Math 6510 - Homework 2
Due in class on 9/23/14

1. Assume that $M$ and $N$ are submanifolds of Euclidean space and that $f : M \to N$ is a diffeomorphism. Show that $f$ determines a diffeomorphism between $TM$ and $TN$.

2. Recall that $M(n)$ is the space of $n \times n$ matrices and is naturally identified with $\mathbb{R}^{n^2}$. Let $SL(n) = \{ A \in M(n) | \det A = 1 \}$. Show that $SL(n)$ is a differentiable submanifold and show that the tangent space at the identity is the subspace of all matrices of trace zero.

3. Let $M = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 | x_0^2 + x_1^2 = x_2^2 + x_3^2 = 1 \}$. Show that $M$ is a differentiable submanifold of $\mathbb{R}^n$. Given an explicit description of $TM$ and show that it is diffeomorphic to $M \times \mathbb{R}^2$. Can you give another description of this manifold?

4. Let $M$ be a differentiable manifold. Recall that $v : C^\infty(M) \to \mathbb{R}$ is a derivation at $x \in M$ if

   (a) $v(f + \lambda g) = v(f) + \lambda v(g)$ for all $f, g \in C^\infty(M)$ and $\lambda \in \mathbb{R}$;

   (b) $v(fg) = f(x)v(g) + v(f)g(x)$.

The space of all derivations at $x$ is a vector space. Show that it is naturally isomorphic to $T_xM$. Here is an outline of how to do it.

   (a) If $f$ is zero in a neighborhood of $x$ use (b) to show that $v(f) = 0$. You can use the that for any open sets $U$ and and $V$ with $\bar{V} \subset U$ there exists a $\phi \in C^\infty(M)$ with support in $U$ and that is $\equiv 1$ on $V$. Use such a $\phi$ to decompose $f$ into the product of two smooth functions that are zero at $x$.

   (b) If $f \equiv 1$ use (b) to show that $v(f) = 0$ and then use (a) to show that $v(f) = 0$ for all constant functions $f$.

   (c) Combine the previous two statements to show that if $f$ is constant in a neighborhood of $x$ then $v(f) = 0$.

   (d) If $f = g$ on a neighborhood of $x$ show that $v(f) = v(g)$.

   (e) Reduce the statement to the following special case: If $M = \mathbb{R}^n$ and $x = 0$ then every derivation is of the form

   $$v(f) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(0).$$

   Use the following calculus fact. If $f : C^\infty(\mathbb{R}^n)$ with $f(0) = 0$ then in a neighborhood of 0

   $$f(x) = \sum_{i=1}^{n} x_i g_i(x)$$

   where the $g_i$ are smooth functions with $\frac{\partial f}{\partial x_i}(0) = g_i(0)$. 
