $\psi$ is an isomorphism.
8. Assume that all of $G_i \cong \mathbb{Z}$. Find homomorphisms

$$\phi_{i,j} : G_i \to G_j$$

such that $\tilde{G} \cong \mathbb{Q}$.

**Topology**

Let $X$ be a topological space and

$$X_0 \subset X_1 \subset X_2 \subset \ldots$$

a collection of open subspaces such that

$$X = \bigcup X_i.$$  

The $X_i$ are an *exhaustion* of $X$. Let

$$t_{i,j} : X_i \to X_j$$

be the inclusion maps. Fix a basepoint $x_0 \in X_0$. Let $G_i = \pi_1(X_i,x_0)$ and $\phi_{i,j} = (t_{i,j})_*$ be the induced homomorphisms.

9. Show that $\pi_1(X,x_0) \cong \tilde{G}$.

**Added on March 19th**

We would like to construct a space $X$ with $\pi_1(X,x_0) \cong \mathbb{Q}$. The construction will parallel the construction the group $\tilde{G}$.

Let $X_0, X_1, X_2, \ldots$ be (path-connected) topological spaces and for each $i < j$ let

$$\phi_{i,j} : X_i \to X_j$$

be an injective, continuous maps with $\phi_{i,j}(X_i)$ open in $X_j$ such that if $i < j < k$ then

$$\phi_{i,k} = \phi_{j,k} \circ \phi_{i,j}.$$  

Let $\mathcal{X}$ be the disjoint union of the $X_i$. We define a relation on $\mathcal{X}$ as we did for the collection of groups $\mathcal{G}$. If $x_i \in X_i$, $x_j \in X_j$ then $x_i \sim x_j$ if

- $i = j$ and $x_i = x_j$;
- $i < j$ and $\phi_{i,j}(x_i) = x_j$;
- $j < i$ and $\phi_{j,i}(x_j) = x_i$.

10. Show that $\sim$ is an equivalence relation.

Let $X = \mathcal{X}/\sim$ be the quotient space.
11. Show that the homeomorphisms

\((\phi_{i,j})_*: \pi_1(X_i,x_i) \to \pi_1(X_j,x_j)\)

satisfy the conditions above on groups.

12. Let \(\overrightarrow{G}\) be the group constructed there and show that \(\pi_1(X,x_0)\) is isomorphic to \(\overrightarrow{G}\). To simplify this problem (so that you can apply problem 7) you can assume that the \((\phi_{i,j})_*\) are injective.

13. Let

\[ B^2 = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 < 1\} \]

be the open disk and let

\[ W = S^1 \times B^2. \]

Then \(W\) is a solid torus. As \(W\) and \(S^1\) are homotopy equivalent \(\pi_1(W,w_0) \cong \mathbb{Z}\). Show that for any integer \(n \in \mathbb{Z}\) there is an injective, continuous map

\[ \phi_n: W \to W \]

such that the homomorphism \((\phi_n)_*\) is multiplication by \(n\).

14. Use the previous problem (and problems 8 and 9) to construct a space \(X\) with \(\pi_1(X,x_0) \cong \mathbb{Q}\).

In problem 7 we assume that the homomorphisms \(\phi_{i,j}\) are injective. While this condition is sufficient there is actually a weaker condition that is both necessary and sufficient. Namely we can assume that if \(\psi_i(g_i) = id\) then there exists a \(j > i\) such that \(\phi_{i,j}(g_i) = id\) in \(G_j\).

In the topological setup one can check that this condition always holds using that homotopies are compact. In particular, if \(X\) is the topological space constructed in problem 10 and \([f] \in \pi_1(X,x_0)\) is trivial then not only is \([f] \in \pi_1(X_i,x_i)\) for some \(i\) but there is a \(j > i\) such that \([f]\) is trivial in \(\pi_1(X_j,x_j)\). This is exactly the algebraic condition of the previous paragraph.