Recall that $O(n)$ is the group of $n \times n$ matrices $A$ with $AA^T = I$ and that it is a differentiable manifold. Let $G(n) = \mathbb{R}^n \times O(n)$ where $(v_0, A_0) \cdot (v_1, A_1) = (v_0 + A_0 v_1, A_0 A_1)$. For $T = (v, A) \in G$ define (in abuse of notation) $T : \mathbb{R}^n \to \mathbb{R}^n$ by $Tx = Ax + v$.

1. If $T_0, T_1 \in G(n)$ then we can multiply them as elements of $G(n)$ and compose them as maps of $\mathbb{R}^n$. Show that $T_1 \cdot T_0 = T_1 \circ T_0$.

2. A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry if $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R}^n$. Show that every isometry of $\mathbb{R}^n$ is represented by an element in $G(n)$.
   (a) Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry. Show that there exist $T_0 \in G$ such that $T_0 \circ S(0) = 0$.
   (b) Let $e_i = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^n$ where the 1 is in the $i$th place. Find a $T_1 \in G$ such that $T_1 \circ T_0 \circ S(e_i) = e_i$ and $T_1 \circ T_0 \circ S(0) = 0$.
   (c) Show that $T_1 \circ T_0 \circ S(x) = x$ and therefore $S = (T_1 T_0)^{-1}$.

3. Let $U \subset \mathbb{R}^n$ be open and connected and $\phi : U \to \mathbb{R}^n$ an isometry onto its image. Show that $\phi$ is the restriction of an element in $G(n)$.

4. For all $x \in \mathbb{R}^n$ the tangent space $T_x \mathbb{R}^n$ can be canonically identified with $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth map. Show that $f$ is an isometry if for all $x \in \mathbb{R}^n$ and $v \in T_x \mathbb{R}^n$ then $|f_x(x)v| = |v|$. In particular, if $f_x(x) \in O(n)$ for all $x \in \mathbb{R}^n$ show that $f$ is an isometry and conclude that $f_x(x) \equiv A$ for some $A \in O(n)$.

5. Let $T_s$ be a smooth path in $G$ with $T_0 = I$. For each $x \in \mathbb{R}^n$, $\alpha_x(s) = T_s(x)$ is a smooth path. Let $V(x) = \alpha'_x(0)$. Then $V(x)$ is a vector field on $\mathbb{R}^n$. Show that $V(x) = Ax + v$ where $A$ is a skew-symmetric $n \times n$ matrix and $v \in \mathbb{R}^n$.

6. Given a vector field $V(x) = Ax + v$ of the above form show that there exists a flow $\phi_t$ for $V$ defined on all of $\mathbb{R}^n$ and for all time $t$. Further show that $\phi_t \in G(n)$. Here is one way to do this. Let $U \subset \mathbb{R}^n$ an open set with compact closure. Then $\phi_t$ exists for $t \in (-\epsilon, \epsilon)$. We'll show that $\phi_t$ is the restriction to $U$ of a path in $G(n)$.

   For $x \in U$, let $v \in T_x U$ and let $h_v(t) = |(\phi_t)_*(x)v|^2$. Since $\phi_0(x) = x$ we have that $h_v(0) = |v|^2$.

   We want to show that $h_v$ is constant and then $\phi_t$ is an isometry by (4).

   (a) We first calculate $h'_v(0)$. Let $B(t) = (\phi_t)_*(x)$. Show that

   $h_v(t) = v^T B(t)^T B(t) v$.

   (b) Let $\dot{B}$ be the derivative of $B(t)$ at $t = 0$. Using the fact that we can write $\phi_t(x) = x + t\psi_t(x)$ show that $\dot{B} = A$. Conclude that

   $h'_v(0) = v^T (\dot{B}^T B(0) + B(0)^T \dot{B}) v = v^T (A^T I + IA)v = 0$.

   (c) To calculate $h'_v(s)$ we replace $U$ with $W = \phi_s(U)$, $x$ with $y = \phi_s(x)$, $v$ with $w = (\phi_s)_*(x)v$ and the flow with $\phi_s \circ \phi_t \circ \phi_s^{-1}$. (Note that this last composition changes the domain of the flow. Where $U$ and $W$ intersect the two flows are equal.) We can then define $h_w$ as above. Show that $h'_v(t) = h'_w(0)$ and conclude that $h'_v \equiv 0$ and therefore $\phi_t$ is an isometry.
(d) Show that \( \phi_t \) can be extended to a flow of \( V \) on all of \( \mathbb{R}^n \) for all time.

7. Note that \( T_s \) is a smooth path in \( G(n) \) so its derivative at \( s = 0 \) determines a tangent vector \( \dot{T} \) in \( T_I(G(n)) \). Use (4) and (5) to show that the vector field \( V \) is determined by \( \dot{T} \).

8. Let \( g(n) \) be all vector fields of the above form. Show that \( g(n) \) is a vector space of dimension \( \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} \) and that the natural map from \( T_I G(n) \) to \( g(n) \) is an isomorphism.

9. Let \( S = (w, B) \in G(n) \). Define map \( ad_B : g(n) \to g(n) \) as follows. Given \( V \in g(n) \) there exists a path \( T_s \) in \( G(n) \) whose derivative when \( s = 0 \) is \( V \). Let \( \dot{T}_s = ST_sS^{-1} \) and let \( ad_B(V) \) be the time zero derivative of this path. Show that \( ad_B \) is well defined and linear. In particular if \( V(x) = Ax + v \) show that

\[
ad_B(V)(x) = BAB^{-1}x - BAB^{-1}w + Bv.
\]

10. Now let \( n = 2 \) and define a basis for \( g(2) \) by \( V_1(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} x \), \( V_2(x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \) and \( V_3(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Using (8) show that in this basis

\[
ad_B = \begin{pmatrix} \det B & 0 \\ -\det Bw^\perp & B \end{pmatrix}
\]

where \( w^\perp = V_1(w) \) is a \( \pi/2 \)-counter clockwise rotation of \( w \). Note that this is a \( 3 \times 3 \) matrix and in this way \( G(2) \) can be represented as a group of matrices. (With more work we could do this for any \( G(n) \).)