Math 6510 - Homework 6
Due in class on 12/9/14

1. Let $T \in \mathcal{T}^k(V)$ and $S \in \mathcal{T}^l(V)$ be tensors on a vector space $V$ with $\text{Alt}(S) = 0$. Show that $\text{Alt}(T \otimes S) = \text{Alt}(S \otimes T) = 0$. This is in Spivak and I sketched how to prove this in class but it is in good exercise in the linear algebra we are using to try to write down a complete proof on your own.

2. $\Lambda(M)$ is the vector space of smooth vector fields on $M$. Let $\bar{T} : \Lambda(M) \times \cdots \times \Lambda(M) \to C^\infty(M)$ be a function such that $\bar{T}(V_1, \ldots, fV_i + gV_i', \ldots, V_k) = f\bar{T}(V_1, \ldots, V_i, \ldots, V_k) + g\bar{T}(V_1, \ldots, V_i', \ldots, V_k)$. Show that there exists a $T \in \mathcal{T}^k(M)$ with $T = \bar{T}$. In particular, you should show that if $V_1, \ldots, V_k$ and $W_1, \ldots, W_k$ are vector fields with $V_i(x) = W_i(x)$ then $\bar{T}(V_1, \ldots, V_k)(x) = T(W_1, \ldots, W_k)(x)$.

   (a) First assume that there is a neighborhood $U$ of $x$ such that $V_i|_U = W_i|_U$. You can use that fact that there is a smooth function $\psi : M \to \mathbb{R}$ with $\text{supp}(\psi) \subset U$ and $\psi(x) = 1$.

   (b) Let $E_1, \ldots, E_n$ be smooth vector fields that are a basis for each tangent space in $U$. Assume that $V_i(x) = 0$ and show that $T(V_1, \ldots, V_k)(x) = 0$ by using the fact that there are smooth functions $f_j : M \to \mathbb{R}$ with $V_i = \sum f_j E_j$ on $U$.

   (c) Use (a) and (b) to finish the proof.

3. Let $\beta$ be a $k$-form on the product manifold $M \times N$. We say that $\beta$ is tangent to $M$ if $\beta(\ldots, V_i, \ldots) = 0$ when $V$ is tangent to $N$.

   For the product manifold $M \times I$, where $I$ is an interval, let $\pi_I : M \times I \to I$ be the projections to $I$. If $\omega$ is a $k$-form on $M \times I$ show that there exists a $k$-form $\alpha$ and a $k - 1$-form $\eta$ on $M \times I$ such that

   $$\omega = \alpha + \pi_I^* dt \wedge \eta$$

   with $\alpha$ and $\eta$ tangent to $M$.

4. Let $G = \text{Isom}^+(\mathbb{R}^2)$ be the group of orientation preserving isometries of $\mathbb{R}^2$. We will view points in $\mathbb{R}^2$ as column vectors. If $T \in G$ then $T$ is of the form $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$

   where $A \in SO(2)$. Define vector fields on $\mathbb{R}^2$ by $V_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$, $V_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $V_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let $\mathfrak{g}$ be the vector space spanned by $V_1$, $V_2$ and $V_3$. That is $\mathfrak{g}$ is the vector space of vector fields on $\mathbb{R}^2$ that are linear combinations of the $V_i$.

   (a) Given $T \in G$ and $V \in \mathfrak{g}$ show that $T_*V \in \mathfrak{g}$ and therefore $T_* : \mathfrak{g} \to \mathfrak{g}$ is a linear map.

   (b) Using the $V_i$ as a basis for $\mathfrak{g}$ write $T_*$ as a $3 \times 3$-matrix and show that $G$ is a subgroup of $GL(n)$. 

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