Introductory topics in Kleinian groups and hyperbolic 3-manifolds

Convex hull problems

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1. Show that a discrete subgroup of $\text{Isom}(\mathbb{H}^3)$ acts properly discontinuously on $\mathbb{H}^3$.

   For any two points $p_1$ and $p_2$ in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ there is a unique geodesic with endpoints $p_1$ and $p_2$. A set $K$ in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ is convex if whenever $p_1$ and $p_2$ are contained in $K$ then this geodesic is also in $K$.

2. If $K$ is a closed convex set in $\mathbb{H}^3 \cup \hat{\mathbb{C}}$ show that for every $p \in \mathbb{H}^3$ there is a unique ball centered at $p$ that intersects $K$ in exactly one point. If $p \in \hat{\mathbb{C}}$ show that there is a unique horoball centered at $p$ that intersects $K$ in exactly one point. Note that we allow the ball and horoball to be single point.

   Define a map $\pi_K : \mathbb{H}^3 \cup \hat{\mathbb{C}} \longrightarrow K$ by setting $\pi_K(p)$ to be the point of intersection given in the previous problem. The map $\pi_K$ is the nearest point retraction onto $K$.

3. Show that $\pi_K$ is continuous and $\pi_K(p) = p$ if and only if $p \in K$.

4. If $K$ is $\Gamma$-invariant show that $\pi_K$ commutes with the action of $\Gamma$.

   The convex hull, $CH(\Lambda)$, of a set $\Lambda$ is the smallest closed convex set that contains $\Lambda$.

5. Show that the convex hull is well defined.

   The limit set $\Lambda = \Lambda(\Gamma)$ of Kleinian group $\Gamma$ is the smallest, non-empty, closed $\Gamma$-invariant subset of $\hat{\mathbb{C}}$.

6. Show that $CH(\Lambda)$ is $\Gamma$-invariant.

   The domain of discontinuity, $\Omega = \Omega(\Gamma)$, for $\Gamma$ is the complement of the limit set. That is $\Omega = \hat{\mathbb{C}} \setminus \Lambda$. 

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7. Use the nearest point retraction to show that $\Gamma$ acts properly discontinuously on $\Omega$. 