Notes for Math 3210, Midterm 2

Limits. Let $\{a_n\}$ be a sequence. Then

$$\lim a_n = a$$

if for all $\epsilon > 0$ there exists an N such that if $n > N$ then $|a_n - a| < \epsilon$. If no such a exists then the sequence is *divergent*. The sequence a_n is *Cauchy* if for all $\epsilon > 0$ there exists an $N > 0$ such that if $n, m > N$ then $|a_n - a_m| \leq \epsilon$.

Theorem 0.1 *A sequence is convergent if and only if it is Cauchy.*

Theorem 0.2 *Every bounded sequence of real numbers has a convergent subsequence.*

Theorem 0.3 *Suppose $a_n \to a$, $b_n \to b$, c is a real number and k a natural number. Then*

1. $ca_n \to ca$;
2. $a_n + b_n \to a + b$;
3. $a_n b_n \to ab$;
4. $a_n/b_n \to a/b$ *if* $b \neq 0$ *and* $b_n \neq 0$ *for all* n;
5. $a_n^k \to a^k$;
6. $a_n^{1/k} \to a^{1/k}$ *if* $a_n \geq 0$ *for all* n.

If A is a subset of $\mathbb{R}$ the $a = \sup A$ if $a \geq x$ for all $x \in A$ and $a' \geq x$ for all $x \in A$ then $x \leq y$. We define $\inf A$ be reversing the inequalities. If we allow $+\infty$ and $-\infty$ the $\sup A$ and $\inf A$ always exist.

Let $\{a_n\}$ be a sequence and define $i_n = \inf\{a_k : k \geq n\}$ and $s_n = \sup\{a_k : k \geq n\}$. Then

$$\liminf a_n = \lim i_n$$

and

$$\limsup a_n = \lim s_n.$$

If $x \neq 1$ then

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Continuity. Let $f : D \longrightarrow \mathbb{R}$ be a function defined on a domain $D \subset \mathbb{R}$. Then

$$\lim_{x \to a} f = b$$

if for all $\epsilon > 0$ there exists a $\delta > 0$ such that if for all $x \in D$ with $0 < |x - a| < \delta$ then $|f(x) - b| < \epsilon$. The function f is *continuous* at a if

$$\lim_{x \to a} f = f(a)$$

There is a theorem similar to Theorem 0.3 for limits of functions.