1. Let
\[ M = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \cdot x = y \cdot y = 1, x \cdot y = 0\} \]
where \( x \cdot y \) is the usual dot product on vectors in \( \mathbb{R}^3 \). Show that \( M \) is a submanifold of \( \mathbb{R}^3 \times \mathbb{R}^3 \). What is the dimension of \( M \)?

2. Let \( M, N \) and \( X \) be differentiable manifolds and \( Z \subset X \) a differentiable submanifold. Given \( x \in N \) let \( \iota_x : M \rightarrow M \times N \) be the inclusion map. Let \( F : M \times N \rightarrow X \) be differentiable and let \( f_x = F \circ \iota_x \). If \( M \) is compact and \( f_x \) is transverse to \( Z \) show that there is a neighborhood \( U \) of \( x \) in \( N \) such that if \( y \in U \) then \( f_y \) is transverse to \( Z \).

3. Let \( V(x) = \sum f_i(x) \frac{\partial}{\partial x_i} \) be a smooth vector field on \( \mathbb{R}^n \) and define \( \omega \in \Omega^{n-1}(\mathbb{R}^n) \) by \( \omega(x)(v_1, \ldots, v_{n-1}) = \det(V(x) v_1 \cdots v_{n-1}) \) where the right hand side is the determinant of the matrix of column vectors \( V(x), v_1, \ldots, v_{n-1} \). Show that \( d\omega = \sum \frac{\partial f_i}{\partial x_j} dx_1 \wedge \cdots \wedge dx_n \).

4. Let \( M \) be a differentiable manifold. Prove that its tangent bundle \( TM \) and and its cotangent bundle are isomorphic as smooth vector bundles.

5. Let \( W \) be a vector field on a smooth manifold \( M \) and assume that \( V \) has a flow on that is defined on all of \( M \) and for all time. Let \( V \) be another vector field on \( M \) such that \( V - W \) has compact support. Show that \( V \) has a flow on all of \( M \) defined for all time.

6. Let \( M = \mathbb{R}^2 \setminus \{(-1,0),(1,0)\} \). Let \( \iota_+ : S^1 \rightarrow M \) be a diffeomorphism from \( S^1 \) to the circle of radius 1 centered at \((1,0)\) and similarly define \( \iota_- : S^1 \rightarrow M \) with \((-1,0)\) the center of the circle. Define a map \( \phi : \Omega^1(M) \rightarrow \mathbb{R}^2 \) by
\[
\phi(\omega) = \left(\int_{S^1}(\iota_+)^*\omega, \int_{S^1}(\iota_-)^*\omega\right).
\]
Show that \( \phi \) is surjective and conclude that there is a surjective homomorphism from \( H^1(M) \) to \( \mathbb{R}^2 \). (There is, in fact, an isomorphism but you do not need to show this.)