Let \([[0, 1], \mathcal{M}, m]\) be the standard Lebesgue measure space on \([0, 1]\) and let \(L^2([0, 1]) = L^2(m)\) (using the notation from Rudin).

1. Define \(T : L^2([0, 1]) \to L^2([0, 1])\) by \((Tf)(t) = tf(t)\). Show that:
   (a) \(T\) is bounded and symmetric with \(\|T\| = 1\);
   (b) has no eigenvalues;
   (c) and \(T - \lambda I\) is surjective if and only if \(\lambda \not\in [0, 1]\).

Now let \([[0, 1]^n, \mathcal{M}, m]\) be Lebesgue measure on the cube in \(\mathbb{R}^n\) and let \(L^2([0, 1]^n) = L^2(m)\). We want to find an orthonormal basis for \(L^2([0, 1]^n)\). Our strategy will be the same as it was for \(L^1([0, 1])\).

We can also define Fourier series for periodic functions on \(\mathbb{R}^n\). (If you like you can simplify things by assuming \(n = 2\).)

We need a bit of notation. If \(\xi \in \mathbb{Z}^n\) and \(x \in \mathbb{R}^n\) then \(\xi \cdot x = \xi_1x_1 + \cdots + \xi_nx_n\). Let \(|\xi|_\infty = \max\{|\xi_1|, \ldots, |\xi_n|\}\).

A function \(f : \mathbb{R}^n \to \mathbb{C}\) is periodic if \(f(x) = f(x + \xi)\) for all \(x \in \mathbb{R}^n\) and \(\xi \in \mathbb{Z}^n\). Let \(C(\mathbb{T}^n)\) be continuous periodic functions on \(\mathbb{R}^n\). Of course, any function in \(C(\mathbb{T}^n)\) can be restricted to a function in \(L^2([0, 1]^n)\). We want to show that the functions \(e_\xi(x) = e^{2\pi i \xi \cdot x}\) are an orthonormal basis for \(L^2([0, 1]^n)\).

2. Show that \((e_{\xi_0}, e_{\xi_1}) = 1\) if \(\xi_0 = \xi_1\) and \((e_{\xi_0}, e_{\xi_1}) = 0\) if \(\xi_0 \neq \xi_1\).

3. Show that \(e_\xi(x)e_\xi(y) = e_\xi(x + y)\).

4. (Optional) Let
   \[D_N(x) = \sum_{k=-N}^{N} e^{2\pi ikx}\]
   and show that
   \[D_N(x) = \frac{\sin(\pi(2N+1)x)}{\sin(\pi x)}\].

5. (Optional) Let
   \[K_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} D_k(x)\]
   and show that
   \[K_N(x) = \frac{1}{N} \left(\frac{\sin(N\pi x)}{\sin(\pi x)}\right)^2\].
6. Let
\[ K_N(x) = \frac{1}{N^n} \prod_{j=1}^{n} \sum_{M=0}^{N-1} \sum_{k=-M}^{M} e^{2\pi i k x_j}. \]
Show that
\[ K_N(x) = \frac{1}{N^n} \prod_{j=1}^{n} \left( \frac{\sin(N \pi x_j)}{\sin(\pi x_j)} \right)^2 \]
and that \( K_N(x) \) is a finite linear combination of \( e_{\xi}(x) \). Conclude that
\[ f_N(x) = \int_{[0,1]^n} f(y) K_N(x-y) dm(y) \]
is a finite linear combination of \( e_{\xi}(x) \).

7. Periodic functions \( \Phi_k \in C(\mathbb{T}^n) \) form an approximate identity if
(a) \( \int_{[0,1]^n} \Phi_k(x) dm(x) = 1; \)
(b) \( \sup_n \int_{[0,1]^n} |\Phi_k(x)| dm(x) < \infty; \)
(c) For all \( \delta > 0, \int_{1/2 > |x| > \delta} |\Phi_k(x)| dm(x) \to 0. \)
If \( \Phi_k \) is an approximate identity and \( f \) is continuous show that
\[ \int_{[0,1]^n} f(y) \Phi_k(x-y) dm(y) \to f(x) \]
uniformly.

8. Show that \( K_N(x) \) is an approximate identity and therefore if \( f \) is in \( C(\mathbb{T}^n) \) show that \( f_N \) converges to \( f \) uniformly. Conclude that the \( e_{\xi} \) are an orthonormal basis for \( L^2([0,1]^n) \).

**Hint:** If \([a, b]\) is an interval show that
\[ \int_{[a,b] \times [-1/2,1/2]^{n-1}} K_N(x) dm(x) = \int_{a}^{b} K_N(x) dm(x) \]
and therefore for all \( \delta > 0 \) we have
\[ \lim_{N \to \infty} \int_{[\delta,1/2] \times [-1/2,1/2]^{n-1}} K_N(x) dm(x) \to 0 \]
with the same statement holding if we replace \([\delta,1/2]\) with \([-1/2,-\delta]\). Use the fact that the set of \( x \) with \( 1/2 > |x| > \delta \) can be covered by \( 2n \) sets of the form \([\delta',1/2] \times [-1/2,1/2]^{n-1}\) or \([-1/2,-\delta'] \times [-1/2,1/2]^{n-1}\) for some \( \delta' > 0 \) to show that \( K_N(x) \) satisfies (c).

9. Let \( H^s_\xi \) be sequences \( u_{\xi} \) indexed by \( \xi \in \mathbb{Z}^n \) such that
\[ \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^{s} |u_{\xi}|^2 < \infty. \]
Show that if \( s \geq |n/2| + 1 \) and \( u \in H^s_\xi \) then the series \( \sum_{\xi} |u_{\xi}| \) converges. Conclude that \( \sum_{\xi} u_{\xi} e_{\xi}(x) \) converges to a continuous function. (**Hint:** Write \( |u_{\xi}| = (1 + |\xi|^2)^{-s/2}(1 + |\xi|^2)^{s/2}|u_{\xi}| \) and apply the Cauchy-Schwarz inequality. You will use the assumption that \( s \geq |n/2| + 1 \) to show that \( \sum (1 + |\xi|^2)^{-s} \) converges.)
If \( s \geq |n/2| + m + 1 \) show \( \sum_{\xi} u_{\xi} e_{\xi}(x) \) has partial derivatives of all order \( \leq m. \)
10. We can make $H^n_s$ into a Hilbert space by defining the inner product

$$(u, v)_s = \sum_{\xi \in \mathbb{Z}^n} (1 + |\xi|^2)^s u_{\xi} \overline{v_{\xi}}.$$

If $s > t$ show that the inclusion of $H^n_s$ in $H^n_t$ is a compact operator.