1. Find a differentiable atlas for $S^1 \times S^1$.

2. Show that if $M$ and $N$ are differentiable manifolds then $M \times N$ is a differentiable manifold.

3. Find a differentiable atlas of $\mathbb{R}$ such that the identity map is not smooth.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isomorphism. Show that there exists $c_1, c_2 \in \mathbb{R}^+$ such that $c_1|v| \leq |Tv| \leq c_2|v|$.

5. Let $U, V \subset \mathbb{R}^n$ be open and $f : U \to V$ a smooth homeomorphism such that $f_*(x)$ is an isomorphism for some $x \in U$. Show that $f^{-1}$ is differentiable at $y = f(x)$ and that $(f^{-1})_*(y) = (f_*(x))^{-1}$.

Here is an outline of how to prove this: To simplify notation let $g = f^{-1}$ and $T = (f_*(x))^{-1}$. We want show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|w| < \delta$ then

$$\frac{|g(y + w) - (g(y) + Tw)|}{|w|} < \epsilon.$$ 

If $w$ is small then $y + w \in V$ then there exists $v \in \mathbb{R}^n$ such that $x + v = g(y + w)$.

(a) Show that

$$-(g(y + w) - (g(y) + Tw)) = T(f(x + v) - (f(x) + f_*(x)v))$$

and conclude that

$$\frac{|g(y + w) - (g(y) + Tw)|}{|w|} < \|T\| \frac{|f(x + v) - (f(x) + f_*(x)v)|}{|w|}$$

where $\|T\| = \sup_{|v| \leq 1} |Tv|$.

(b) To finish the proof we need to show that there exists a neighborhood $V_0$ of $y$ and a $c > 0$ such that $|w| \geq c|v|$ for all $w$ with $y + w \in V_0$. Assume this is false and show that there is a sequence $\lambda_i w_i$ and $\sigma_i v_i$ with $|w_i| = |v_i| = 1$, $g(y + \lambda_i w_i) = x + \sigma_i v_i$ and $\lambda_i / \sigma_i \to 0$. On the other hand use the fact that $f$ is differentiable at $x$ to show that $\lim inf \lambda_i / \sigma_i > 0$ to obtain a contradiction and show that $c$ can be chosen with $2c = \lim inf \lambda_i / \sigma_i > 0$.

(c) Use (a) and (b) to show that if $w \in V_0$ then

$$\frac{|g(y + w) - (g(y) + Tw)|}{|w|} < \|T\| \frac{|f(x + v) - (f(v) + f_*(x)v)|}{c |v|}$$

and conclude the proof that $g = f^{-1}$ is differentiable at $y$.

6. For $i = 0, 1$ let $U_i \subset \mathbb{R}^k$ be open and $\phi_i : U_i \to \mathbb{R}^n$ smooth, injective maps such that $\phi_0(U_0) = \phi_1(U_1)$ and $(\phi_i)_*(x)$ is injective for all $i \in U_i$. Show that there exists a diffeomorphism $f : U_0 \to U_1$ such that $\phi_0 = \phi_1 \circ f$.